Answers to even numbered exercises

1.1

20. $h = -2$.
22. $h = -5/3$.
23. True, false, true, true.
24. True, false, false, false.
34. $T_1 = 20$, $T_2 = 27.5$, $T_3 = 30$, $T_4 = 22.5$.

1.2

20 (a) $h = 9, k \neq 6$ **(b)** $h \neq 9$, any k **(c)** $h = 9$, $k = 6$.
21. False, false, true, true, false.
22. False, false, true, false, true.
24. No, since there will be a row of zeroes in the coefficient matrix to the left of this pivot.
26. Back substitution will produce a unique solution.
28. There should be a pivot in each column of the coefficient matrix, but not in the right-hand column of the augmented matrix.
30. $x + y + z = 1$ and $x + y + z = 2$.
32. About half as $n \to \infty$ (I think, but don't really care).
34. Matlab exercise.

1.3

23. True, false, true, true, false ($\boldsymbol{u}$ and $\boldsymbol{v}$ could be collinear).
24. True, true, false, true, true.

1.4

16. The echelon form of the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 0 & 0 & b_1 + 2b_2 + b_3 \end{array}\right].$$

Thus there is a solution if and only if $b_1 + 2b_2 + b_3 = 0$.
23. False, true, false (true if we replace the words 'augmented matrix' by 'coefficient matrix'), true, true, true.
24. True, true, true, true, false, true.

1.5

23. True, false, false, false, false (since they don't say what $\boldsymbol{p}$ is).
24. False, true, false, true, true (assuming some solution exists).
26. See 24(e).

1.7

21. True (assuming they mean ONLY the trivial solution), false, true, true.
22. True, false, true, false.
24. The second row must have all zeroes.
26. In other words $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are linearly independent. Then the echelon form has exactly one row of zeroes.
28. 5 (if there were a row of zeroes in the echelon form of the matrix, which we'll call A, then there would be no solution to $A\boldsymbol{x} = \boldsymbol{b}$ for some $\boldsymbol{b}$).
30. n.
34. True (really stupid question !).
36. False, whenever $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_4$ are already linearly dependent.
38. True. Any subset of a linearly independent set of vectors is linearly independent. Equivalently, any superset of a linearly dependent set of vectors is linearly dependent.

1.8

21. True, false, true, true, true.
22. True, true, false (it's an 'existence' question), true, true.

1.9

4.

$$\begin{bmatrix} \cos(-\pi/4) & -\sin(-\pi/4) \\ \sin(-\pi/4) & \cos(-\pi/4) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

23. True, true, false, false, false.
24. False, true, true, false, true.
32. m (see Theorem 12).

2.1

15. False, false, true, true, false.
16. False (true without the + signs), True, False, False, True.
22. In general, the columns of an $m \times n$ matrix M are linearly dependent if and only if there is a non-zero vector $\boldsymbol{x} \in \mathbb{R}^n$ such that $M\boldsymbol{x} = \mathbf{0}$.

So suppose the columns of B are linearly dependent. Thus there exists a non-zero vector $\boldsymbol{x}$ such that $B\boldsymbol{x} = \mathbf{0}$. Multiply both sides of this equation on the left by A, and we have $A(B\boldsymbol{x}) = A \cdot \mathbf{0} = \mathbf{0}$. But matrix multiplication is associative, so $A(B\boldsymbol{x}) = (AB)\boldsymbol{x}$. Thus $(AB)\boldsymbol{x} = \mathbf{0}$ so, by the same reasoning as before, the columns of AB must be linearly dependent.
24. Let $\boldsymbol{b}$ be given and multiply both sides of the equation $AD = I_m$ on the right by $\boldsymbol{b}$. This yields $(AD)\boldsymbol{b} = \boldsymbol{b}$. By associativity of matrix multiplication, the left-hand side of this equals $A(D\boldsymbol{b})$. But then we have indeed a solution to $A\boldsymbol{x} = \boldsymbol{b}$, namely $\boldsymbol{x} = D\boldsymbol{b}$.

2.2

9. True (in the sense that a right-inverse must always be a left-inverse too, and vice versa), False, False (e.g.: $A = \begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix}$), True, True.
10. False, True, True, True, False.
12. Row reduction $A \sim I$ corresponds to left-multiplication by A^{-1} (thought of as a product of elementary matrices). Performing the same row reduction on B thus results in $A^{-1}B$, v.s.v.
32. The matrix is not invertible, since the row operations $R_2 \mapsto R_2 - 4R_1$, $R_3 \mapsto R_3 + 2R_1$, $R_3 \mapsto R_3 - 2R_2$ take it to the echelon form $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

2.3

11. True, True, False (unless A is invertible), True, True.
12. True, True, True, False, True.

3.1

39. True, False (a subtle matter of terminology, since in the text the (i, j)-th cofactor is defined to be the number $C_{i,j} = (-1)^{i+j} \det A_{ij}$).
40. False, False (true if we replace 'sum' by 'product').

3.2

27. True (since by 'row replacement operation' he means adding to a row some multiple of another : see page 197 and Theorem 3(a) on page 192), True, True, False.
28. True, False, False, False.
32. $\det(rA) = r^n(\det A)$.

3.3

26. A typical vector $\boldsymbol{v}$ in the set $\boldsymbol{p} + S$ is of the form $\boldsymbol{v} = \boldsymbol{p} + \boldsymbol{s}$, for some vector $\boldsymbol{s} \in S$. Applying T and using linearity we have $T(\boldsymbol{v}) = T(\boldsymbol{p} + \boldsymbol{s}) = T(\boldsymbol{p}) + T(\boldsymbol{s})$, which is just a typical element of $T(\boldsymbol{p}) + T(S)$ since, by definition, $T(S) = \{T(\boldsymbol{s}) : \boldsymbol{s} \in S\}$.
32. Let T_1, T_2 be the names of the tetrahedra with sides $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ and $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ respectively. By the formula for the volume of a tetrahedron given in the text, we have that $\text{Vol}(T_1) = 1/6$, since it has perpendicular height one and its base is an equilateral triangle of side-length one, thus of area $1/2$.

Now the linear transformation defined by $T(\boldsymbol{e}_i) = \boldsymbol{v}_i$, $i = 1, 2, 3$ transforms T_1 to T_2. By definition, the matrix of this transformation is $M_T = [\boldsymbol{v}_1 \ \ \boldsymbol{v}_2 \ \ \boldsymbol{v}_3]$, i.e.: the 3×3 matrix whose columns are the $\boldsymbol{v}$-vectors. By the geometric definition of determinant, we have that $\text{Vol}\ T_2 =$
$|\det M_T| \cdot (\text{Vol}\ T_1)$. Thus, by what we noted at the outset, it follows that

$$\text{Vol}(T_2) = \pm\frac{1}{6}\begin{vmatrix} | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \\ | & | & | \end{vmatrix},$$

the sign depending on whether the determinant is positive or negative.

4.1

4. I will try so say this in words. Draw any line $\mathcal{L}$ in the plane not passing through $(0,0)$. Pick any point P on the line and let $\boldsymbol{v}$ be the vector $\vec{OP}$. Consider $2\boldsymbol{v}$. This is the vector $\vec{OQ}$, where Q is the point along the line through O and P, which is twice as far away from O as is P and in the same direction. Clearly, this point is not on your line $\mathcal{L}$, thus proving that $\mathcal{L}$ is not a subspace of $\mathbb{R}^2$.

20 (a) You need to know that a sum of two continuous functions is continuous, as is a scalar multiple of a continuous function (see Adams, Section 1.4, Theorem 6).
(b) Suppose $f(a) = f(b)$ and $g(a) = g(b)$. Then, clearly, $(f + g)(a) = (f + g)(b)$. Also $(cf)(a) = (cf)(b)$, so the set of functions under consideration is closed under addition and scalar multiplication, hence a subspace of $C[a, b]$.

23. False, False (the opposite is true), False, True, False (they're digital, according to the waffle at the start of the chapter).
24. True, True, True (of itself),
False, though it is *isomorphic* to a subspace of $\mathbb{R}^3$, for example the subspace of all vectors whose z-component is zero,
False, since it doesn't say what $\boldsymbol{u}$ and $\boldsymbol{v}$ are. The statement would be true if it read instead : (ii) for any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in H, it is also the case that $\boldsymbol{u} + \boldsymbol{v}$ is in H (iii) if $\boldsymbol{u}$ is in H then so is $c\boldsymbol{u}$, for any scalar c.

36. $\boldsymbol{y}$ is in Col(A) if and only if the system $A\boldsymbol{x} = \boldsymbol{y}$ has a solution. So run the command $A\backslash\boldsymbol{y}$ and see if you get an error message.

4.2

25. True, False, True, True (if he means that the equation is consistent for EVERY $\boldsymbol{b}$), True, True.
26. True, True, False, True, True, True (don't bother yet as to why).
30. Let $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ be any two vectors in the range of T and c any scalar. We must show that both $\boldsymbol{w}_1 + \boldsymbol{w}_2$ and $c\boldsymbol{w}_1$ are in the range of T.

Since both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are in the range of T there exist, by definition, vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in V such that

$$T(\boldsymbol{v}_1) = \boldsymbol{w}_1, \qquad T(\boldsymbol{v}_2) = \boldsymbol{w}_2.$$

But T is linear, thus

$$T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2) = \boldsymbol{w}_1 + \boldsymbol{w}_2$$

and

$$T(c\boldsymbol{v}_1) = cT(\boldsymbol{v}_1) = c\boldsymbol{w}_1.$$

This equations show that both $\boldsymbol{w}_1 + \boldsymbol{w}_2$ and $c\boldsymbol{w}_1$ are in the range of T, as desired.

4.3

4. The matrix with these three vectors as its columns can be Gauss-reduced to the diagonal matrix

$$\begin{bmatrix} 2 & 1 & -7 \\ 0 & 1 & 1 \\ 0 & 0 & 12 \end{bmatrix}.$$

Thus, these three vectors are linearly independent and form a basis for $\mathbb{R}^3$.
10. The matrix can be Gauss-reduced to

$$\begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ 0 & 1 & -4 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}.$$

Take x_3 and x_5 as free variables, and perform back substitution to get that a general element of the nullspace can be written as

$$x_3 \cdot \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \cdot \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

The two vectors above then form a basis for the nullspace.
21. False, False, True, False, True.
22. False, True, True, False, False (rather the corresponding columns in A itself).
30. Let A be the $n \times k$ matrix which has these vectors as its columns. If these vectors formed a basis for $\mathbb{R}^n$ then, in particular, they would be linearly independent. This would mean that Nul(A) would contain only the zero vector. But since A has more columns than rows, there will remain at least one free variable after Gauss elimination on A, and thus Nul(A) contains non-zero vectors (see Section 1.5, for example, though I don't know what theorem exactly he wants you to use (and it doesn't matter !)).
36, 38. Matlab exercises.

4.4

10. $\begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix}$.

15. True, False, False ($\mathbb{P}_3 \cong \mathbb{R}^4$).

16. True, False (other way round), True (namely, if the plane passes through the origin).

4.5

6. Write out the subspace more explicitly as

$$\left\{ a \cdot \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix} + b \cdot \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix} + c \cdot \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

Here the first vector is just -3 times the third one. So the space has dimension 2, and a basis is

$$\left\{ \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

14. Both are 3-dimensional.

19. True, False (unless it goes through the origin), False, False (unless S contains exactly n vectors), True.

20. False (see 4.1.24(d)), False (rather the number of FREE variables), False, False (see 19(d)), True.

29. True, True, True.

30. False, True, False.

4.6

17. True, False (unless B was obtained from A without any row interchanges), True, False (unless A is square), ?? (don't know what he means and don't care !).

18. False (see 4.3.22(e)), False, True, True, True.

30. They must be equal, since consistency means that $\boldsymbol{b}$ is a linear combination of the columns of A, hence adding $\boldsymbol{b}$ as a column to the matrix does not increase the dimension of its column space.

4.7

10. $\begin{bmatrix} 2 & 3 \\ -5 & -8 \end{bmatrix}$ and $\begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$ respectively.
11. False, True.
12. True, False (rather it satisfies $[\boldsymbol{x}]_{\mathcal{C}} = P[\boldsymbol{x}]_{\mathcal{B}}$).

5.1

6. Compute

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = (-2) \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Thus, yes, it is an eigenvector, with corresponding eigenvalue -2.
21. False ($\boldsymbol{x}$ must be non-zero), True, True, True, False (row operations can change the eigenvalues of a matrix).
22. False (it's true if $\boldsymbol{x}$ is not the zero vector), False (opposite true), True, False, True.

5.2

18. $h = 6$.
20. We know that for any matrix M it holds that $\det M = \det M^T$. Let λ be a scalar. Then

$$\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I).$$

Thus $\det(A - \lambda I) = 0$ if and only if $\det(A^T - \lambda I) = 0$. In other words, λ is an eigenvalue of A if and only if it is an eigenvalue of A^T, v.s.v.
21. False, False, True, False (rather -5 is then an eigenvalue).
22. False (the volume equals $|\det A|$), False, True,
False : as an example, take $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. This is diagonal, so its only eigenvalue is $\lambda = 1$. The row replacement $R_2 \mapsto R_2 - R_1$ produces the matrix $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$. One can check that the characteristic polynomial for this matrix is $\lambda^2 - \lambda + 1$, so there are two complex eigenvalues $\lambda_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{3}i\right)$.

24. Similarity means that there exists an invertible matrix P such that $B = P^{-1}AP$. Then

$$\det B = \det(P^{-1}AP) = (\det P^{-1})(\det A)(\det P) = \left(\frac{1}{\det P}\right)(\det A)(\det P) = \det A, \quad \text{v.s.v.}$$

30. Blah ...

5.3

21. False (D must be diagonal), True, False, False.
22. False (true if we add the words 'linearly independent'), False (converse true), True, False.

5.7

6. The solution is

$$\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{x}_1(t) \\ \boldsymbol{x}_2(t) \end{bmatrix} = \begin{bmatrix} 5e^{-t} - 2e^{-2t} \\ 5e^{-t} - 3e^{-2t} \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t}.$$

The origin is an attractor and the direction of greatest attraction is along the line $2y = 3x$.

6.1

6. $\frac{5}{49}\begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$.
8. 7.
19. True, true, true, false (rather Row(A) is orthogonal to Nul(A)), true.
20. True, false (rather $|c|$), true, true, true.
24. We have

$$||\boldsymbol{u} \pm \boldsymbol{v}||^2 = (\boldsymbol{u} \pm \boldsymbol{v}) \cdot (\boldsymbol{u} \pm \boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{u} + \boldsymbol{v} \cdot \boldsymbol{v} \pm 2(\boldsymbol{u} \cdot \boldsymbol{v}) = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 \pm 2(\boldsymbol{u} \cdot \boldsymbol{v}).$$

When we add, the terms $\pm 2(\boldsymbol{u} \cdot \boldsymbol{v})$ cancel and we're left with the right-hand side.
26. W is the nullspace of the 1×3 matrix with the single row $\boldsymbol{u}^T$. So he's probably referring to some theorem that says that the nullspace of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$. Geometrically, W is a plane through the origin with normal vector $\boldsymbol{u}$. Its equation is $5x - 6y + 7z = 0$.
34. Blah ...

6.2

23. True, true, false, true, false (rather $||\boldsymbol{y} - \hat{\boldsymbol{y}}||$).
24. False, false (true if $||\boldsymbol{u}_i|| = 1$ for each $i = 1, ..., p$), true, true, true.

6.3

21. True, true, false, true, true.
22. True, true, true, false (rather $\text{proj}_W\, \boldsymbol{y}$), false (true when $n = p$).

6.4

17. False (true if $c \neq 0$), true, true.
18. True, true, true.

6.5

17. True, true, false (rather $\geq$), true, true.
18. True, false, true, false (true if $A^T A$ is invertible), ?? (don't understand what he means by 'reliable'), True (I guess).

6.6

4. $y = \frac{1}{10}(43 - 7x)$.
10 (a) The model is $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} e^{-.02t_1} & e^{-.07t_1} \\ e^{-.02t_2} & e^{-.07t_2} \\ e^{-.02t_3} & e^{-.07t_3} \\ e^{-.02t_4} & e^{-.07t_4} \\ e^{-.02t_5} & e^{-.07t_5} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} M_A \\ M_B \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}.$$

7.1

2. Not symmetric.
4. Symmetric.
6. Not symmetric.
24. Check directly that $A\boldsymbol{v}_1 = 10\boldsymbol{v}_1$ and $A\boldsymbol{v}_2 = \boldsymbol{v}_2$. Thus we have at least two eigenvalues, $\lambda_1 = 10$ and $\lambda_2 = 1$. The matrix A is symmetric, so we know it must be orthogonally diagonalisable. Therefore, there must be a

third eigenvector $\boldsymbol{v}_3$, which is orthogonal to both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. But, since we're working in $\mathbb{R}^3$, there is only one possibility for such a vector, up to a scalar multiple, namely we can take

$$\boldsymbol{v}_3 = \boldsymbol{v}_1 \times \boldsymbol{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\vec{i} + \vec{j} - 4\vec{k}.$$

Thus $\boldsymbol{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$ must be an eigenvector. Now check directly that $A\boldsymbol{v}_3 = \boldsymbol{v}_3$. Thus the eigenvalue here is also $\lambda_2 = 1$, so this eigenspace is two-dimensional.

We then have an orthogonal diagonalisation

$$A = PDP^T,$$

where

$$D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} | & | & | \\ \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \\ | & | & | \end{bmatrix}$$

and $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ are the normalised eigenvectors, i.e.:

$$\boldsymbol{u}_1 = \frac{\boldsymbol{v}_1}{||\boldsymbol{v}_1||} = \frac{\boldsymbol{v}_1}{3}, \quad \boldsymbol{u}_2 = \frac{\boldsymbol{v}_2}{||\boldsymbol{v}_2||} = \frac{\boldsymbol{v}_2}{\sqrt{2}}, \quad \boldsymbol{u}_3 = \frac{\boldsymbol{v}_3}{||\boldsymbol{v}_3||} = \frac{\boldsymbol{v}_3}{\sqrt{18}}.$$

Thus finally

$$P = \begin{bmatrix} -2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & -4/\sqrt{18} \end{bmatrix}.$$

26. True, true, false, true.
28.

$$(A\boldsymbol{x}) \cdot \boldsymbol{y} = (A\boldsymbol{x})^T \boldsymbol{y} = (\boldsymbol{x}^T A^T)\boldsymbol{y} \overset{A=A^T}{=} (\boldsymbol{x}^T A)\boldsymbol{y} = \boldsymbol{x}^T(A\boldsymbol{y}) = \boldsymbol{x} \cdot (A\boldsymbol{y}), \quad \text{v.s.v.}$$