Att lämna in senast fredagen den 6 oktober, kl 1700

REGLER : Q.1 and **Q.2** are worth 30 points each. The other questions are worth 20 points each. Thus a total of 180 points are available. A score of 130 points will correspond to $11\frac{1}{3}$ points for the course (see points system as described on homepage).

1. Find an explicit formula for the terms of the sequence (u_n) which satisfy the recurrence relation

$$2u_n = 11u_{n-1} - 12u_{n-2} + 4^n + n + 1, \quad \forall\, n \geq 2,$$
$$u_0 = 1, \quad u_1 = 2.$$

2. You are referred to the network in Fig.1 on the handout.

(i) Take away all the arrows and find a minimal weight spanning tree in the resulting undiretced graph.
(ii) Find a shortest path from s to t.
(iii) Find a maximal flow from s to t and a corresponding minimal cut.

3. Let $(F_n)_{n=0}^{\infty}$ denote the Fibonacci numbers, i.e.:

$$F_0 = F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}\ \forall\, n \geq 2.$$

(i) Prove that, for every $n \geq 0$, F_n equals the number of ways to write the integer n as a sum of 1:s and 2:s, where the ordering is important. For example, $F_3 = 5$ and here are the 5 different ways to write 4 as a sum of 1:s and 2:s :

$$1+1+1+1 \quad 1+1+2 \quad 1+2+1 \quad 2+1+1 \quad 2+2$$

(ii) Deduce that, for each $n \geq 0$,

$$F_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}.$$

(Hint : Consider the number of ways of writing n where the number of 1:s is fixed).

4 (see 19.7.5 in Biggs) Let C_n denote the cycle graph with n vertices labelled $0, 1, ..., n-1$. Let $f_n(k)$ denote the number of ways to color the vertices with k (or fewer) colors. By splitting the set of colorings into two parts, according as vertices 0 and 2 do or do not have the same color, show that

$$f_n(k) = (k-1)f_{n-2}(k) + (k-2)f_{n-1}(k).$$

Deduce that

$$f_n(k) = (k-1)[(k-1)^{n-1} + (-1)^n], \quad \forall\, n \geq 3.$$

5. Let G be any simple graph. Let $\overline{G}$ denote its' complement, i.e.: the simple graph with the same set of nodes as G and whose edges are precisely those missing from G. There is an exercise in Biggs which asks one to show that, for any simple graph G with n vertices,

$$\chi(G) \cdot \chi(\overline{G}) \leq n.$$

(i) Give an example to show that this need not be true.
(ii) Prove that, on the other hand, it is always true that

$$\chi(G) + \chi(\overline{G}) \leq n+1.$$

(Hint : Use induction on n).

6 (i) For each $n > 0$, give an example of a simple graph G with $2n$ vertices and n^2 edges such that G has no triangles.
(ii) Let $n > 0$. Prove that if G is a simple graph with $2n$ vertices and more than n^2 edges, then G contains a triangle.
(Hint : Use induction on n).

7 (i) Exhibit an explicit isomorphism between the graphs in Figs. 2a and 2b on the handout. (Both are the so-called *Petersen graph* P).

For the rest of the exercise, use Fig. 2a.

(ii) Exhibit cycles of lengths 5,6,8 and 9 in P.
(iii) An *edge coloring* of a graph is a coloring of its edges such that two edges which share a vertex always get different colors.

Prove that P has no Hamilton cycle in the following steps :

(a) Explain why a graph in which every vertex has degree 3 and which contains a Hamilton cycle can be edge-colored with 3 colors.

(b) Explain why the edges of P cannot be colored with 3 colors.

8. Suppose you're given twelve enkronor and told that one of them is defective, hence either slightly lighter or heavier than the rest. You're not told, however, which coin it is or whether it's heavier or lighter. Describe how to find the defective coin via three weighings on a weighing scale.