Torsdag 1/2

Before solving the exercises in Section 1.7, it is worth noting the following fact. Recall that in class we defined *linear independence* of vectors $v_1, ..., v_k$ as meaning that every vector that can be expressed as a linear combination of them can be done so in only one way. Now we have :

Proposition *Let A be an $m \times n$ matrix. The following statements are all equivalent to the columns of A being linearly independent vectors in $\mathbf{R}^m$:*

(i) the linear transformation $x \mapsto Ax$ from $\mathbf{R}^n$ to $\mathbf{R}^m$ is injective
(ii) for each $b \in \mathbf{R}^m$ the equation $Ax = b$ has at most one solution
(iii) the only solution to $Ax = 0$ is $x = 0$
(iv) every column of an echelon form of A contains a pivot.

PROOF : The equivalence of our definition and (ii) follows from the fact that the matrix equation $Ax = b$ expresses b as a linear combination of the columns of A. Statements (i) and (ii) are equivalent by definition. That (ii) is equivalent to (iii) follows from Theorem 6, p.53. One then just needs to observe that (iii) is equivalent to (iv).

1.7.6 The following sequence of row operations

$$\begin{aligned} R_3 &\mapsto 4R_3 + R_4, \\ R_4 &\mapsto 4R_4 + 5R_1, \\ R_3 &\mapsto R_3 - 3R_2, \\ R_4 &\mapsto R_4 + R_2, \\ R_3 &\leftrightarrow R_4 \end{aligned}$$

reduces the matrix to the echelon form

$$\begin{pmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 28 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since every column contains a pivot, we deduce that the columns of the original matrix are linearly independent in this case.

1.7.8 This is a 3×4 matrix. As shown in class, since $3 < 4$, this means that the corresponding linear transformation from $\mathbf{R}^4$ to $\mathbf{R}^3$ cannot be injective. By part (i) of the above Proposition, it follows that the columns of the matrix are not linearly independent.

If one wants, one can go further and by the usual Gauss elimination process, find numbers x_1, x_2, x_3, x_4, not all zero, such that

$$x_1 \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 7 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

1.7.22 **(a)** True.
(b) False. For example take

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v_2 \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$$

We have two vectors with three entries each, but $v_2 = 2v_1$ so the two vectors are not linearly independent.
(c) True. What is stated is that there are some constants c_1 and c_2 such that $z = c_1 x + c_2 y$. To put it another way, the vector $v := (c_1, c_2, 1)^T$ solves $Av = 0$ where A is the matrix with columns x, y and z.
(d) False. This is just a reformulation of **(b)**.

1.7.34 This is true. For any constant c, the vector $(0, 0, c, 0)^T$ solves $Ax = 0$ where A is the 4×4 matrix whose columns are v_1, v_2, v_3 and v_4.

1.7.36 False. This could be true, but doesn't have to be, since the set $\{v_1, v_2, v_4\}$ could already be linearly dependent. What IS always true is that a finite set of vectors is linearly independent if none of them can be expressed as a linear combination of the others.

1.7.38 True. In general, any subset of a linearly independent set of vectors is still linearly independent.

1.8.13 T is a reflection in the origin. $Tu = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$ and $Tv = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$.

1.8.14 T is a contraction by a factor of 2. We have $Tu = \begin{pmatrix} 5/2 \\ 1 \end{pmatrix}$ and $Tv = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

1.8.15 T is a projection onto the y-axis. $Tu = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $Tv = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$.

1.8.16 T is a reflection in the line $y = x$. We have $Tu = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $Tv = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

1.8.20 If you think about it for a minute, you'll see that this is just another way of saying that the columns of A are v_1 and v_2, i.e.: $A = \begin{pmatrix} -2 & 5 \\ 7 & -3 \end{pmatrix}$.

1.8.21 **(a)** True.
(b) False. Rather $\mathbf{R}^3$ is the codomain.
(c) False, unless A is surjective. Note the subtle difference in the meaning of the terms *range* and *codomain* : see p.73-4 for the definitions.
(d) True : this is Theorem 10, p.83. Note that there is a very confusing sentence on p.77 which reads

'Every matrix transformation is a linear transformation. Important examples of linear transformations that are not matrix transformations will be discussed in Chapters 4 and 5'.

This text suggests that the answer to the present exercise should be FALSE. However, the text is rubbish in the context of linear transformations from $\mathbf{R}^n$ to $\mathbf{R}^m$ or, more generally (in the language of Chapter 4), in the context of finite-dimensional vector spaces.

Since infinite-dimensional vector spaces are not discussed at all in this course (in which context the text above becomes valid), please ignore this text completely.

1.8.32 The point here is that the map $f(x) = |x|$ is not a linear map from $\mathbf{R}$ to $\mathbf{R}$. Namely, it is not true for arbitrary real numbers x and y that

$$|x + y| = |x| + |y|.$$

Indeed, this equality holds if and only if x and y have the same sign.

For the present exercise, we get a counterexample to T being linear by choosing input vectors (x_1, x_2) and (y_1, y_2) in which x_2 and y_2 have opposite signs.

1.9.6 It is already given that

$$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix},\quad T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}.$$

Thus the matrix of T is $\begin{pmatrix}1&3\\0&1\end{pmatrix}$.

1.9.8 The first reflection takes a point (x,y) to $(-x,y)$ and the second then takes this to $(y,-x)$. Thus $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}y\\-x\end{pmatrix}$. In particular,

$$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\-1\end{pmatrix}\text{ and }T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}.$$

So the matrix of T is $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$.

1.6.6 Write the reaction as

$$x_1(\mathrm{Na_3PO_4})+x_2(\mathrm{Ba(NO_3)_2})\to x_3(\mathrm{Ba_3(PO_4)_2})+x_4(\mathrm{NaNO_3}).$$

Equating the number of atoms of, respectively, Na, P, Ba, N and O gives the following system of 5 equations in the 4 unknowns x_1,x_2,x_3,x_4 :

$$\begin{aligned}3x_1&=x_4,\\x_1&=2x_3,\\x_2&=3x_3,\\2x_2&=x_4,\\4x_1+6x_2&=8x_3+3x_4.\end{aligned}$$

This is a pretty easy system to solve (use Gauss elimination if you like). The general solution has x_4 as a free parameter and is given by

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=x_4\cdot\begin{pmatrix}1/3\\1/2\\1/6\\1\end{pmatrix}.$$

Thus, since we are only interested in solutions in which each x_i is an integer, the simplest way to balance the equation is to set

$$x_1=2,\quad x_2=3,\quad x_3=1,\quad x_4=6.$$

1.6.12 **(a)** Equating the inward and outward traffic flows in each of the 4 junctions A,B,C and D gives respectively the following four equations in the unknowns $x_1, ..., x_5$:

$$\begin{aligned} x_1 &= 40 + x_4 + x_3, \\ 200 &= x_1 + x_2, \\ x_2 + x_3 &= 100 + x_5, \\ x_4 + x_5 &= 60. \end{aligned}$$

The augmented matrix for this system is

$$\begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 40 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{pmatrix}.$$

The sequence of row operations

$$R_2 \mapsto R_2 - R_1, \quad R_3 \mapsto R_3 - R_2, \quad R_4 \mapsto R_4 + R_3, \quad R_3 \mapsto -R_3$$

produces the echelon form

$$\begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can choose x_3 and x_5 as free parameters, and the general solution is then

$$\text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\} + \begin{pmatrix} 100 \\ 100 \\ 0 \\ 60 \\ 0 \end{pmatrix}.$$

(b,c) If that road is closed then $x_4 = 0$. Then $x_5 = 60$ and the general solution now involves only one free parameter and can be written as

$$\text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} + \begin{pmatrix} 40 \\ 160 \\ 0 \\ 0 \\ 60 \end{pmatrix}.$$

Here x_3 is the free parameter. Assuming x_3 must be non-negative (a one-way road system), the minimum value of x_1 equals 40, and is achieved by taking $x_3 = 0$. In that case, the unique full solution is

$$x_1 = 40, \quad x_2 = 160, \quad x_3 = x_4 = 0, \quad x_5 = 60.$$

Tillägg

I wish to finish the discussion of the three facts I wrote on the board at the end of the last lecture. Recall what these were :

If A is an $m \times n$ matrix, then the corresponding linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$

(i) cannot be surjective if $n < m$,
(ii) cannot be injective if $n > m$,
(iii) is injective if and only if it is surjective, when $n = m$.

Already (i) was discussed in class. Note that (iii) follows from the argument I used in solving exercise **1.4.34** on Tuesday, since this argument applies to any number of dimensions, not just three.
Part (ii) is explained by similar arguments. Multiplication by A is injective if and only if (see the Proposition at the beginning of this document) the equation $Ax = 0$ has only the trivial solution $x = 0$. But if $n < m$ then the matrix A has more columns than rows, so the RREF form of A must have free parameters. Thus there will unavoidably be infinitely many solutions to $Ax = 0$.

Intuitively, what part (ii) says is that you can't map a higher dimensional space in a 1-1 manner onto a lower dimensional space, by a *rigid* mapping like a linear transformation. Some of the dimensions have to get 'squashed to a point', and then you're not 1-1. If you want a picture, think of mapping a plane onto a line.

Tisdag 20/2

4.5.22 Standardbasen till $\mathbf{P}_3$ är

$$\boldsymbol{e}_1 = 1, \quad \boldsymbol{e}_2 = t, \quad \boldsymbol{e}_3 = t^2, \quad \boldsymbol{e}_4 = t^3.$$

I termer av standardbasen ges de fyra Laguerre polynomen av

$$\boldsymbol{e}_1, \quad \boldsymbol{e}_1 - \boldsymbol{e}_2, \quad 2\boldsymbol{e}_1 - 4\boldsymbol{e}_2 + \boldsymbol{e}_3, \quad 6\boldsymbol{e}_1 - 18\boldsymbol{e}_2 + 9\boldsymbol{e}_3 - \boldsymbol{e}_4.$$

Dessa representeras alltså med koordinatvektorerna

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ -18 \\ 9 \\ -1 \end{pmatrix}.$$

Laguerre polynomen utgör alltså en bas för $\mathbf{P}_3$ om och endast om dessa fyra vektorer spänner upp $\mathbf{R}^4$, dvs om och endast om 4×4 matrisen A vars kolonner är dessa vektorer är inverterbar. Vi har att

$$A = \begin{pmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Matrisen är triangulär och därmed inverterbar (med determinant $1 \cdot (-1) \cdot 1 \cdot (-1) = 1$).

4.6.4 Echelonformen B har tre rader skilda från nollvektorn så $\operatorname{rank}(B) = \operatorname{rank}(A) = 3$. A har 6 rader och därför är $\dim \operatorname{Nul}(A) = 6- \operatorname{rank}(A) = 6 - 3 = 3$. De tre första raderna i antingen B eller A utgör en bas för $\operatorname{Row}(A)$. Kolonner nr. 1,2 och 4 innehåller pivotelementen, och dessa kolonner i A utgör en bas för $\operatorname{Col}(A)$. Vi hittar en bas till $\operatorname{Nul}(A)$ genom att lösa $B\boldsymbol{x} = \mathbf{0}$ via baksubstitution. Här är $\boldsymbol{x} = (x_1, x_2, ..., x_6)^T$ och x_3, x_5, x_6 är de fria variablerna. Vi kollar att

$$\begin{aligned} x_4 &= x_5 + 2x_6, \\ x_2 &= x_3 - 7x_5 - 3x_6, \\ x_1 &= 2x_3 - 9x_5 - 2x_6, \end{aligned}$$

och därmed ges den allmänna lösningen till $B\boldsymbol{x} = \mathbf{0}$ av

$$\left\{ x_3 \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \cdot \begin{pmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_6 \cdot \begin{pmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} : x_3, x_5, x_6 \in \mathbf{R} \right\}.$$

De tre vektorerna ovan utgör då en bas för $\mathrm{Nul}(A)$.

4.6.15/16 För en $m \times n$ matris gäller å ena sidan att (Sats 14, s.265)

$$\mathrm{rank}\ A + \dim\ \mathrm{Nul}\ A = n,$$

och å andra sidan att

$$\mathrm{rank}\ A \leq \min\{m, n\}.$$

Dessa medför att den minsta möjliga dimensionen av $\mathrm{Nul}(A)$ är $n - \min\{m, n\}$, dvs noll om $n \leq m$ och $n - m$ om $n > m$.

Svaren till dessa två uppgifter är därmed 2 resp. 0.

4.6.18 (a) Sant.
(b) Det är lite otydligt vad som menas här. Men vad som är SANT är följande : om en viss grupp av rader i matrisen är linjärt oberoende (resp. beroende) och vi utför radoperationer på dessa, som inte blandar in andra rader, så förblir dessa rader linjärt oberoende (resp. beroende) i den framtagna matrisen.
(c) Sant.
(d) Sant (per definition).
(e) Sant : detta är en omformulering av vad jag sa på föreläsningen, att när man utför radoperationer på en matris så ändras inte dess radrum.

Torsdag 1/3

5.1.18 The eigenvalues are 4,0 and -3, i.e.: the entries on the main diagonal of the matrix. This is the case for any triangular matrix (Theorem 1, p.306), a fact which is well worth knowing. Note that it is a consequence (see the proof of the theorem) of the fact that the determinant of a triangular matrix is the product of the diagonal entries (Theorem 2, p.189).

5.1.22 (a) False. $\boldsymbol{x}$ could be the zero vector. Otherwise, the claim is true.
(b) False. A single eigenspace can have dimension greater than one.
(c) Ignore, we haven't discussed stochastic matrices.
(d) False. The claim is true for certain matrices, for example triangular ones.
(e) True. For an eigenvalue λ, the corresponding eigenspace is the nullspace of the matrix $A - \lambda I_n$.

5.1.27 (see also 5.2.20) λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$. Any matrix has the same determinant as its transpose (Theorem 5, p.196). Now $(A - \lambda I_n)^T = A^T - (\lambda I_n)^T = A^T - \lambda I_n$, since λI_n is a diagonal, and hence symmetric, matrix.

Thus $\det(A - \lambda I_n) = 0$ if and only if $\det(A^T - \lambda I_n) = 0$. In other words, λ is an eigenvalue of A if and only if it is an eigenvalue of A^T, v.s.v.

5.2.14 I'll do more than what is asked, i.e.: I'll find all eigenvalues and eigenvectors, and hence diagonalise the matrix. This is the kind of basic thing you need to be able to do.

We have

$$A - \lambda I_3 = \begin{pmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{pmatrix}.$$

The characteristic polynomial $p(\lambda)$ for A is the determinant of this matrix. This is most easily computed by a cofactor expansion along the second row. Thus we find that

$$p(\lambda) = (1-\lambda) \cdot \begin{vmatrix} 5-\lambda & 3 \\ 6 & -2-\lambda \end{vmatrix} =$$

$$(1-\lambda)\left[(5-\lambda)(-2-\lambda) - 6 \cdot 3\right] = (1-\lambda)(\lambda^2 - 3\lambda - 28) = (1-\lambda)(\lambda-7)(\lambda+4).$$

The eigenvalues of A are the solutions of $p(\lambda) = 0$, i.e.: there are three distinct eigenvalues $\lambda_1 = 1$, $\lambda_2 = 7$, $\lambda_3 = -4$. Let us now find the corresponding eigenvectors :

$\lambda_1 = 1$: We must compute the nullspace of

$$A - I_3 = \begin{pmatrix} 4 & -2 & 3 \\ 0 & 0 & 0 \\ 6 & 7 & -3 \end{pmatrix}.$$

The row operations $R_2 \leftrightarrow R_3$, $R_2 \mapsto 2R_2 - 3R_1$, $R_2 \mapsto \frac{1}{5}R_2$ take the matrix to the echelon form

$$\begin{pmatrix} 4 & -2 & 3 \\ 0 & 4 & -3 \\ 0 & 0 & 0 \end{pmatrix},$$

from which we deduce that the λ_1-eigenspace is spanned by the vector $\boldsymbol{v}_1 := (-3, 6, 8)^T$.

$\lambda_2 = 7$: We must compute the nullspace of

$$A - 7I_3 = \begin{pmatrix} -2 & -2 & 3 \\ 0 & -6 & 0 \\ 6 & 7 & -9 \end{pmatrix}.$$

The row operations $R_2 \leftrightarrow R_3$, $R_2 \mapsto R_2 + 3R_1$, $R_3 \mapsto R_3 - 6R_2$, $R_2 \mapsto -R_2$ take the matrix to the echelon form

$$\begin{pmatrix} -2 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

from which we deduce that the λ_2-eigenspace is spanned by the vector $\boldsymbol{v}_2 := (3, 0, 2)^T$.

$\lambda_3 = -4$: We must compute the nullspace of

$$A + 4I_3 = \begin{pmatrix} 9 & -2 & 3 \\ 0 & 5 & 0 \\ 6 & 7 & 2 \end{pmatrix}.$$

The row operations $R_2 \leftrightarrow R_3$, $R_2 \mapsto 3R_2 - 2R_1$, $R_3 \mapsto 5R_3 - R_2$, $R_2 \mapsto \frac{1}{25}R_2$ take the matrix to the echelon form

$$\begin{pmatrix} 9 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

from which we deduce that the λ_3-eigenspace is spanned by the vector $\boldsymbol{v}_3 := (-1, 0, 3)^T$.

Finally, we can now diagonalise A. We have that

$$P^{-1}AP = D$$

where

$$P = \begin{pmatrix} | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \\ | & | & | \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Thus in this exercise,

$$P = \begin{pmatrix} -3 & 3 & -1 \\ 6 & 0 & 0 \\ 8 & 2 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

5.3.14 Since there are only two eigenvalues, we'll be able to diagonalise if and only if one of the eigenspaces is 2-dimensional. So let's see :

$\lambda_1 = 4$: We must compute the nullspace of

$$A - 4I_3 = \begin{pmatrix} 0 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

The row operations $R_1 \leftrightarrow R_2$, $R_3 \mapsto 2R_3 + R_2$, $R_2 \mapsto -\frac{1}{2}R_2$ take the matrix to the echelon form

$$\begin{pmatrix} 2 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

from which we deduce that the λ_1-eigenspace is one-dimensional and spanned by the vector $\boldsymbol{v}_1 := (1, -2, 0)^T$.

$\lambda_2 = 5$: We must compute the nullspace of

$$A - 5I_3 = \begin{pmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

The row operations $R_2 \mapsto R_2 + 2R_1$, $R_1 \mapsto -R_1$ take the matrix to the echelon form

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

from which we deduce that the λ_2-eigenspace is two-dimensional and spanned by the vectors $\boldsymbol{v}_2 := (0,1,0)^T$ and $\boldsymbol{v}_3 = (-2,0,1)^T$.

Since we have a two-dimensional eigenspace, we can diagonalise A. Thus

$$P^{-1}AP = D$$

where

$$P = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

5.3.21 **(a)** False. The word *diagonal* should be inserted between 'some' and 'matrix' for the right definition of diagonalisability.
(b) True. One can take P to be the matrix whose columns are such a basis of eigenvectors.
(c) False. If all the eigenvalues are distinct then A is definitely diagonalisable. This is sometimes true even when there are multiple eigenvalues, but not always. Note that, counting multiplicities, there are always n complex eigenvalues for an $n \times n$ matrix.
(d) False. A matrix is invertible if and only if zero is not an eigenvalue. A simple example of a diagonalisable matrix with zero as one of its eigenvalues is any diagonal matrix with a zero on the diagonal, e.g.: $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

5.3.22 **(a)** False. Any matrix has infinitely many eigenvectors, because the eigenvectors are grouped into eigenspaces. What is TRUE is that A is diagonalisable if it has n *linearly independent* eigenvectors.

(b) False, but the converse is true : see 5.3.21(c) above.
(c) True. See class notes. This is where we got the eigenvalue equation from.
(d) False. These are quite different concepts.

5.3.26 Yes it is possible. The third eigenspace will either be one- or two-dimensional, and A will be diagonalisable if and only if the latter case holds.

5.3.28 An $n \times n$ matrix has n linearly independent eigenvectors if and only if it is diagonalisable (see 5.3.22(a) above). Thus we can reformulate the question as :

'Show that if a square matrix A is diagonalisable, then so is A^T'

So suppose A is diagonalisable, i.e.: suppose there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

Transpose both sides. Thus

$$A^T = (PDP^{-1})^T = (P^{-1})^T D^T P^T = Q\Delta Q^{-1},$$

where we have set $Q := (P^{-1})^T$ and $\Delta := D^T$. Obviously Q is invertible since P is, and Δ is diagonal since the transpose of a diagonal matrix is itself. Thus A^T is also diagonalisable, v.s.v.

5.4.14 The basis should consist of eigenvectors of A. The characteristic equation for A is

$$0 = p(\lambda) = (5 - \lambda)(1 - \lambda) - 21 = \lambda^2 - 6\lambda - 16 = (\lambda - 8)(\lambda + 2),$$

so we have the distinct eigenvalues $\lambda_1 = 8$, $\lambda_2 = -2$. One then checks that the corresponding eigenspaces are spanned by the vectors $\boldsymbol{v}_1 := (1, -1)^T$ and $\boldsymbol{v}_2 := (3, 7)^T$.

Thus we can take $\mathcal{B} := \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Tisdag 6/2

6.3.22 (a) Sant. Jag bevisade detta på föreläsningen i måndags : om $\boldsymbol{x} \in W^\perp$ så innebär det att $\boldsymbol{x} \cdot \boldsymbol{w} = 0$ för alla $\boldsymbol{w} \in W$. Men om $\boldsymbol{x}$ självt är också i W så följer speciellt att $\boldsymbol{x} \cdot \boldsymbol{x} = 0$. Men $\boldsymbol{x} \cdot \boldsymbol{x} = ||\boldsymbol{x}||^2$ så $||\boldsymbol{x}|| = 0$, dvs $\boldsymbol{x} = \boldsymbol{0}$.
(b) Sant. Varje term har formen

$$\left(\frac{\boldsymbol{y} \cdot \boldsymbol{u}_i}{||\boldsymbol{u}_i||^2} \right) \boldsymbol{u}_i,$$

som är ortogonalprojektionen av $\boldsymbol{y}$ på det 1-dimensionella underrummet som spänns upp av $\boldsymbol{u}_i$.
(c) Sant. Följer från entydigheten av ortogonaldecompositionen.
(d) Falskt. Det är snarare själva $\text{proj}_W \boldsymbol{y}$ (som jag har betecknat med $\hat{\boldsymbol{y}}_W$) som är den bästa uppskattningen till $\boldsymbol{y}$ inom W.
(e) Falskt. En sådan matris U satisfierar $U^T U = I_n$, som jag visade i samband med uppgift **6.2.24** (se Theorem 6, p.390). Om $n = p$ så innebär detta att $U^T = U^{-1}$ och då gäller att $UU^T = I_n$ också. Så påståendet är faktiskt sant i fallet $n = p$.

I allmänhet, dock, det som gäller ges av Theorem 10, p.399, nämligen att $UU^T\boldsymbol{x}$ är lika med projektionen av $\boldsymbol{x}$ på underrummet till $\boldsymbol{R}^n$ som spänns upp av kolonnerna till U. Alltså $UU^T\boldsymbol{x} = \boldsymbol{x}$ om och endast om $\boldsymbol{x}$ ligger i detta underrum, dvs om och endast om $\boldsymbol{x}$ ligger i $\text{Col}(U)$.