Att lämna in torsdag den 29 september

REGLER : Du får full poäng för korrekta lösningar till 6 valfria uppgifter. Du kan få ett bonus av upp till 30 procent genom att lösa 8 valfria uppgifter.

1. Solve the recurrence relation

$$2a_n = 9a_{n-1} - 4a_{n-2} + 4^n + n \quad \forall\, n \geq 2,$$
$$a_0 = 1, \quad a_1 = 1.$$

2. For each $n \geq 0$, let D_n be the number of Dyck paths from $(0,0)$ to $(2n+2,0)$ which have no peaks of height two. Show that $D_n = C_n$, the n:th Catalan number, for every n.

(Tips : Establish a 1-1 correspondence between these special paths of length $2n+2$ and all Dyck paths of length $2n$).

3. For each $n \geq 0$ let E_n be the number of ways of connecting $2n$ points in the plane lying in a horizontal line by n nonintersecting arcs, each of which connects two of the points and lies above them. Show that $E_n = C_n$ for every n.

4. Determine the number of ways of representing 97111014 as a product of four positive integers, each greater then one, if no attention is paid to the internal order of the integers in these representations.

5. Determine all solutions to the Diophantine equation

$$18x + 47y = 3000.$$

In particular, determine all solutions for which $x > 0$ and $y > 0$.

6. Let n be a positive integer. Prove that $n! + 1$ and $(n+1)! + 1$ are always relatively prime.

7. Determine with proof all integers n for which $n^4 + n^2 + 1$ is prime.

8. The *least common multiple* of two positive integers a and b is the smallest positive integer n which is divisible by both a and b.

Use the FTA to explain why, for any two positive integers a and b,

$$\mathrm{LCM}(a,b) \times \mathrm{GCD}(a,b) = a \cdot b.$$

9. Let p be a prime and let $0 < i < p$. Explain why the binomial coefficient $\binom{p}{i}$ is a multiple of p. Give an example to show this result does not hold if p is not a prime.

10. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be any polynomial with integer coefficients (i.e.: $a_i \in \mathbf{Z}$ for $i = 0, ..., n$) and degree at least 1 (i.e.: $n \geq 1$). Prove that there are infinitely many integers t for which $p(t)$ is not prime.