Homework 2 : Solutions

1. The homogeneous equation is

$$2u_n - 11u_{n-1} + 12u_{n-2} = 0.$$

The characteristic equation for this is

$$2x^2 - 11x + 12 = 0,$$

which factorises as

$$(2x-3)(x-4) = 0,$$

and hence has the roots $x = 3/2, x = 4$. Hence the general solution to the homogeneous equation is

$$u_n^h = C_1 \cdot \left(\frac{3}{2}\right)^n + C_2 \cdot 4^n.$$

Since 4^n is already a solution to the homogeneous equation, our guess for a particular solution should have the form

$$u_n^p = A \cdot n \cdot 4^n + Bn + C.$$

Substituting into the recurrence relation, the requirement on A is that

$$A \cdot \left[2n4^n - 11(n-1)4^{n-1} + 12(n-2)4^{n-2}\right] = 4^n, \tag{1}$$

whereas the requirement on B and C is that

$$2[Bn + c] - 11[B(n-1) + C] + 12[B(n-2) + C] = n + 1. \tag{2}$$

From (1) we deduce that $A = 4/5$ and from (2) that $B = 1/3$, $C = 16/9$. Hence the general solution to our recurrence relation is

$$u_n = C_1 \cdot \left(\frac{3}{2}\right)^n + \left(C_2 + \frac{4n}{5}\right) \cdot 4^n + \left(\frac{n}{3} + \frac{16}{9}\right).$$

It remains to insert the initial conditions :

$$n = 0 \Rightarrow u_0 = 1 = C_1 + C_2 + \frac{7}{9},$$

$$n = 1 \Rightarrow u_1 = 2 = \frac{3C_1}{2} + 4\left(C_2 + \frac{4}{5}\right) + \left(\frac{1}{3} + \frac{16}{9}\right).$$

Solving, we obtain $C_1 = 2/25$, $C_2 = -193/225$. Hence the final answer is

$$u_n = \frac{2}{25} \cdot \left(\frac{3}{2}\right)^n + \left(\frac{4n}{5} - \frac{193}{225}\right) \cdot 4^n + \left(\frac{n}{3} + \frac{16}{9}\right).$$

2. I label the nodes of the graph as follows : s to the left, t to the right as in the diagram ; reading downwards in each column of four from left to right : $a, b, c, d,$, then e, f, g, h, then i, j, k, l, then m, n, o, p.

(i) Use BFS, starting, say, from the vertex s, to build up the following sequence of edges in a MST :

$$\{s,b\}, \{b,f\}, \{f,a\}, \{f,i\}, \{i,m\}, \{a,e\}, \{s,c\}, \{c,g\}, \{f,j\},$$
$$\{j,o\}, \{o,t\}, \{o,k\}, \{k,h\}, \{h,d\}, \{h,l\}, \{t,n\}, \{t,p\}.$$

The total weight of this tree is $1+2+2+3+2+4+4+2+4+5+2+3+2+1+3+4+5 = 49$.

(ii) Använd BFS för att bygga upp ett träd med följande sekvens av kanter :

$$\{s,b\}, \{b,f\}, \{s,c\}, \{f,a\}, \{f,i\}, \{c,g\}, \{s,d\}, \{d,h\}, \{f,j\},$$
$$\{i,m\}, \{a,e\}, \{h,k\}, \{h,l\}, \{k,o\} \text{ eller } \{j,o\}, \{o,t\} \text{ eller } \{m,t\}.$$

If $\{o,t\}$ is the last edge chosen, then the unique path back to s is $t-o-k-h-d-s$. If instead $\{m,t\}$ is chosen last, then the unique path back to s is $t-m-i-f-b-s$.

Both these paths have total length 14.

(iii) Starting with the null flow $f \equiv 0$, one can find the following sequence of f-augmenting paths from s to t :

$$\begin{aligned}
s-a-e-i-m-t, &\quad \epsilon = 2,\\
s-b-f-j-n-t, &\quad \epsilon = 1,\\
s-c-g-k-o-t, &\quad \epsilon = 2,\\
s-d-h-l-p-t, &\quad \epsilon = 1,\\
s-a-f-j-n-t, &\quad \epsilon = 2,\\
s-c-h-l-p-t, &\quad \epsilon = 1.
\end{aligned}$$

This yields the following maximal flow with $|f| = 10$

Edge	Flow	Edge	Flow		
$\{s,a\}$	4	$\{e,i\}$	2	$\{k,o\}$	2
$\{s,b\}$	1	$\{f,i\}$	0	$\{l,o\}$	0
$\{s,c\}$	4	$\{f,j\}$	3	$\{l,p\}$	3
$\{s,d\}$	1	$\{g,j\}$	0	$\{m,t\}$	2
$\{a,e\}$	2	$\{g,k\}$	2	$\{n,t\}$	3
$\{a,f\}$	2	$\{h,k\}$	0	$\{o,t\}$	2
$\{b,f\}$	1	$\{h,l\}$	3	$\{p,t\}$	3
$\{b,g\}$	0	$\{i,m\}$	2		
$\{c,g\}$	2	$\{j,m\}$	0		
$\{c,h\}$	2	$\{j,n\}$	3		
$\{d,h\}$	1	$\{j,o\}$	0		

The corresponding minimal cut is $S = \{s,a,d,e,i\}$, $T =$ rest of them. Its' capacity is given by

$$c(S,T) = c(s,b) + c(s,c) + c(a,f) + c(d,h) + c(i,m) = 1+4+2+1+2 = 10, \quad \text{v.s.v..}$$

3 (i) Let A_n denote the number of ways of writing n as a sum of 1:s and 2:s. Clearly $A_0 = A_1 = 1$. So assume $n \geq 2$ and we must verify that $A_n = A_{n-1} + A_{n-2}$. Divide the ways of writing n as a sum of 1:s and 2:s into two types :

TYPE I : Those in which the first summand is a 1. Then the remaining summands sum to $n-1$, and there are A_{n-1} choices for them.

TYOE II : The first summand is a 2. Then the remaining summands sum to $n-2$ so there are A_{n-2} choices for them.

The addition principle yields the desired result.

(ii) Any expression of n as a sum of 1:s and 2:s can include at most $\lfloor n/2 \rfloor$ twos. We claim that, for each $k \in \{0,1,...,\lfloor \frac{n}{2} \rfloor\}$, the number of such expressions including exactly k twos is $\binom{n-k}{k}$. But this is clear. For if there are k twos, then there are $n-k$ summands in all, and we just have to choose which k of these are the twos.

4 (i) First suppose nodes 0 and 2 get the same colour. Ignore node 1 and identify nodes 0 and 2. This leaves us with the cycle C_{n-2}, which is to be coloured with k colours. By definition, there are $f_{n-2}(k)$ ways to do this. Once this colouring has been chosen, there remain $k-1$ ways to colour the reinserted node 1 (it must get a different colour from its two neighbours). Thus, by MP, there are $(k-1)f_{n-2}(k)$ possible colourings in which nodes 0 and 2 get the same colour.

Next suppose nodes 0 and 2 get different colours. First ignore node 1 again. The remaining nodes form a C_{n-1} and, since nodes 0 and 2 are assumed to get different colours, this remaining cycle is to be coloured under exactly the same rules as at the outset. Thus there are $f_{n-1}(k)$ possible colourings of all the nodes except node 1. Reinserting node 1, it can then be coloured in $k-2$ ways, as it must get a different colour from both 0 and 2. By MP, there are in total $(k-2)f_{n-1}(k)$ possible colourings in which nodes 0 and 2 get different colours.

The above two paragraphs, plus the addition principle, suffice to prove the result.

(ii) In fact the formula holds for all $n \geq 2$ if we interpret C_2 as meaning a single edge. A single edge can be coloured with k colours in $k(k-1)$ ways, as the two vertices must get different colours. Observe that the formula also gives $f_2(k) = (k-1)[(k-1)+1] = k(k-1)$.

For $n = 3$ the formula states that

$$f_3(k) = (k-1)[(k-1)^2 - 1] = (k-1)[k^2 - 2k] = k(k-1)(k-2),$$

which is also true, since all the vertices in a C_3 must get different colours. Now we proceed by induction on n to show that the formula holds for all $n \geq 2$. Since we've already established the initial cases $n = 2, 3$, all that remains is to insert the formulas for $f_{n-2}(k)$ and $f_{n-1}(k)$ and verify that

$$(k-1)[(k-1)^{n-1} + (-1)^n] = (k-1)[(k-1)^{n-3} + (-1)^{n-2}] + (k-1)[(k-1)^{n-2} + (-1)^{n-1}].$$

This is a straightforward algebra exercise and is left to the reader to check.

5 (i) The complete bipartite graph $K_{n,n}$ has $2n$ vertices and n^2 edges. Being bipartite, it has no odd cycles whatsoever, never mind triangles.

(ii) For $n = 1$ the claim is vacuous, as a graph on two vertices can't have more than one edge anyway. For $n = 2$, if a graph on 4 vertices contains more than 4 edges, then it is either K_4 or K_4 minus a single edge. One readily checks that removing a single edge from K_4 inevitably leaves some remaining triangles (two of them, in fact).

Now to the induction step. Suppose the result holds for some n, and consider a graph on $2n+2$ vertices. We can assume there's at least one edge in G. Isolate two vertices, call them v and w, which are joined by an edge in G. Let H denote the subgraph spanned by the remaining $2n$ vertices. The induction hypothesis implies that H contains no more than n^2 edges. But now, since $\{v, w\}$ is an edge in G, if v and w had a common neighbour x in H then $\{v, w, x\}$ would be a triangle in G. Thus every vertex in H is a neighbour of at most one of v and w, and thus G contains no more than $2n$ edges between H and these two vertices. All in all, we thus have

(a) no more than n^2 edges within H,

(b) no more than $2n$ edges between H and v, w,

(c) one edge $\{v, w\}$.

So the total number of edges in G is at most $n^2 + 2n + 1 = (n+1)^2$, v.s.v.

6 (i) Let G have 3 vertices a, b, c and two edges $\{a, b\}$ and $\{b, c\}$. Then check that $\chi(G) = \chi(\overline{G}) = 2$, so $\chi(G)\chi(\overline{G}) = 4 > 3$.

(ii) The proof is by induction on n. The theorem is easily checked to be true for $n = 2$. Suppose it's true for n and let G be a graph on $n+1$ vertices. Let v be any vertex and G_v the graph on n vertices got by removing v and all edges through it. By the induction hypothesis,

$$\chi(G_v) + \chi(\overline{G_v}) \leq n + 1. \tag{3}$$

We must prove that

$$\chi(G) + \chi(\overline{G}) \leq n + 2. \tag{4}$$

First, since G contains only one more vertex than G_v, we have that $\chi(G) \leq \chi(G_v) + 1$. Similarly for the complements. Hence (4) will certainly be satisfied if we have strict inequality in (3). Hence we may assume that we have equality in (3). In this case it suffices to prove that either $\chi(G) = \chi(G_v)$ or $\chi(\overline{G}) = \chi(\overline{G_v})$. Call this condition (*).

Pick any $n+1$ colors which suffice to color both G_v and $\overline{G_v}$, and color both using these colors, such that no color is used on both graphs. Now the n vertices of G_v can be partitioned into 2 subsets X and Y, namely those joined to v in G and $\overline{G}$ respectively. Now at least one of our $n+1$ colors, call it C, is neither used among the vertices of X in the coloring of G_v, nor among the vertices of Y in the coloring of $\overline{G_v}$. Hence we obtain colorings of both G and $\overline{G}$ if we extend the colorings of G_v and $\overline{G_v}$ by coloring v with color C. Since all $n+1$ colors were used to color G_v and $\overline{G_v}$, the color C will already have appeared in the coloring of one of them. This implies that condition (*) is satisfied.

7 (i) The vertices of P in the pentagonal representation were given to you as follows : 1,2,3,4,5 on the outer pentagon, starting at the top and moving clockwise. 6,7,8,9,10 on the inside attached to 1,2,3,4,5 respectively.

One can now label the vertices in the hexagonal representation as follows, for example (there are many possible isomorphisms) : 1,2,3,4,9,6 on the outer hexagon, starting top left; 7,5,8 on the three diagonals $(2,9)$, $(1,4)$ and $(3,6)$ respectively; 10 in the middle.

With these labellings, the two graphs are isomorphic.

(ii) In the pentagonal representation, with the same labelling as in **(i)**, $(1,2,3,4,5,1)$ is a simple C_5, $(1,2,3,8,10,5,1)$ is a simple C_6, $(1,2,3,8,6,9,4,5,1)$ is a simple C_8 and $(1,2,3,8,6,9,7,10,5,1)$ is a simple C_9.

(iii) Let G be such a graph. Since every vertex of G has degree 3 and the sum of the degrees of all the vertices must be an even number (eq.(40) in lecture notes), it follows that G has an even number of vertices. Let $\mathcal{C}$ be a Hamilton cycle in G. Then $\mathcal{C}$ contains an even number of edges and hence can be colored using only 2 colors. Then, since every vertex in G has degree 3, the remaining edges (i.e.: those not on this cycle) must form a complete matching of G, and hence can all be colored with the same third color. Hence G is 3-edge-colorable, v.s.v.

(iv) Let's try to color P with three colors A,B,C, using the pentagonal representation. Because of P's rotational symmetry, there is no loss of generality in assuming that the outer pentagon is colored as

$$\begin{aligned} A \text{ on } \{1,2\} \text{ and } \{3,4\}, \\ B \text{ on } \{2,3\} \text{ and } \{4,5\}, \\ C \text{ on } \{5,1\}. \end{aligned}$$

Then the colors to be assigned to the 5 inward edges are uniquely determined, namely : $\{2,7\}$, $\{3,8\}$ and $\{4,9\}$ must all get C, $\{5,10\}$ must get A and $\{1,6\}$ must get B.

But then both $\{6,8\}$ and $\{6,9\}$ must get A, so we're screwed !

8. No time to draw the decision tree, so instead I'll describe it. Label the coins 1,...,12. The notations E,L,R will denote that the result of a weighing was, respectively, 'equal', 'left side heavy', 'right side heavy'.

STEP 1 : (1,2,3,4) v. (5,6,7,8).

- If E, go to Step 2E.
- If L, go to Step 2L.
- If R, follow the same chain as if L, just interchange (1,2,3,4) and (5,6,7,8).

STEP 2E : (1,2,3) v. (9,10,11).

- If E, go to Step 3EE.
- If L, go to Step 3EL.
- If R, go to Step 3ER.

STEP 2L : (1,2,5,9) v. (3,6,10,11).

- If E, go to Step 3LE.
- If L, go to Step 3LL.
- If R, go to Step 3LR.

STEP 3EE : 1 v. 12. E is impossible. If L, then 12 is light. If R, then 12 is heavy.

STEP 3EL : 9 v. 10. If E, then 11 is light. If L, then 10 is light. If R, then 9 is light.

STEP 3ER : 9 v. 10. If E, then 11 is heavy. If L, then 9 is heavy. If R, then 10 is heavy.

STEP 3LE : 7 v. 8. If E, then 4 is heavy. If L, then 8 is light. If R, then 7 is light.

STEP 3LL : 1 v. 2. If E, then 6 is light. If L, then 1 is heavy. If R, then 2 is heavy.

STEP 3LR : 5 v. 9. If E, then 3 is heavy. L is impossible. If R, then 5 is light.