Att lämna in senast onsdagen den 18 oktober, kl 1700

REGLER : Questions 1,3 and 6 are worth 30 points. Every other question is worth 20 points. Full marks for 130 points.

1 (see 11.4.14 in Grimaldi) Determine whether each of the graphs in Fig. 1 is planar or not. If it is, redraw the graph as a plane graph. If it isn't, explain what kind of K_5 or $K_{3,3}$ obstruction arises.

2 (see 14.4.18 in Grimaldi) Let G be a plane graph such that
(i) G divides the plane into 53 regions
(ii) each of these regions, including the infinite region, has at least five edges in its boundary.
Prove that G has at least 82 vertices.

3 (i) If $G = (X, Y, E)$ is a bipartite graph and $A \subseteq X$, recall that $\Gamma(A) := \{y \in Y : \{a, y\} \in E \text{ for some } a \in A\}$. Define the *deficiency* of A, denoted $\delta(A)$, as $\delta(A) := |A| - |\Gamma(A)|$. Define the deficiency of G to be the maximum of the deficiencies of all subsets of X, and denote it δ_G.

Now use Hall's theorem to deduce that the maximum size of a matching in G is $|X| - \delta_G$.

(ii) Suppose that
(a) $\deg(x) \geq 4$ for all $x \in X$,
(b) $\deg(y) \leq 5$ for all $y \in Y$,
(c) $|X| \leq 10$.
Prove that $\delta_G \leq 2$.

4. A prime number p such that $p + 2$ is also a prime is called a *prime twin*. One of the most famous open problems in all of mathematics asks whether there are infinitely many prime twins. So if you can answer that question, you get a 5 for the course directly !

Less ambitiously, call a prime number p a *prime triplet* if both $p + 2$ and $p + 4$ are also primes. State and prove a theorem giving a complete list of prime triplets.

5. Find the general solution of the Diophantine equation

$$18x + 29y = 2500.$$

Also write down all solutions for which both x and y are positive.

6. Compute the remainder when

$$(5^{74} + 7^{111} + 3)^{35}$$

is divided by 108.

7. Describe a colouring of the natural numbers with three colours (i.e.: each number is coloured red, green or blue) such that there are no monochromatic solutions to the equation

$$x + 2y = 4z.$$

(Hint : Think powers of 2.)

REMARK : If you can describe such a 3-colouring of the reals $\mathbf{R}$, or prove that none exists, I will personally give you all the money in my bank account !!