Demonstration exercises for week 2

1 (11.5.1 in Biggs) Calculate $\phi(n)$ and $\mu(n)$ for each n in the range $95 \leq n \leq 100$.

(Obs! The *Möbius function* $\mu(n)$ has not yet been discussed in class so will first be defined by the instructor).

2 (11.5.4 in Biggs) Show that if $1 \leq x \leq n$ then

$$\mathrm{GCD}(x, n) = \mathrm{GCD}(n - x, n).$$

Hence prove that the sum over all integers x such that $1 \leq x \leq n$ and $\mathrm{GCD}(x, n) = 1$ equals $\frac{1}{2}n\phi(n)$.

3 (11.4.1 in Biggs) Find the number of ways of arranging the letters A,E,M,O,U,Y in a sequence in such a way that neither of the words ME nor YOU occur.

4 (8.1.8 in Grimaldi) Determine the number of integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 19$$

such that $-5 \leq x_i \leq 10$ for $i = 1, 2, 3, 4$.

5 (8.2.7 in Grimaldi) If 13 cards are dealt from a standard deck of 52, what is the probability that these 13 cards include (a) at least one card from each suit (b) exactly one void (for example, no clubs) (c) exactly two voids ?

6 (see 19.1.3 in Biggs) Give a combinatorial argument for why

$$d_n = (n - 1)(d_{n-1} + d_{n-2})$$

for all $n \geq 3$. Hence deduce that

$$d_n = nd_{n-1} + (-1)^n$$

for all $n \geq 2$.

7 (10.1.2(c) in Grimaldi) Find the unique solution of the recurrence relation

$$3a_{n+1} - 4a_n = 0 \ \forall\, n \geq 0, \quad a_1 = 5.$$

8 (10.2.1(e) in Grimaldi) Solve the recursion relation

$$a_n + 2a_{n-1} + 2a_{n-2} = 0 \;\; \forall\, n \geq 2, \quad a_0 = 1, \quad a_1 = 3.$$

Express the final answer without any complex numbers.

9 (19.2.3 in Biggs and 10.2.11 in Grimaldi) Let q_n denote the number of binary words of length n which contain no two consecutive zeroes. Explain why

$$q_1 = 2, \quad q_2 = 3, \quad q_n = q_{n-1} + q_{n-2} \;\; \forall\, n \geq 3.$$

Now let r_n denote the number of binary words of length n without consecutive zeroes and such that, in addition, the first and last bits are not both zeroes. Find and solve a recurrence relation for r_n.