Inlämningsuppgift 1 : Lösningar

Q.1 (i) One wins if one picks either 3,4,5 or 6 correct numbers. There are, for example, $\binom{6}{3} \cdot \binom{33}{3}$ ways to pick exactly 3 correct numbers, since one must pick 3 of the 6 correct numbers and 3 of the 33 wrong numbers. Similarly for the other possibilities .... Hence, the total number of ways one can win is

$$\sum_{k=3}^{6} \binom{6}{k} \cdot \binom{33}{6-k} = 20 \cdot 5456 + 15 \cdot 528 + 6 \cdot 33 + 1 \cdot 1 = 117239.$$

(ii) There are $\binom{39}{6}$ possible sets of winning numbers and, of these, $\binom{20}{6}$ consist entirely of odd numbers. Hence, the probability that all six winning numbers are odd is

$$\frac{\binom{20}{6}}{\binom{39}{6}} \approx 0.012.$$

Q.2 (i) 26 indistinguishable objects (the goals) are to be distributed among 10 distinguishable cells (the matches). The number of ways this can be done is

$$\binom{26+10-1}{10-1} = \binom{35}{9}.$$

(ii) First of all, there are $\binom{10}{2}$ choices for the two goalless matches. Next, there are $\binom{8}{2}$ choices for the two matches in each of which one goal was scored. Finally, 24 goals are to be distributed among the remaining 6 matches, but each must get at least 2 goals, so we can freely distribute only $24 - 6 \cdot 2 = 12$ goals. This can be done in $\binom{12+6-1}{6-1} = \binom{17}{5}$ ways.

Putting everything together, and by the multiplication principle, the total number of possible distributions of the entire 26 goals is

$$\binom{10}{2} \cdot \binom{8}{2} \cdot \binom{17}{5} = 7,796,880.$$

Q.3 Let $n \geq 4$ and let $a_1 a_2 \cdots a_n$ be any such binary word (so $a_i \in \{0,1\}$). Put $a_0 := 1$ and $a_{n+1} := 0$. Now the point is that the fact that there are exactly two occurrences of the pattern '01' is equivalent to there being exactly 5 values of $i \in \{0, 1, ..., n\}$ such that $a_i \neq a_{i+1}$. Hence the number of possible words is just equal to the number of ways of choosing these 5 values of i from the set $\{0, 1, ..., n\}$, namely $\binom{n+1}{5}$.

Example : $n = 15$ and I choose $i = 2, 5, 7, 8, 13$. The corresponding binary word is

$$110001101111100.$$

Q.4 3^n is just the number of ternary words (i.e.: words in the alphabet $\{0, 1, 2\}$) of length n. A typical term in the sum is $\binom{n}{k} \cdot \binom{n-k}{l}$ for some pair (k, l) such that $0 \leq k \leq n$ and $0 \leq l \leq n - k$. This term simply represents the number of length-n ternary strings consisting of k zeroes and l ones, since one may first choose the locations of the zeroes (k locations from n) and then those of the ones (l locations from the remaining $n - k$).

Q.5 (i) Take

$$X_n := \{m : 1 \leq m \leq n \text{ and } m = 4^k u \text{ for some } k \geq 0 \text{ and } u \text{ odd}\}.$$

Clearly, X_n contains no two elements of which one is twice the other. We have

$$X_n = \bigsqcup_{k=0}^{\lfloor \log_4 n \rfloor} X_n^k,$$

where X_n^k is defined as X_n except that we fix a value for k. In addition, we clearly have

$$|X_n^k| = \lceil \frac{\lceil n/4^k \rceil}{2} \rceil.$$

Hence, asymptotically, it is clear that

$$\frac{|X_n|}{n} \to \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{4^k} = \frac{1}{2}\times\frac{4}{3} = \frac{2}{3}, \quad \text{v.s.v.}$$

(ii) Note that there was a small error in the hint, as will become clear from the argument below. For each $k \geq 0$ let

$$X^k := \{4^k u : u \text{ is odd or } u/2 \text{ is odd}\}.$$

Clearly the sets X^k partition $\mathbf{N}$. Fix $n > 0$. Let $X^{k,n} := X^k \cap \{1, ..., n\}$. In the notation of part **(i)** we have that

$$X^{k,n} = X_n^k \sqcup 2 \cdot \overline{X}_n^k, \tag{1}$$

where $\overline{X}_n^k$ is the 'lower half' of X_n^k. Let $A_n \subseteq \{1, ..., n\}$ be a set not consisting of any two elements of which one equals twice the other. Write

$$A_n = \bigsqcup_{k=0}^{\lfloor \log_4 n \rfloor} A_n^k,$$

where $A_n^k := A_n \cap X^k$. From (1) and the requirement on A_n we easily deduce that, for each k,

$$|A_n^k| \leq \frac{2}{3}|X^{k,n}| + 1, \tag{2}$$

since for each $x \in \overline{X}_n^k$, at most one of x and $2x$ can lie in A_n. Summing (2) over k we find that

$$|A_n| \leq \frac{2n}{3} + 1 + \lfloor \log_4 n \rfloor,$$

from which the desired result follows.

Q.6 Let q_n denote the number of possible stacks of n chips. Clearly $q_1 = 4$ (any of the 4 colors will do) and $q_2 = 15$ (of the $4 \cdot 4 = 16$ possible 2-chip stacks only the stack consisting of two blue chips is disallowed). We shall explain why, for each $n \geq 3$,

$$q_n = 3q_{n-1} + 3q_{n-2}. \tag{3}$$

When solving this recurrence and inserting initial conditions, it will simplify the algebra to extend (3) to $n = 2$, which requires us to define $q_0 := 1$.

The explanation of (3) is as follows : consider a permissible stack of n chips. If the first chip is blue, then the second must be in one of the 3 other colors, and there then remain q_{n-2} ways to insert the remaining $n-2$ chips in the stack. Hence, there are $3q_{n-2}$ stacks of height n whose first chip is blue. If the first chip is not blue, then it can come in any of 3 colors, and there then remain q_{n-1} ways to choose the remaining $n-1$ chips. Thus there are $3q_{n-1}$ stacks of height n whose first chip is not blue. This discussion explains (3).

We proceed to solve the recurrence. The characteristic equation is

$$a^2 - 3a - 3 = 0,$$

which has the two roots

$$a = \frac{3 \pm \sqrt{21}}{2}.$$

Hence the general solution of (3) is

$$q_n = K_1 \left(\frac{3 + \sqrt{21}}{2} \right)^n + K_2 \left(\frac{3 - \sqrt{21}}{2} \right)^n.$$

Inserting $n = 0$ we must have

$$1 = K_1 + K_2, \tag{4}$$

while inserting $n = 1$ we must have

$$4 = \left(\frac{3 + \sqrt{21}}{2} \right) K_1 + \left(\frac{3 - \sqrt{21}}{2} \right) K_2. \tag{5}$$

Solving (4) and (5) gives

$$K_1 = \frac{\sqrt{21} + 5}{2\sqrt{21}}, \quad K_2 = \frac{\sqrt{21} - 5}{2\sqrt{21}}.$$

Hence the final answer is

$$q_n = \frac{1}{2^{n+1}\sqrt{21}} \times \left[(5 + \sqrt{21}) \cdot (3 + \sqrt{21})^n - (5 - \sqrt{21}) \cdot (3 - \sqrt{21})^n \right].$$

Q.7 Let X be the set of all permutations of $1, 2, ..., 100$. Thus $|X| = 100!$. We have to remove the following three subsets of X :

$$A := \{\pi \in X : \pi(1) \text{ is next to } \pi(2)\},$$
$$B := \{\pi \in X : \pi(1) \text{ is next to } \pi(3)\},$$
$$C := \{\pi \in X : \pi(2) \text{ is next to } \pi(3)\}.$$

Hence the final answer will be $|X| - |A \cup B \cup C|$. By the sieve principle,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

In addition, by symmetry,

$$|A| = |B| = |C|$$

and

$$|A \cap B| = |A \cap C| = |B \cap C|.$$

Now for the computations. First, I claim that $|A| = 2 \cdot 99!$. For we can consider the pair $1, 2$ as a single number (like they're glued together), so that we're left with 99 numbers to permute freely. Every such permutation gives rise to two permutations of the original 100 numbers in which 1 is next to 2, since the internal order of 1 and 2 still needs to be determined.

Similarly, $|A \cap B| = 2 \cdot 98!$. Since we first glue $1, 2$ and 3 together as a single number, leaving us with 98 numbers to permute freely. If 1 is to be next to 2 and also next to 3, then there are two possibilities for the internal order of 1,2 and 3, namely 312 or 213.

Finally, $|A \cap B \cap C| = 0$ since there's no way to place three objects in a line so that each is next to the other.

Putting everything together, the final answer is

$$100! - 6 \cdot 99! + 6 \cdot 98! = (99 \cdot 100 - 6 \cdot 99 + 6) \cdot 98! = 9312 \cdot 98! = 96 \cdot 97 \cdot 98!$$

Obs! I'll leave it as an exercise to explain 'directly' why the final answer is $96 \cdot 97 \cdot 98!$.