Inlmningsuppgift 1 : Lsningar

1. Lt A, B, C resp. D beteckna antalet tips rader som har exakt 10,11,12 resp. 13 rätt. Vi söker $A + B + C + D$. Men, t.ex., $A = C(13,3) \cdot 2^3 = 2288$ eftersom vi måste först välja vilka 3 rader som ska fyllas i fel och har 2 val för hur vi ska fylla i var och en av dessa tre rader.

På samma sätt har vi att $B = C(13,2) \cdot 2^2 = 312$, $C = C(13,1) \cdot 2^1 = 26$ och $D = C(13,0) \cdot 2^0 = 1$. Lägger vi ihop så far vi att svaret på uppgiften är

$$2288 + 312 + 26 + 1 = 2627$$

olika tips rader.

2. Twenty-four indistinguishable objects (the goals) are to be placed in 21 distinguishable cells (the players). The number of ways this can be done is

$$\binom{24+21-1}{21-1} = \binom{44}{20}.$$

3. There is one term corresponding to each non-empty subset of $\{1, ..., n\}$, namely the intersection of the corresponding A_i. There are 2^n subsets of an n-element set, hence $2^n - 1$ non-empty subsets.

4. Let x be any element of X. We can describe an explicit one-to-one correspondence between 'odd' and 'even' subsets of X as follows (A denotes a subset of X) :

$$A \leftrightarrow A \cup \{x\} \ \text{ if } x \notin A,$$
$$A \leftrightarrow A - \{x\} \ \text{ if } x \in A.$$

5. The sum simplifies to $n \cdot 2^{n-1}$. There are many ways to prove this. I will present just two proofs, which I think are nice (and quite different from one another !).

First Proof :

For each subset of $\{1, ..., n\}$ list its' elements. The sum counts the total number of numbers in the entire list. On the other hand, each of the numbers from 1 to n occurs in exactly half the subsets, that is 2^{n-1} subsets, and hence appears 2^{n-1} times in our list. Hence, the total number of numbers in the list is also $n \cdot 2^{n-1}$, v.s.v.

SECOND PROOF :

The binomial theorem can be written as

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k.$$

Differentiate both sides w.r.t. x. We get that

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k \cdot \binom{n}{k} \cdot x^{k-1}.$$

If we now put $x = 1$, then we get exactly what we want !!

6 (i) Denote by q_n the number of possibilities when there are n spaces. Observe that $q_0 = q_1 = 0$ and $q_2 = 1$. Now let $n \geq 3$. If we start with a motorcycle, then this takes two places and there are q_{n-2} possible ways to fill the remaining places. On the other hand, if we start with a car, then this takes three places and there are q_{n-3} possililities for the remaining places.

Thus we obtain the recurrence relation

$$q_n = q_{n-2} + q_{n-3}, \quad \forall\, n \geq 3.$$

(ii) For example, one can state that there are two different models of motorcycle, three different models of car, that a cycle takes up one place and a car two.

The recurrence is solved in the usual way. The characteristic equation is

$$x^2 = 2x + 3,$$

which has the two solutions $x = 3$, $x = -1$. Thus the general solution is

$$q_n = C_1 \cdot 3^n + C_2 \cdot (-1)^n.$$

We now insert the initial oonditions :

$$n = 1 \Rightarrow q_1 = 2 = 3C_1 - C_2, \tag{1}$$
$$n = 2 \Rightarrow q_2 = 7 = 9C_1 + C_2. \tag{2}$$

Solving (1) and (2) we get $C_1 = \frac{3}{4}$, $C_2 = \frac{1}{4}$. Thus the answer is

$$q_n = \frac{1}{4}\left(3^{n+1} + (-1)^n\right).$$

7 (i) I think the hint given in the book basically solves the problem.
(ii) Let X denote the set of all non-negative integer solutions to $a+b+c = 21$, and define three subsets of X as follows :

$$\begin{aligned} A &:= \{(a,b,c) \in X : a \geq 11\}, \\ B &:= \{(a,b,c) \in X : b \geq 8\}, \\ C &:= \{(a,b,c) \in X : c \geq 13\}. \end{aligned}$$

Then the quantity we must compute is

$$|X| - |A \cup B \cup C|. \tag{3}$$

By part **(i)** we have that

$$|X| = \binom{23}{2} = 253.$$

By the sieve principle, we know that

$$\begin{aligned} |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| \\ -|A \cap C| - |B \cap C| + |A \cap B \cap C|. \end{aligned} \tag{4}$$

Each of the terms on the HL of (4) can also be computed using part **(i)**. Take $|A|$ for example. Put $a_1 := a - 11$. Then $|A|$ is just the number of non-negative integer solutions to $a_1 + b + c = 21 - 11 = 10$. According to part **(i)**, we thus have

$$|A| = \binom{12}{2} = 66.$$

Similarly,

$$\begin{aligned} |B| = C(15,2) = 105, \quad |C| = C(10,2) = 45, \\ |A \cap B| = C(4,2) = 6, \quad |B \cap C| = C(2,2) = 1, \\ |B \cap C| = |A \cap B \cap C| = 0. \end{aligned}$$

Substituting everything into (3) and (4), we find that the answer to the question is

$$253 - (66 + 105 + 45) + (6 + 1) = 44,$$

i.e.: there are 44 possible solutions.

Bonus problems

8. The formula for the number of regions is

$$1 + \binom{n}{2} + \binom{n}{4}.$$

PROOF : Before we start drawing chords, the circle is one region. We get one extra region
(a) for each chord and
(b) for each crossing between a pair of chords.

The number of chords is just the number of pairs of points, which is just $\binom{n}{2}$. And there is one crossing for each quadruple of points, so the number of those is just $\binom{n}{4}$. Hey presto !

9 (i) To get a recurrence relation for A_n, we count the number of AP:s among $\{1, ..., n\}$ which include the number n. This is just the number of triples $(n - 2i, n - i, n)$ such that $0 < 2i < n$.

If n is even, say $n = 2k$, then $2i < n \Rightarrow i < k \Rightarrow 1 \leq i \leq k - 1$. Thus,

$$A_{2k} = A_{2k-1} + (k - 1). \tag{5}$$

If n is odd, say $n = 2k + 1$, then $2i < n \Rightarrow i \leq k$. Thus,

$$A_{2k+1} = A_{2k} + k. \tag{6}$$

(ii) From (5) and (6) we find that, for all $n \geq 1$, both odd and even,

$$A_{n+2} = A_n + n.$$

If n is even, say $n = 2k$, we thus have that

$$A_{2k} = 2 \cdot \sum_{i=1}^{k-1} i = k(k-1) = \frac{n^2 - 2n}{4}, \tag{7}$$

whereas if $n = 2k + 1$ is odd, then

$$A_{2k+1} = \sum_{i=0}^{k-1} (2i+1) = k(k-1) + k = k^2 = \frac{n^2 - 2n + 1}{4}. \tag{8}$$

A single formula valid for all n is thus

$$A_n = \lfloor \frac{n^2 - 2n}{4} \rfloor.$$

ALTERNATIVE APPROACH :

We can get a formula for A_n without using the recurrence relation. We note that an AP (a, b, c) is completely determined by the choice of a and c, and that the difference between these two numbers must be even, i.e.: they have the same parity. Hence the number of AP:s among $\{1, ..., n\}$ equals the number of ways to choose a pair of numbers from the set which have the same parity. I'll leave it for you to work out the details of how this leads one to the formulae (7) and (8).

10. Let the numbers be

$$a_1, a_2, \cdots a_{n^2}, a_{n^2+1}.$$

Suppose there is no decreasing subsequence of length $n + 1$. Then we must show that instead there is an increasing subsequence of this length. For each $i = 1, 2, ..., n^2 + 1$, let L_i be the length of the longest decreasing subsequence which begins with a_i. By assumption, each L_i is one of the numbers $1, 2, ..., n$. But there are $n^2 + 1$ of them so, by the Pigeonhole principle, there is some collection of $n + 1$ of the L_i which are all equal, say

$$L_{i_1} = L_{i_2} = \cdots = L_{i_{n+1}}, \quad \text{where } i_1 < i_2 < \cdots < i_{n+1}.$$

But then, if you think for a few seconds, you'll realise that the subsequence

$$a_{i_1}, a_{i_2}, \cdots, a_{i_{n+1}}$$

is an increasing subsequence of length $n+1$. The proof is complete.

11. The maximum size of a sum-free subset of $\{1, ..., n\}$ is $\lceil n/2 \rceil$. Two examples of a sum-free set of this size are

(i) all the odd numbers between 1 and n,
(ii) the numbers from $\lceil n/2 \rceil$ up to n.

One can see that one can't do any better as follows : let A be a sum-free subset of $\{1, ..., n\}$. Let m be the largest number in A. Let

$$B = \{m - a : a \in A,\ a \neq m\}.$$

Then $|B| = |A| - 1$. But since A is sum-free, the sets A and B must be disjoint subsets of $\{1, ..., n\}$. Thus

$$n \geq |A| + |B| = 2|A| - 1 \Rightarrow |A| \leq \frac{n+1}{2} \Rightarrow |A| \leq \lceil \frac{n}{2} \rceil,$$

since $|A|$ is, a priori, an integer.