Torsdag 7/9

1 (6.4.2 in Biggs) Show that, in any set of 12 integers, there are two whose difference is a multiple of 11.

2 (10.2.1 in Biggs) In Dr. Cynthia Angst's calculus class 32 of the students are boys. Each boy knows 5 of the girls in the class and each girl knows 8 of the boys. How many girls are in the class ?

3 (10.5.2 in Biggs) How many 4-letter words can be made from an alphabet of 10 symbols if there are no restrictions on spelling except that no letter can be used more than once.

4 (10.7.4 in Biggs) In how many ways can we place 8 rooks on a chessboard so that none can attack any other ?

5 (10.7.5 in Biggs) Suppose there are m girls and n boys in a class. What is the number of ways of arranging them in a line so that all the girls are together ?

6 (1.3.7 in Grimaldi) A committee of 12 is to be selected from 10 men and 10 women. In how many ways can the selection be carried out if

(a) there are no restrictions
(b) there must be 6 men and 6 women
(c) there must be an even number of women
(d) there must be more women than men
(e) there must be at least 8 men.

Fredag 8/9

1 (11.1.7 in Biggs) Prove the identity

$$\binom{s-1}{0} + \binom{s}{1} + \cdots + \binom{s+n-2}{n-1} + \binom{s+n-1}{n} = \binom{s+n}{n}.$$

2 (11.2.2 in Biggs) Show that when 3 indistinguishable dice are thrown there are 56 possible outcomes. What is the number of possible outcomes if n indistinguishable dice are thrown ?

3 (11.3.2 (iv) in Biggs) Expand $(3 + 4x)^6$ as a polynomial in x.

Torsdag 14/9

1 (11.5.1 in Biggs) Calculate $\phi(n)$ and $\mu(n)$ for each n in the range $95 \leq n \leq 100$.

(OBS! The *Möbius function* $\mu(n)$ has not yet been discussed in class so will first be defined by the instructor).

2 (11.5.4 in Biggs) Show that if $1 \leq x \leq n$ then

$$\mathrm{GCD}(x, n) = \mathrm{GCD}(n - x, n).$$

Hence prove that the sum over all integers x such that $1 \leq x \leq n$ and $\mathrm{GCD}(x, n) = 1$ equals $\frac{1}{2}n\phi(n)$.

3 (11.4.1 in Biggs) Find the number of ways of arranging the letters A,E,M,O,U,Y in a sequence in such a way that neither of the words ME nor YOU occur.

4 (8.1.8 in Grimaldi) Determine the number of integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 19$$

such that $-5 \leq x_i \leq 10$ for $i = 1, 2, 3, 4$.

5 (8.2.7 in Grimaldi) If 13 cards are dealt from a standard deck of 52, what is the probability that these 13 cards include (a) at least one card from each suit (b) exactly one void (for example, no clubs) (c) exactly two voids ?

6 (see 19.1.3 in Biggs) Give a combinatorial argument for why

$$d_n = (n - 1)(d_{n-1} + d_{n-2})$$

for all $n \geq 3$. Hence deduce that

$$d_n = nd_{n-1} + (-1)^n$$

for all $n \geq 2$.

Torsdag 21/9

1 (12.1.1 in Biggs) Compute the Stirling numbers $S(n,k)$ for all values of n and k with $n \leq 8$.

2 (12.1.2 in Biggs) Give direct (i.e.: without use of the recurrence relation) proofs of the identities

$$S(n,2) = 2^{n-1} - 1, \qquad S(n,n-1) = \binom{n}{2}.$$

3. In this exercise, $p(n,k)$ denotes the number of partitions of a positive integer n into k parts and $p(n) = \sum_k p(n,k)$ is the total number of partitions of n.

(i) Evaluate $p(8)$.
(ii) Find a formula for $p(n,2)$.
(iii) Explain why $p(n, n-k) = p(k)$ when $k \leq n/2$.

4 (10.1.2(c) in Grimaldi) Find the unique solution of the recurrence relation

$$3a_{n+1} - 4a_n = 0, \ \ \forall\, n \geq 0, \quad a_1 = 5.$$

5 (19.2.3 in Biggs and 10.2.11 in Grimaldi) Let q_n denote the number of binary words of length n which don't contain the pattern 00. Explain why

$$q_1 = 2, \quad q_2 = 3, \quad q_n = q_{n-1} + q_{n-2} \;\forall\, n \geq 3.$$

Express q_n in terms of the Fibonacci numbers, and hence give an explicit formula for q_n.

6 (i) Let $n \geq 3$. Find a recurrence relation for the number of ways to park motorcycles and compact cars in a row of n spaces if each cycle requires two spaces and each car needs three.

(OBS! All cycles are identical in appearance, as are all cars, and we want to use up all the n spaces).

(ii) How would you reformulate the question so that the number q_n of ways to fill n spaces satisfied

$$q_1 = 2, \quad q_2 = 7, \quad q_n = 2q_{n-1} + 3q_{n-2} \;\; \forall\, n > 2.$$

Solve this latter recurrence relation.

Fredag 22/9

1. Solve the recurrence relation

$$\begin{aligned} a_n = 4a_{n-1} - 4a_{n-2} + n^2 + 1, \quad \forall\, n \geq 2, \\ a_0 = a_1 = 1. \end{aligned}$$

2. Solve the recurrence relation

$$\begin{aligned} a_n = 4a_{n-1} - 4a_{n-2} + 2^n, \quad \forall\, n \geq 2, \\ a_0 = a_1 = 1. \end{aligned}$$

Torsdag 28/9

1 (15.2.1 in Biggs) Prove that the following two graphs are not isomorphic :

Figures omitted

2 (15.3.1) Is it possible that the following lists are the degrees of all the vertices of a simple graph ? If so, give a pictorial representation of such a graph.

$$(i)\ 2,2,2,3 \qquad (ii)\ 1,2,2,3,4$$
$$(iii)\ 2,2,4,4,4 \qquad (iv)\ 1,2,3,4.$$

3 (15.4.3 in Biggs) Find a Hamilton cycle in the graph formed by the vertices and edges of an ordinary cube.

4 (15.6.2 in Biggs) Determine the chromatic numbers of the following graphs

Picture omitted

5 (15.7.1 in Biggs) Find orderings of the vertices of the cube graph for which the greedy algorithm requires 2,3 and 4 colors respectively.

6 (15.8.5 and 15.8.6 in Biggs) The *k-cube* Q_k is the graph whose vertices are all binary words of length k and whose edges join words which differ in exactly one position. Show that

(i) Q_k is a regular graph of degree k,

(ii) Q_k is bipartite,

(iii) Q_k has a Hamilton cycle for all $k \geq 2$.

Torsdag 5/10

1. Let $f(n)$ be the number of isomorphism classes of trees on n nodes. Compute $f(n)$ for $n = 1, 2, ..., 6$, drawing all trees.

(OBS! f is a difficult function to compute, but seems to grow roughly exponentially.).

2. A *labelled graph* on n nodes is a graph on n nodes where the nodes have been explicitly labelled from 1 to n. Two labelled graphs are said to be *isomorphic as labelled graphs*, if they are isomorphic graphs, and the given labellings exhibit an explicit isomorphism between them.

There is a theorem of Cayley (19th century) that the number of isomorphism classes of labelled trees on n nodes is n^{n-2}. Verify this for $n = 4$.

3 18.5.11 in Biggs) Find a maximal flow and minimal cut in the network below

Figure omitted

Torsdag 12/10

1 (13.1.1 in Biggs) Without doing any long multiplication, show that

(i) $1234567 \times 90123 \equiv 1 \pmod{10}$,

(ii) $2468 \times 13579 \equiv -3 \pmod{25}$.

2 (13.1.2 in Biggs) Use the method of casting out nines to show that two of the following equations are false. What can be said of the third ?

(i) $5783 \times 40162 = 233256846$,

(ii) $9787 \times 1258 = 12342046$,

(iii) $8901 \times 5743 = 52018443$.

3 (se 13.4.1 i Biggs) Ange alla inverterbara element tillsammans med deras inverser i $\mathbf{Z}_{11}$, $\mathbf{Z}_{15}$ och $\mathbf{Z}_{16}$.

4. Ange resten vid divsion med 19 av

$$(7^{143} + 13^{261})^{11}.$$

Fredag 13/10

1. Genom att tänka modulo 8, bevisa att den Diofantiska ekvationen

$$x^2 - 5y^2 = 122$$

saknar helt och hållet heltalslösningar.

2. Ange den allmänna lösningen till den Diofantiska ekvationen

$$45x + 128y = 5000.$$

Ange speciellt alla lösningar där både $x > 0$ och $y > 0$.

3. Ange alla heltalslösningar (om sådana finns) till kongruenserna

$$45x \equiv 7 \pmod{128},$$
$$45x \equiv 7 \pmod{129}.$$

4. Ange resten vid division med 128 som lämnas av

$$(3^{195} + 11^{129} + 7)^{63}.$$

5. Ange alla heltalslösningar (om sådana finns) till kongruenserna

$$x^2 + 3x + 7 \equiv 0 \pmod{11},$$
$$x^2 + 3x + 8 \equiv 0 \pmod{11}.$$