Homework 1 : Solutions

OBS! Theorems, Propositions, etc refer to my lecture notes.

1 (i) We count money in units of 100,000 crowns, hence there are 150 units of cash to be distributed among 15 applicants. The money units are of course indistinguishable (all anyone cares about is HOW MUCH money they get) whereas the applicants equally obviously are not. Thus by Proposition 9, the number of ways the cash can be distributed is $\binom{150+15-1}{15-1} = \binom{164}{14}$.

(ii) For each university, 50 cash units are to be distrbuted among 5 applicants, so this can be done in $\binom{50+5-1}{5-1} = \binom{54}{4}$ ways. By MP, the number of ways to distribute all the cash is $\binom{54}{4}^3$.

(iii) First of all, there are $\binom{15}{7}$ ways to choose who gets paid. Once this choice has been made, each recipient must first get 5 cash units. This leaves 115 cash units to be distributed freely among 7 recipients, leaving $\binom{115+7-1}{7-1} = \binom{121}{6}$ possibilities.

By MP, the number of possible ways to carry out the funding is $\binom{15}{7} \times \binom{121}{6}$.

2. The point is the following :

(i) $C(n,k) = C(n,n-k)$
(ii) $C(n,k)$ is a strictly increasing function of k for $0 \le k \le n/2$.

Now (i) is just Proposition 5(ii). There are several ways to verify (ii). Just as easy as anything else is to use Proposition 3. Thus one needs to show that if $0 \le k < (n-1)/2$, then

$$\frac{n!}{k!(n-k)!} < \frac{n!}{(k+1)!(n-k-1)!}$$

Cancelling as much as possible from this inequality, it reduces to

$$\frac{1}{n-k} < \frac{1}{k+1} \Leftrightarrow n-k > k+1 \Leftrightarrow k < (n-1)/2, \quad \text{v.s.v.}$$

Then (i) and (ii) imply that if a fair coin is tossed 6 million times, the most likely outcome is exactly 3 million heads. This is because the number of ways to get k heads is just $C(6000000, k)$ for any $0 \leq k \leq 6000000$.

Btw, note that if n is odd, then when the coin is tossed n times, it is equally likely that the number of heads will be $(n-1)/2$ as $(n+1)/2$.

2 bonus. If a fair dice is tossed n times, the number of ways to get exactly k ones is $C(n, k) \cdot 5^{n-k}$, as there are $C(n, k)$ ways of choosing which tosses yield the ones, and 5 possibilities for the outcome of each remaining toss. It is reasonable to expect that this function of k, for a fixed n, reaches a maximum when k is as close as possible to $n/6$. So let's see, when is

$$C(n, k) \cdot 5^{n-k} < C(n, k+1) \cdot 5^{n-k-1} \ ?$$

Using Prop. 3 and cancelling as much as possible as before, this inequality reduces to

$$\frac{5}{n-k} < \frac{1}{k+1} \Leftrightarrow 5(k+1) < n-k \Leftrightarrow k < \frac{n-5}{6}.$$

This means that a maximum will be reached for $k = \lceil \frac{n-5}{6} \rceil$, and if $(n-5)/6$ is an integer, then there will be exactly the same probability of obtaining $(n+1)/6$ ones.

3 (i) The LHS counts the number of ways of choosing k balls from $n+m$ of them. Divide these $n+m$ balls into two groups of n and m respectively. Then the ways to choose the k balls can be divided into $k+1$ types : namely, for any i with $0 \leq i \leq k$, we can combine a choice of i balls from the first n with a choice of $k-i$ balls from the last m. This and MP explains the sum on the RHS.

(ii) This is a bit more subtle. The LHS counts the number of ways to choose n balls from N. Imagine the N balls being numbered $1, ..., N$. Then for any k with $1 \leq k \leq N$, the k:th term on the RHS counts the number of ways to choose the n balls in such a way that the r:th ball chosen, in increasing order, is ball number k. This is because, in such a case, $r-1$

balls must be chosen from the first $k-1$, and this combined with a choice of $n-r$ balls from the last $N-k$.

4. There are $C(9,5) = 126$ five-element subsets of A. Now the sum of any five elements of $\{1, 2, ..., 29\}$ is at least $1+2+3+4+5 = 15$ and at most $29+28+27+26+25 = 135$. Thus there are a priori $(135-15)+1 = 121 < 126$ possible sums for the elements of a 5-element subset of $\{1, 2, ..., 29\}$. Thus, by the Pigeonhole Principle, there must be some overlap (at least five of them, in fact) amongst the sums of the 5-element subsets of A.

5. First note that $s_1 = 3$ since any digit will do. I claim that

$$s_n = 2s_{n-1} + 2s_{n-2} \quad \text{for any } n \geq 3. \tag{1}$$

Note that (1) can also be made to hold when $n = 2$ if we set $s_0 := 1$, because then it gives $s_2 = 8$, which is correct since of the $3^2 = 9$ possible 2-digit words, only 11 is not allowed.

To prove (1) we divide the admissable words of length n into two types :

TYPE I : Those that begin with a 1. Then the second digit must be 0 or 2, giving two possibilities. Then the remaining $n-2$ digits must be chosen according to the same rules as at the outset, leaving s_{n-2} possibilities for these. By MP, there are $2s_{n-2}$ possible words of Type I.

TYPE II : The first digit is 0 or 2, thus two possibilities. Then there are no extra restrictions on the reamining $n-1$ digits, so s_{n-1} possibilities for these. By MP, there are $2s_{n-1}$ words of Type II.

The addition principle now yields (1). This is a standard 2nd order linear homogeneous recurrence with constant coefficients. The characteristic equation is

$$x^2 = 2x + 2,$$

which has roots $1 \pm \sqrt{3}$. Thus the general form of a solution to (1) is

$$s_n = C_1(1+\sqrt{3})^n + C_2(1-\sqrt{3})^n.$$

Inserting the initial conditions we get

$$s_0 = 1 = C_1 + C_2,$$
$$s_1 = 3 = (1+\sqrt{3})C_1 + (1-\sqrt{3})C_2.$$

Solving these yields

$$C_1 = \frac{\sqrt{3}+2}{2\sqrt{3}} = \frac{(1+\sqrt{3})^2}{4\sqrt{3}}, \quad C_2 = \frac{\sqrt{3}-2}{2\sqrt{3}} = -\frac{(1-\sqrt{3})^2}{4\sqrt{3}}.$$

Thus the final answer can be written nicely as

$$s_n = \frac{1}{4\sqrt{3}}\left[(1+\sqrt{3})^{n+2} - (1-\sqrt{3})^{n+2}\right], \quad \forall\, n \geq 1.$$

5 bonus Let a_n denote the number of admissable words of length n which start with a 0. Note that if an admissable word starts with a 1, then there are no extra conditions on the remaining $n-1$ bits, hence t_{n-1} such words. In particular, this means that

$$a_n = t_n - t_{n-1}. \tag{2}$$

Now let us divide the admissable words of length n into three types :

TYPE I : Those that begin with a 1. As stated above, there are t_{n-1} such words.

TYPE II : Those that begin 01. Then the third bit is forced to be 1 also. There are no additional restrictions on the remaining $n-3$ bits, hence t_{n-3} such words.

TYPE III : Those that begin 00. The first zero has no impact, so there are a_{n-1} such words, hence $t_{n-1} - t_{n-2}$ of them by (2).

Adding, we obtain that

$$t_n = t_{n-1} + t_{n-3} + (t_{n-1} - t_{n-2}) = 2t_{n-1} - t_{n-2} + t_{n-3}.$$

Note that solving this recurrence involves solving the cubic equation $x^3 = 2x^2 - x + 1$.

6. Let X denote the set of all 15-digit decimal numbers. Thus $|X| = 9 \cdot 10^{14}$ as there are 9 choices for the leading digit (which can't be 0) and 10 choices for each remaining digit.

Now we need to sieve out from X the subsets A, B and C, which consist of those numbers not containing any 2's, 3's or 5's respectively. We compute $|A \cup B \cup C|$ using eq.(15) in the notes. First,

$$|A| = |B| = |C| = 8 \cdot 9^{14},$$

as there is now one less choice for each digit. Similarly,

$$|A \cap B| = |A \cap C| = |B \cap C| = 7 \cdot 8^{14},$$

as there are then two less choices for each digit, and finally

$$|A \cap B \cap C| = 6 \cdot 7^{14}.$$

Putting everything together, we find that the number of 15-digit numbers containing at least one occurrence of each of 2,3 and 5 is

$$9 \cdot 10^{14} - 3 \cdot 8 \cdot 9^{14} + 3 \cdot 7 \cdot 8^{14} - 6 \cdot 7^{14} = 439246619377026.$$

7 (i) List the elements of A in increasing order as $a_1 < a_2 < \cdots < a_n$. Then

$$\begin{aligned} & a_1 + a_1 < a_1 + a_2 < \cdots < a_1 + a_n < \\ & < a_2 + a_n < a_3 + a_n < \cdots < a_n + a_n, \end{aligned} \tag{3}$$

which proves that $|A + A| \geq 2n - 1$. Alternatively, one could for example note that

$$a_1 + a_1 < a_1 + a_2 < a_2 + a_2 < a_2 + a_3 < \cdots < a_{n-1} + a_n < a_n + a_n.$$

(ii) I claim that

$$|A + A| = 2 \cdot |A| - 1 \tag{4}$$

if and only if A is a so-called *arithmetic progression*, i.e.: if and only if all the differences $a_{i+1} - a_i$ are the same, for $i = 1, 2, ..., n - 1$. Clearly, an arithmetic progression satisfies (4). Conversely, suppose A has n elements, listed in increasing order as above, and satisfies (4). Consider the sums $a_2 + a_j$ for $j = 2, 3, ..., n - 1$. There are $n - 2$ of them and they must all coincide with members of the chain (3). The smallest of them is strictly greater than the second term in the chain, namely $a_1 + a_2$. And the largest of them is strictly less than the $(n+1)$:st term in the chain, namely $a_2 + a_n$. It follows that

$$a_2 + a_j = a_1 + a_{j+1}, \quad \text{for } j = 2, 3, ..., n-1,$$

and hence that

$$a_{j+1} - a_j = a_2 - a_1 \quad \text{for } j = 2, 3, ..., n-1, \text{ v.s.v.}$$