TMA 055 : Diskret matematik

Tentamen 140106

Lösningar

F.1 *Step 0* : Since $3 \cdot 4 \equiv 1 \pmod{11}$, the first congruence can be rewritten as

$$x \equiv 6 \pmod{11}.$$

Since $2 \cdot 8 \equiv 1 \pmod{15}$, the second congruence can be rewritten as

$$x \equiv 6 \pmod{15}.$$

Since $3 \cdot 11 \equiv 1 \pmod{16}$, the third congruence can be rewritten as

$$x \equiv 11 \equiv -5 \pmod{16}.$$

Step 1 : We compute the inverse of $15 \cdot 16$ modulo 11. Since $15 \cdot 16 \equiv 4 \cdot 5 \equiv -2 \pmod{11}$, we seek a solution to

$$-2a_1 \equiv 1 \pmod{11},$$

and a solution is $a_1 = 5$.

Step 2 : Compute the inverse of $11 \cdot 16$ modulo 15. Since $11 \cdot 16 \equiv -4 \cdot 1 \equiv -4 \pmod{15}$, we must solve

$$-4a_2 \equiv 1 \pmod{15}.$$

A solution is $a_2 = -4$.

Step 3 : Compute the inverse of $11 \cdot 15$ modulo 16. Since $11 \cdot 15 \equiv (-5) \cdot (-1) \equiv 5 \pmod{16}$, we must solve

$$5a_3 \equiv 1 \pmod{16}.$$

A solution is $a_3 = -3$.

Step 4 : A solution to the three congruences is given by

$$\begin{aligned} x = 6 \cdot a_1 \cdot (15 \cdot 16) + 6 \cdot a_2 \cdot (11 \cdot 16) - 5 \cdot a_3 \cdot (11 \cdot 15) \\ = 6 \cdot 5 \cdot 15 \cdot 16 - 6 \cdot 4 \cdot 11 \cdot 16 + 5 \cdot 3 \cdot 11 \cdot 15 \\ = 5451. \end{aligned}$$

Step 5 : The general solution is

$$x = 5451 + (11 \cdot 15 \cdot 16) \cdot n = 5451 + 2640n,$$

where n is an arbitrary integer. In particular, the smallest positive solution is $x = 171$, got by taking $n = -2$.

F.2 The homogeneous equation is

$$u_n - 5u_{n-1} + 4u_{n-2} = 0.$$

The characteristic equation for this is

$$x^2 - 5x + 4 = 0,$$

which factorises as

$$(x-1)(x-4) = 0,$$

and hence has the roots $x = 1$ and $x = 4$. Hence the general solution to the homogeneous equation is

$$u_n^h = C_1 + C_2 \cdot 4^n.$$

Since C_1 is already a solution to the homogeneous equation, our guess for a particular solution should have the form

$$u_n^p = An^2 + Bn.$$

Substituting into the recurrence relation, we obtain $A = -1/6$ and $B = -11/18$. Hence the general solution to our recurrence relation is

$$u_n = C_1 + C_2 \cdot 4^n - \frac{n^2}{6} - \frac{11n}{18}.$$

It remains to insert the initial conditions :

$$n = 0 \Rightarrow u_0 = 1 = C_1 + C_2,$$
$$n = 1 \Rightarrow u_1 = 2 = C_1 + 4C_2 - \frac{1}{6} - \frac{11}{18}.$$

Solving, we obtain $C_1 = 11/27$, $C_2 = 16/27$. Hence the final answer is

$$u_n = \frac{11}{27} + \frac{16}{27} \cdot 4^n - \frac{n^2}{6} - \frac{11n}{18}.$$

F.3 $99 = 3^2 \cdot 11$ so $\phi(99) = \phi(3^2) \cdot \phi(11) = (3^2 - 3)(11 - 1) = 6 \cdot 10 = 60$. Hence, Euler's Theorem states that, if n is an integer relatively prime to 99, then

$$n^{60} \equiv 1 \pmod{99}.$$

Note that both 5 and 17 are relatively prime to 99. Hence (all congruences are modulo 99)

$$5^{183} = (5^{60})^3 \cdot 5^3 \equiv 1^3 \cdot 125 \equiv 125 \equiv 26,$$

and

$$17^{121} = (17^{60})^2 \cdot 17 \equiv 1^2 \cdot 17 \equiv 17.$$

Thus,

$$(5^{183} + 17^{121})^{59} \equiv (26 + 17)^{59} = 43^{59}.$$

Since 43 is a prime, it is also relatively prime to 99, so we can apply Euler's theorem again and deduce that

$$43^{59} = 43^{60-1} \equiv 43^{-1}.$$

So it remains to compute the inverse of 43 modulo 99. Vi hittar inversen genom Euklides algoritm. Framåt får vi

$$\begin{aligned} 99 &= 2 \cdot 43 + 13, \\ 43 &= 3 \cdot 13 + 4, \\ 13 &= 3 \cdot 4 + 1, \\ 4 &= 4 \cdot 1 + 0. \end{aligned}$$

Bakåt får vi då

$$\begin{aligned} 1 &= 13 - 3 \cdot 4 \\ &= 13 - 3 \cdot (43 - 3 \cdot 13) \\ &= 10 \cdot 13 - 3 \cdot 43 \\ &= 10 \cdot (99 - 2 \cdot 43) - 3 \cdot 43 \\ &= 10 \cdot 99 - 23 \cdot 43. \end{aligned}$$

Från den sista raden härleder vi att $43^{-1} \equiv -23 \equiv 76 \pmod{99}$.

So the answer is 76.

F.4 (i) Clearly, $\chi(G) \geq 3$ since G contains many triangles. In fact, $\chi(G) \geq 4$ because the vertex b is at the centre of a wheel formed by v, a, e, f, c. This 5-cycle requires three colours, and then a fourth is needed for b. Similarly, f is at the centre of a 5-cycle formed by b, e, i, j, c. On the other hand, the graph is plane, hence $\chi(G) \leq 4$, by the Four-Colour Theorem. It follows that $\chi(G) = 4$.

If we apply the greedy algorithm with the nodes ordered $w, j, i, h, g, f, e, d, c, b, a, v$, then we get a 4-coloring, namely (the colors are $1, 2, 3, 4$)

v	2	f	1
a	1	g	1
b	3	h	2
c	4	i	3
d	3	j	2
e	2	w	1

(ii) Apply Dijkstra's algorithm to build up the following tree

Step	Choice of edge	Labelling
1	$\{v, c\}$	$c := 3$
2	$\{v, a\}$	$a := 4$
3	$\{a, b\}$	$b := 4$
4	$\{b, f\}$	$f := 8$
5	$\{a, d\}$	$d := 8$
6	$\{a, e\}$	$e := 8$
7	$\{d, g\}$	$g := 10$
8	$\{d, h\}$	$h := 11$
9	$\{c, j\}$	$j := 11$
10	$\{f, i\}$	$i := 12$
11	$\{i, w\}$	$w := 21$

Hence the shortest path from v to w is the path $v \to a \to b \to f \to i \to w$ and has length 21.

F.5 (i) There are $\binom{20}{4}\binom{16}{6}$ ways to choose which four throws give a 1 and which six give either 2 or 3. For any such assignment, the probability of the corresponding outcome is

$$\left(\frac{1}{6}\right)^4 \times \left(\frac{1}{3}\right)^6 \times \left(\frac{1}{2}\right)^{10}.$$

Hence, the answer is

$$\binom{20}{4} \times \binom{16}{6} \times \left(\frac{1}{6}\right)^4 \times \left(\frac{1}{3}\right)^6 \times \left(\frac{1}{2}\right)^{10}.$$

(ii) The idea is to show that every pair of numbers in the sequence are relatively prime. Since each has at least one prime factor, this immediately proves the existence of infinitely many primes. That the numbers are pairwise relatively prime is easily shown by a double induction argument. The idea is as follows : let $m \leq n$ be given and suppose p is a common prime divisor of $2^{2^m} + 1$ and $2^{2^n} + 1$. Then p also divides their difference, namely $2^{2^m}(2^{2^{n-m}} + 1)$. The only prime divisor of the first factor here is $p = 2$, which is impossible since all the numbers in our sequence are odd. Hence p must also divide $2^{2^{n-m}} + 1$. So now you see where the induction comes in ....

(iii) Let v be any node of G. Let $\mathcal{X}$ be its' set of neighbours. Then $|\mathcal{X}| \geq d$. Let $\mathcal{Y}$ be the collection of all neighbours to the nodes of $\mathcal{X}$, other than v. Since g has girth 5, no two nodes in $\mathcal{X}$ are joined to one another (this would give a cycle of length 3), nor do they share any neighbours in $\mathcal{Y}$ (this would give a cycle of length 4). The first assertion implies that $\mathcal{Y}$ is disjoint from $\mathcal{X}$. The second implies, since all nodes have degree at least d, that $|\mathcal{Y}| \geq (d-1)|\mathcal{X}| \geq d(d-1)$.

Hence the number of vertices in G is at least $1 + |\mathcal{X}| + |\mathcal{Y}| \geq 1 + d + d(d-1) = d^2 + 1$, v.s.v.