Homework 3 : Solutions

1 (i) This is not planar. There is a $K_{3,3}$ obstruction. For example, remove the four corner vertices, contracting the edges through them, e.g.: removing a means that the edges $\{a, b\}$ and $\{a, e\}$ are replaced by the single edge $\{b, e\}$. The remaining graph has 6 vertices b, c, e, f, h, i and is a $K_{3,3}$, with b, h, f as the red group and c, e, i as the blue group.

(ii) This is planar. Here is an attempt to describe a plane redrawing of it :

(a) drag the edge $\{a, e\}$ outside f, the edge $\{a, c\}$ outside b, and the edge $\{e, c\}$ outside d.

(b) then drag the edge $\{e, b\}$ outside f, the edge $\{a, d\}$ outside b and the edge $\{c, f\}$ outside d.

(iii) This is not planar. There is a $K_{3,3}$ obstruction. Remove f so that each of the paths $e - f - b$ and $g - f - b$ is contracted into a single edge. Then remove a and all its adjacent edges. The remaining graph has 6 vertices b, c, d, e, g, h and is a $K_{3,3}$, with b, d, h as the red group and c, e, g as the blue group.

2. Let V, E, R denote the number of vertices, edges and regions for G. Euler's formula states that

$$V - E + R = 2. \tag{1}$$

Consider the sum

$$\sum_{\text{regions r}} n_r,$$

where n_r is the number of edges along the boundary of the region r. We are told that $n_r \geq 5$ for every region, hence the sum is at least $5R = 5 \cdot 53 = 265$. On the other hand, since the graph is plane, every edge is counted twice in the sum, so the sum is $2E$. It follows that $2E \geq 265 \Rightarrow E \geq 133$. Thus from (1) we obtain that $V = 2 + E - R \geq 2 + 133 - 53 = 82$, v.s.v.

3 (i) It is clear that there is no matching of size greater than $|X| - \delta_G$, as for any subset A of X, at most $\Gamma(A)$ of its vertices can be matched.

Conversely, let $G = (X, Y, E)$ be bipartite with deficiency $\delta_G \geq 0$. Define a new bipartite graph $G^* = (X^*, Y^*, E^*)$ as follows :

(a) $X^* = X$,
(b) $Y^* = Y \sqcup Z$, where $|Z| = \delta_G$,
(c) $E^* = E \sqcup \{\{x, z\} : x \in X,\ z \in Z\}$.

In words, we add δ_G vertices to Y and join each of these new vertices to every vertex in X. The point is that the resulting graph has deficiency zero, so by Hall's theorem it has a perfect matching for X. But in this matching, at most $|Z| = \delta_G$ of the vertices of X will be matched to a vertex in Z. The rest of the matching thus constitutes a matching of size at least $|X| - \delta_G$ in the original graph G. So we're done !

(ii) Hmmm ... the following proof seems to work for $|X| \leq 14$.

Suppose $\delta_G \geq 3$. This implies the existence of a subset A of X such that

$$|A| \geq |\Gamma(A)| + 3. \tag{2}$$

On the other hand, we have the following sequence of inequalities :

$$4|A| \leq \sum_{x \in A} \deg(x) \leq \sum_{y \in \Gamma(A)} \deg(y) \leq 5|\Gamma(A)|. \tag{3}$$

I'll explain the three inequalities in turn :
- The first one follows from assumption (a).
- The second one follows from the fact that the right-hand sum counts all edges protruding from the vertices of $\Gamma(A)$ whereas the left-hand sum only counts those among these edges which have their other endpoint in A.
- The third one follows from assumption (b).

So from (3) we have that

$$4|A| \leq 5|\Gamma(A)|.$$

Substituting this into (2) yields that $|A| \geq 15$. Thus we obtain a contradiction whenever $|X| \leq 14$, as A is a subset of X.

4. The only prime triplet is $p = 3$. For let $n, n+2, n+4$ be three consecutive odd numbers. Modulo 3 these are congruent to $n, n+2, n+1$

respectively, i.e.: three consecutive numbers (after reordering). But among three consecutive numbers, we always have a multiple of 3, and 3 itself is the only such number which is also a prime.

5. Det är lätt att se att $\mathrm{SGD}(18, 29) = 1$ eftersom 29 är ett primtal. Därför vet vi att det finns heltal x_0, y_0 så att

$$18x_0 + 29y_0 = 1. \tag{4}$$

Vi hittar först en lösning till (4) genom att köra Euklides algoritm fram och tillbaka. Framåt får vi

$$\begin{aligned}
29 &= 1 \cdot 18 + 11, \\
18 &= 1 \cdot 11 + 7, \\
11 &= 1 \cdot 7 + 4, \\
7 &= 1 \cdot 4 + 3, \\
4 &= 1 \cdot 3 + 1, \\
3 &= 3 \cdot 1 + 0.
\end{aligned}$$

Bakåt får vi då

$$\begin{aligned}
1 &= 4 - 3 \\
&= 4 - (7 - 4) \\
&= 2 \cdot 4 - 7 \\
&= 2 \cdot (11 - 7) - 7 \\
&= 2 \cdot 11 - 3 \cdot 7 \\
&= 2 \cdot 11 - 3 \cdot (18 - 11) \\
&= 5 \cdot 11 - 3 \cdot 18 \\
&= 5 \cdot (29 - 18) - 3 \cdot 18 \\
&= 5 \cdot 29 - 8 \cdot 18.
\end{aligned}$$

Därmed har vi hittat lösningen $x_0 = -8$, $y_0 = 5$. Genom att multiplicera dessa med 2500 så får vi en lösning (x_1, y_1) till

$$18x + 29y = 2500, \tag{5}$$

nämligen $x_1 = -20000$, $y_1 = 12500$. Den allmäna lösningen till (5) ges då av

$$x = -20000 + 29n, \tag{6}$$
$$y = 12500 - 18n \tag{7}$$

där n är ett godtyckligt heltal. Vi är nu intresserade av lösningar för vilka både $x > 0$ och $y > 0$.

Å ena sidan

$$x > 0 \Leftrightarrow -20000 + 29n > 0 \Leftrightarrow 29n > 20000 \Leftrightarrow n \geq 690. \tag{8}$$

Å andra sidan

$$y > 0 \Leftrightarrow 12500 - 18n > 0 \Leftrightarrow 18n < 12500 \Leftrightarrow n \leq 694. \tag{9}$$

Från (8) och (9) får vi fem möjligheter för n, nämligen $n = 690, 691, 692, 693, 694$. Till sist sätter vi in dessa åtta värden i (6) och (7) så får vi fem lösningar :

$x = 10,\ y = 80 \quad x = 39,\ y = 62 \quad x = 68,\ y = 44, \quad x = 97,\ y = 26 \quad x = 126,\ y = 8.$

6. $108 = 2^2 \cdot 3^3$ so $\phi(108) = 108 \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) = 36$. Hence, Euler's Theorem states that, if a is an integer relatively prime to 108, then

$$a^{36} \equiv 1 \pmod{108}.$$

Note that both 5 and 7 are relatively prime to 108. Hence (all congruences are modulo 98)

$$5^{74} = (5^{36})^2 \cdot 5^2 \equiv 1^2 \cdot 25 \equiv 25,$$

and

$$7^{111} = (7^{36})^3 \cdot 7^3 \equiv 1^3 \cdot 343 \equiv 19.$$

Thus,

$$(5^{74} + 7^{111} + 3)^{35} \equiv (25 + 19 + 3)^{35} = 47^{35}.$$

Since 47 is a prime, it is also relatively prime to 108, so we can apply Euler's theorem again. Thus

$$47^{35} \equiv 47^{36} \cdot 47^{-1} \equiv 1 \cdot 47^{-1} \equiv 47^{-1}.$$

So it remains to compute the inverse of 47 modulo 108. To do this, we run Euclid back and forth, just as in the previous exercise. Forwards, we get

$$\begin{aligned} 108 &= 2 \cdot 47 + 14, \\ 47 &= 3 \cdot 14 + 5, \\ 14 &= 2 \cdot 5 + 4, \\ 5 &= 1 \cdot 4 + 1, \\ 4 &= 4 \cdot 1 + 0. \end{aligned}$$

Backwards, we then obtain

$$\begin{aligned} 1 &= 5 - 4 \\ &= 5 - (14 - 2 \cdot 5) \\ &= 3 \cdot 5 - 14 \\ &= 3 \cdot (47 - 3 \cdot 14) - 14 \\ &= 3 \cdot 47 - 10 \cdot 14 \\ &= 3 \cdot 47 - 10 \cdot (108 - 2 \cdot 47) \\ &= 23 \cdot 47 - 10 \cdot 108. \end{aligned}$$

From this we can read off that $23 \equiv 47^{-1} \pmod{108}$. So the final answer is 23.

7 (i) Every positive integer n has a unique expression as $2^{f(n)} \cdot u$, where $f(n)$ is a non-negative integer and u is an odd number. The following 3-coloring now works :

'Color n red if $f(n) \equiv 0 \pmod 3$, color n blue if $f(n) \equiv 1 \pmod 3$, and color n green if $f(n) \equiv 2 \pmod 3$.'

I'll show there are no red solutions to $x + 2y = 4z$ - the proof is similar for the other colours. So suppose (x, y, z) was a red solution. Since x, y, z are all red, there exist non-negative integers a, b, c and odd numbers u_1, u_2, u_3 such that

$$x = 2^{3a} u_1, \quad y = 2^{3b} u_2, \quad z = 2^{3c} u_3,$$

and hence the assumption is that

$$2^{3a} u_1 + 2 \cdot 2^{3b} u_2 = 4 \cdot 2^{3c} u_3,$$

in other words that

$$2^{3a} u_1 + 2^{3b+1} u_2 = 2^{3c+2} u_3. \tag{10}$$

Now the number on the right of (10) is green. I claim that that on the left must be red or blue : if so, then the two sides can't be equal - contradiction and we're done !

There are two cases to consider. If $b \geq a$, then write the left side as $2^{3a} u_4$ where $u_4 = u_1 + 2^{3b-3a+1} u_2$. Since u_4 must be odd, we have a red number. If

$b < a$ then instead write the left side as $2^{3b+1}u_5$ where $u_5 = 2^{3a-3b-1}u_1 + u_2$. Since u_5 is odd, we have a blue number, thus establishing our claim.

(ii) Whether or not such a 3-colouring of $\mathbf{R}$ exists is independent of the Zermelo-Fraenkel axioms of set theory[1]. If you have no idea what that means, well don't worry about it for the exam at least !

[1]This was only proven very recently, like in the last couple of years.