**1.** Solve the recurrence relation

$$a_n = 4a_{n-1} - 4a_{n-2} + n^2 + 1, \quad \forall\, n \geq 2,$$
$$a_0 = 1, \quad a_1 = 1.$$

**2.** Solve the recurrence relation

$$a_n = 4a_{n-1} - 4a_{n-2} + 2^n, \quad \forall\, n \geq 2,$$
$$a_0 = 1, \quad a_1 = 1.$$

**3 (12.1.1 in Biggs)** Compute the Stirling numbers $S(n, k)$ for all values of $n$ and $k$ with $n \leq 8$.

**4 (12.1.2 in Biggs)** Give direct (i.e.: without use of the recurrence relation) proofs of the identities

$$S(n, 2) = 2^{n-1} - 1, \qquad S(n, n-1) = \binom{n}{2}.$$

**5.** A permutation $a_1, a_2, ..., a_n$ of the integers $1, 2, ..., n$ is said to be $1-3-2$ *avoiding* if there does not exist any three integers $i, j, k$ such that

$$1 \leq i < j < k \leq n$$

and

$$a_i < a_j > a_k > a_i.$$

Write out all 1-3-2 avoiding permutations of $\{1, 2, ..., n\}$ for $n = 1, 2, 3, 4$. Let $A_n$ denote the number of such permutations. Show that $A_n = C_n$, the $n$:th Catalan number.

(Hint : By considering the position of $n$ in a permutation, show that the $A_n$ satisfy the same recurrence relation as the $C_n$.)

**6.** In this exercise, $p(n, k)$ denotes the number of partitions of a positive integer $n$ into $k$ parts and $p(n) = \sum_k p(n, k)$ the total number of partitions of $n$.

**(i)** Evaluate $p(8)$.
**(ii)** Find a formula for $p(n, 2)$.

**(iii)** Explain why $p(n, n-k) = p(k)$ when $k \leq n/2$.

**7 (see 4.3.1 and 4.3.2 in Biggs)** Without using induction, prove that, for every positive integer $n$, both $n^2 + n$ and $n^2 + 3n$ are even integers and $n^3 + 3n^2 + 2n$ is divisible by 6.

**8.** Let $n$ be a positive integer with prime factorisation

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}.$$

Find a formula for $d(n)$, the number of divisors of $n$. Hence compute $d(3000)$.

**9 (see exercises 9.2 in Biggs)** Use the fundamental theorem of arithmetic (FTA) to prove that $\sqrt{p}$ is an irrational number for any prime $p$.

**10.** Find the general solution to the Diophantine equation

$$23x + 41y = 2000,$$

and write down all solutions for which $x > 0$ and $y > 0$.

**11.** Use FTA to prove that $x = -1$, $y = 0$ and $x = -2$, $y = 0$ are the only solutions to the Diophantine equation

$$y^2 = x^2 + 3x + 2.$$