TMA 055 : Diskret matematik

Tentamen 290806

Lösningar

F.1 Det är lätt att se att $\mathrm{SGD}(20, 33) = 1$ genom att snabbt faktorisera bägge talen : $20 = 2^2 \cdot 5$ och $33 = 3 \cdot 11$. Därför vet vi att det finns heltal x_0, y_0 så att

$$20x_0 + 33y_0 = 1. \tag{1}$$

Vi hittar först en lösning till (1) genom att köra Euklides algoritm fram och tillbaka. Framåt får vi

$$\begin{aligned} 33 &= 1 \cdot 20 + 13, \\ 20 &= 1 \cdot 13 + 7, \\ 13 &= 1 \cdot 7 + 6, \\ 7 &= 1 \cdot 6 + 1, \\ 6 &= 6 \cdot 1 + 0. \end{aligned}$$

Bakåt får vi då

$$\begin{aligned} 1 &= 7 - 6 \\ &= 7 - (13 - 7) \\ &= 2 \cdot 7 - 13 \\ &= 2 \cdot (20 - 13) - 13 \\ &= 2 \cdot 20 - 3 \cdot 13 \\ &= 2 \cdot 20 - 3 \cdot (33 - 20) \\ &= 5 \cdot 20 - 3 \cdot 33. \end{aligned}$$

Därmed har vi hittat lösningen $x_0 = 5$, $y_0 = -3$. Genom att multiplicera dessa med 2000 så får vi en lösning (x_1, y_1) till

$$20x + 33y = 2000, \tag{2}$$

nämligen $x_1 = 10000$, $y_1 = -6000$. Den allmäna lösningen till (2) ges då av

$$x = 10000 - 33n, \tag{3}$$
$$y = -6000 + 20n \tag{4}$$

där n är ett godtyckligt heltal. Vi är nu intresserade av lösningar för vilka både $x > 0$ och $y > 0$.

Å ena sidan

$$x > 0 \Leftrightarrow 10000 - 33n > 0 \Leftrightarrow 33n < 10000 \Leftrightarrow n \leq 303. \tag{5}$$

Å andra sidan

$$y > 0 \Leftrightarrow -6000 + 20n > 0 \Leftrightarrow 20n > 6000 \Leftrightarrow n \geq 301. \tag{6}$$

Från (5) och (6) får vi tre möjligheter för n, nämligen $n = 301, 302, 303$. Till sist sätter vi in dessa tre värden i (3) och (4) så får vi tre lösningar :

$$x = 67,\ y = 20 \quad x = 34,\ y = 40 \quad x = 1,\ y = 60.$$

F.2 The homogeneous equation is

$$u_n - 3u_{n-1} + 2u_{n-2} = 0.$$

The characteristic equation for this is

$$x^2 - 3x + 2 = 0,$$

which factorises as

$$(x-1)(x-2) = 0,$$

and hence has the roots $x = 1$ and $x = 2$. Hence the general solution to the homogeneous equation is

$$u_n^h = C_1 + C_2 \cdot 2^n.$$

Since 1 and 2^n are already solutions to the homogeneous equation, our guess for a particular solution should have the form

$$u_n^p = n(A \cdot 2^n + B).$$

Substituting into the recurrence relation, we obtain $A = 2$ and $B = -1$. Hence the general solution to our recurrence relation is

$$u_n = C_1 + C_2 \cdot 2^n + n \cdot (2^{n+1} - 1).$$

It remains to insert the initial conditions :

$$n = 0 \Rightarrow u_0 = 1 = C_1 + C_2,$$
$$n = 1 \Rightarrow u_1 = 1 = C_1 + 2C_2 + 1 \cdot (2^2 - 1).$$

Solving, we obtain $C_1 = 4$, $C_2 = -3$. Hence the final answer is

$$u_n = 4 - 3 \cdot 4^n + n \cdot (2^{n+1} - 1).$$

F.3 $88 = 2^3 \cdot 11$ so $\phi(88) = \phi(2^3) \cdot \phi(11) = (2^3 - 2^2)(11 - 1) = 4 \cdot 10 = 40$. Hence, Euler's Theorem states that, if n is an integer relatively prime to 88, then

$$n^{40} \equiv 1 \pmod{88}.$$

Note that both 3 and 5 are relatively prime to 88. Hence (all congruences are modulo 88)

$$3^{122} = (3^{40})^3 \cdot 3^2 \equiv 1^3 \cdot 9 \equiv 9,$$

and

$$5^{83} = (5^{40})^2 \cdot 5^3 \equiv 1^2 \cdot 125 \equiv 125 \equiv 37.$$

Thus,

$$(3^{122} + 5^{83} + 35)^{42} \equiv (9 + 37 + 35)^{42} = 81^{42} \equiv (-7)^{42} = 7^{42}.$$

But 7 is also relatively prime to 88, so we can apply Euler's theorem again and deduce that

$$7^{42} = 7^{40} \cdot 7^2 \equiv 1 \cdot 49 \equiv 49.$$

So the answer is 49.

F.4 (i) Clearly, $\chi(G) \geq 3$ since G contains many triangles. In fact, $\chi(G) \geq 4$ because the vertex B is at the centre of a wheel formed by V, A, D, E, C. This 5-cycle requires three colours, and then a fourth is needed for B. On the other hand, the graph is plane, hence $\chi(G) \leq 4$, by the Four-Colour Theorem. It follows that $\chi(G) = 4$.

If we apply the greedy algorithm with the nodes ordered $V, A, B, C, D, E, F, G, H, I, J, W$, then we get a 4-coloring, namely (the colors are $1, 2, 3, 4$)

V	1	F	2
A	2	G	3
B	3	H	1
C	2	I	4
D	1	J	1
E	4	W	2

(ii) Apply Dijkstra's algorithm to build up the following tree

Step	Choice of edge	Labelling
1	$\{V, B\}$	$B := 2$
2	$\{V, A\}$	$A := 3$
3	$\{B, E\}$	$E := 4$
4	$\{V, C\}$	$C := 5$
5	$\{E, D\}$	$D := 6$
6	$\{D, F\}$	$F := 8$
7	$\{E, G\}/\{C, G\}$	$G := 8$
8	$\{C, I\}$	$I := 9$
9	$\{C, J\}$	$J := 11$
10	$\{I, H\}$	$H := 11$
11	$\{I, W\}$	$W := 12$

Hence the shortest path from V to W is the path $V \to C \to I \to W$ and has length 12.

F.5 (i) Think of it this way : there are 1000 pairs of boots to be handed out to 20 people, and each person gets at least 15 pairs. So first give each person 15 pairs. Then there are 700 pairs left which can be distributed freely. The number of ways this can be done is

$$\binom{700 + 20 - 1}{20 - 1} = \binom{719}{19}.$$

(ii) We shall use the inclusion-exclusion principle. Let X denote the set of all functions from $\{1, 2, ..., 8\}$ to $\{1, 2, 3, 4\}$. Let A, B, C resp. D denote the subsets of X consisting of those functions which don't hit 1,2,3 resp. 4. Thus the number we are interested in is $|X \backslash (A \cup B \cup C \cup D)|$. By the multiplication principle, we have

$$\begin{aligned}
|X| &= 4^8, \\
|A| = |B| = |C| = |D| &= 3^8, \\
|A \cap B| = |A \cap C| = |A \cap D| = |B \cap C| = |B \cap D| = |C \cap D| &= 2^8, \\
|A \cap B \cap C| = |A \cap B \cap D| = |A \cap C \cap D| = |B \cap C \cap D| &= 1^8 = 1, \\
|A \cap B \cap C \cap D| &= 0.
\end{aligned}$$

Hence, by I-E we have that

$$|X \backslash (A \cup B \cup C \cup D)| = 4^8 - 4 \cdot 3^8 + 6 \cdot 2^8 - 4 \cdot 1 = 40824.$$

(iii) Note that 101 is a prime. Hence, by Fermat's theorem, for any integer a we have that $a^{100} \equiv 0$ or $1 \pmod{101}$. Thus $16x^{100} \equiv 0$ or $16 \pmod{101}$ and so on. The point being that there are only two possibilities modulo 101 for each of the six terms in the given expression. By MP, there are thus in total no more than $2^6 = 64$ possibilities for the entire sum modulo 101. But there are 101 congruence classes in all, so at least 37 of them are not covered. No number in any of these 37 classes can be written in the given form.