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A CONVOLUTION-THRESHOLDING APPROXIMATION OF GENERALIZED CURVATURE FLOWS*

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Abstract. We construct a convolution-thresholding approximation scheme for the geometric surface evolution in the case when the velocity of the surface at each point is a given function of the mean curvature. Conditions for the monotonicity of the scheme are found and the convergence of the approximations to the corresponding viscosity solution is proved. We also discuss some aspects of the numerical implementation of such schemes and present several numerical results.

Key words. generalized curvature flow, convolution-thresholding scheme, viscosity solution, level-set equation

AMS subject classifications. 65M12, 53C44, 49L25, 35K55

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1. Introduction.

The topic of curvature flows of different types was popular during the last 20 years and is still popular in both pure and applied mathematics. By curvature flow we mean a family $\{\Gamma_t\}_{t\geq 0}$ of hypersurfaces in $\mathbb{R}^n$ depending on time t with local normal velocity equal to the mean curvature or a function of it for generalized curvature flows. The mean curvature in turn denotes here the sum of principal curvatures.

In the three-dimensional case a smooth initial surface can develop singularities after some finite time. There have been several successful attempts to deal with singularities and topological complications: the varifold approach [7], [2], the phase field method [14], [8], and the level-set method. This approach was suggested in the physical literature [26] and was extensively developed for numerical purposes by Osher and Sethian [27]. The main idea of this method is to evolve some continuous function $u : [0,\infty)\times\mathbb{R}^n \mapsto \mathbb{R}$ in such a way that $\Gamma_t \subset \mathbb{R}^n$ would always be a level-set of $u\,(x,t)$, i.e., $\Gamma_t = \{x \in \mathbb{R}^n : u\,(x,t) = 0\}$ for all $t \geq 0$. In the case of the mean curvature flow, the evolution equation for u turns out to be

$$u_t = |Du|\,\mathrm{div}\left(\frac{Du}{|Du|}\right). \tag{1.1}$$

The evolution equation for a function u with each point of a level-set moving along the normal with velocity equal to some function G of the mean curvature is the so-called generalized mean curvature evolution PDE

$$u_t = |Du|\,G\left(\mathrm{div}\left(\frac{Du}{|Du|}\right)\right). \tag{1.2}$$

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This equation is degenerate parabolic. The existence and uniqueness of generalized viscosity solutions (see [12]) to the initial value problem

$$\begin{cases} u_t = |Du|\, G\left(\operatorname{div}\left(\frac{Du}{|Du|}\right)\right) & \text{in } \mathbb{R}^n \times (0,T)\,, \\ u = g\,(x) \in BUC\,(\mathbb{R}^n) & \text{on } \mathbb{R}^n \times \{0\} \end{cases} \tag{1.3}$$

was investigated in [17], [11], [22].

Curvature flows arise naturally in various problems. Among these are the fast reaction–slow diffusion problem [29], [4], [16], [19] and image processing [1].

In the present work we construct a class of approximations of a convolution-thresholding type to the generalized curvature flows. By this we mean the following. Assume that, initially, the surface under consideration is a boundary of a compact set $C \in \mathbb{R}^n$. Take compactly supported functions $\tilde{\rho}_i : \mathbb{R}_+, \longmapsto \mathbb{R}_+, i = 1, 2$ (in fact, one can also take $\tilde{\rho}_i$ with unbounded support decreasing fast for large x). We define $\rho_i : \mathbb{R}^n \mapsto \mathbb{R}_+$,

$$\rho_i\,(x) = \frac{1}{h^{n/2}}\tilde{\rho}_i\left(|x|\,/\sqrt{h}\right),$$

and introduce a convolution

$$M_i\,(C)\,(x,h) = \int_{\mathbb{R}^n} \chi_C\,(y)\,\rho_i\,(x-y)\,dy.$$

Now $M_i\,(C)\,(x,h)$ are functions of x, and we define a new position of the surface as a boundary of the set

$$\mathcal{H}_h C = \{x \in \mathbb{R}^n : F\,(M_1\,(C)\,(x,h)\,, M_2\,(C)\,(x,h)) \geq 0\}\,, \tag{1.4}$$

where F is some (thresholding) function. Next we follow Evans [15] and introduce an operator on the space of bounded functions $\mathbb{B}\,(\mathbb{R}^n)$: $H\,(h) : \mathbb{B}\,(\mathbb{R}^n) \mapsto \mathbb{B}\,(\mathbb{R}^n)$ by

$$[H\,(h)\,u]\,(x) = \sup\,\{\lambda \in \mathbb{R} : x \in \mathcal{H}_h\,[u \geq \lambda]\}\,. \tag{1.5}$$

The purpose of the present study is, for a given function G in (1.3), to find a corresponding thresholding function F in (1.4) so that $H\,(t/m)^m\,g\,(x)$ converges to the unique viscosity solution of (1.3) as $m \to \infty$.

Such a function in the case when G is linear was proposed by Merriman, Bence, and Osher in [25]. This result is often referred to as the Bence–Merriman–Osher method. Rigorous proofs of the convergence of such approximations can be found in [15], [20], and [3]. In this case it is enough to take a thresholding function depending only on one convolution.

Suppose that G is nonlinear. As we show in section 3, in this case one has to use two convolutions M_1 and M_2 and a thresholding function depending on two variables $F\,(M_1, M_2)$. This is necessary to ensure that the operator H is consistent with the PDE in (1.3). We also show how to choose convolution kernels in order to get a monotone H. These two conditions—monotonicity and consistency—are crucial for the convergence.

Using our approach we also suggest a new construction of higher order schemes for the classical curvature flows. The numerical experiments with these schemes show a considerable improvement in the accuracy.

Finite difference approximations for (1.3) have been studied in [27], [31], [13].

Another class of approximation operators, the so-called Matheron filters, comes from image processing. The connection between such operators and the mean curvature evolution PDE (1.2) was established in [10]. This result was then extended in [18] and [9].

Threshold dynamics models, introduced earlier in [21], lead to approximations of the solution of the Cauchy problem to a nonlinear parabolic equation, where the right-hand side can be interpreted as a general elliptic operator on a level set of the solution. This is a generalization of the curvature flow, but it does not entirely include (1.3) as a special case.

Another generalization of the Bence–Merriman–Osher method can be found in [23]. The author suggests an approximation procedure that allows tracking the surface evolution when the velocity of the surface depends also on the coordinates. The convergence of this approximation is also proved.

Outline. This paper is organized as follows. After introducing the basic notions and stating some results for viscosity solutions in section 2, we turn to our method of approximation for such solutions. In section 3, we construct F to get the convergence of the convolution-thresholding approximation to the viscosity solution of (1.3) with a monotone continuous function G. This is the main result of the paper. More precisely, the following local uniform convergence is proved:

$$((H\,(t/m))^m\, g)\,(x) \to u\,(x,t)\,, \qquad m \to \infty,$$

where H is defined by (1.5) and $u\,(x,t)$ is the viscosity solution of (1.3).

We use this construction for numerical calculation for some cases of the generalized curvature flows in $\mathbb{R}^2$ and $\mathbb{R}^3$. Numerical results and two approaches to the implementation are described in section 4.

2. The viscosity solution framework. Consider the nonlinear equation (1.2) on an open set $\Omega \times (0,T)$ with function G continuous and nondecreasing. This is the second order equation with a right-hand side that is monotonic and degenerate elliptic (see [12]) provided that G is nondecreasing and $Du \neq 0$. Viscosity solution to (1.2) was defined by Evans and Spruck in [17] and by Chen, Giga, and Goto in [11]. In our presentation we will use a somewhat more general definition of viscosity solutions introduced by Ishii and Souganidis in [22] to allow for a wider class of functions G in (1.2). For the general degenerate elliptic equation

$$u_t + \mathcal{G}\left(Du, D^2u\right) = 0, \tag{2.1}$$

they introduce a special class of test functions and adapt the definition of viscosity solution for possible singularities of the right-hand side. Representation of (1.3) in the form of (2.1) gives

$$\mathcal{G}(p,X) = -|p|G\left(\frac{1}{|p|}\mathrm{tr}\left(\left(I - \frac{p\otimes p}{|p|^2}\right)X\right)\right).$$

Let us begin by introducing an auxiliary subclass of $C^2\left([0,\infty)\right)$. We say that $f : [0,\infty) \mapsto \mathbb{R}$ lies in $\mathcal{F} \subset C^2$ if $f\,(0) = f'\,(0) = f''\,(0) = 0$, $f''\,(r) > 0$ for $r > 0$ and the following limits hold:

$$\lim \frac{f'\,(|p|)}{|p|}\mathcal{G}\,(p,I) = \lim \frac{f'\,(|p|)}{|p|}\mathcal{G}\,(p,-I) = 0.$$

As was shown in [22], this set of functions is a nonempty cone, provided that the right-hand side lies in $C\left(\left(\mathbb{R}^n \setminus \{0\}\right) \times \mathbb{S}(n)\right)$. The class of test functions $\mathcal{A}(\mathcal{G})$ depends on $\mathcal{G}$ and is defined as follows.

DEFINITION 2.1. *A function ϕ is admissible if it is in $C^2\left(\mathbb{R}^n \times (0,T)\right)$ and if, for each $\acute{z} = \left(\acute{x}, \acute{t}\right)$ where $D\phi\left(\acute{z}\right) = 0$, there is $\delta > 0$, $f \in \mathcal{F}$, and $\omega \in C\left([0,\infty)\right)$ such that $\omega = o(r)$ and for all $(x,t) \in B\left(\acute{z}, \delta\right)$*

$$\left|\phi(x,t) - \phi\left(\acute{z}\right) - \phi_t\left(\acute{z}\right)\left(t - \acute{t}\right)\right| \le f\left(\left|x - \acute{x}\right|\right) + \omega\left(\left|t - \acute{t}\right|\right).$$

Let us also denote by u^* and u_* the upper and lower semicontinuous envelopes of u:

$$u^*(x,t) = \limsup_{(y,s)\to(x,t)} u(y,s), \quad u_*(x,t) = \liminf_{(y,s)\to(x,t)} u(y,s).$$

The definition of viscosity solution follows.

DEFINITION 2.2. *Take an open set $\tilde{\mathcal{O}} \subset \mathbb{R}^n$ and $\mathcal{O} = \tilde{\mathcal{O}} \times (0,T)$. $u : \mathcal{O} \subset \mathbb{R}^n \times (0,T) \mapsto \mathbb{R} \cup \{-\infty\}$ is a viscosity subsolution (supersolution) of* (1.2) *in an open $\mathcal{O}$ if $u^* < \infty$ ($u_* > -\infty$) and for all $\phi \in \mathcal{A}(G)$ and all local maximum (minimum) points (z_0, t_0) of $u^* - \phi$ ($u_* - \phi$),*

$$\begin{cases} \phi_t(z_0,t_0) \le (\ge) \left|D\phi(z_0,t_0)\right| G\left(\operatorname{div}\dfrac{\mathrm{D}\phi(\mathrm{z}_0,\mathrm{t}_0)}{\left|\mathrm{D}\phi(\mathrm{z}_0,\mathrm{t}_0)\right|}\right) & \text{if } D\phi(z) \ne 0, \\ \phi_t(z_0,t_0) \le (\ge) 0 & \text{otherwise.} \end{cases}$$

Consequently, a viscosity solution is a function that is sub- and supersolution simultaneously.

The result by Ishii and Souganidis presented in [22] can be restated in terms of the level-set equation (see [28]) as follows.

THEOREM 2.3. *Assume that G is continuous and nondecreasing. Then the initial value problem* (1.3) *has a unique viscosity solution $u \in BUC\left(\mathbb{R}^n \times (0,T)\right)$.*

In what follows, we also use another result by Ishii and Souganidis [22] concerning locally uniform perturbations of the right-hand side of the equation. One can restate this result in the case of (1.2) as follows (see [28]).

THEOREM 2.4. *Assume that G is continuous and nondecreasing. Suppose also that $\{G_m\}_1^\infty$ is a sequence of continuous, nondecreasing functions on $\mathbb{R}$ and $G_m \to G$ locally uniformly. For any m, let $\mathcal{F}(G) \subset \mathcal{F}(G_m)$ and for any $f \in \mathcal{F}(G)$,*

$$\liminf_{p\to,\ m\to\infty} f'(|p|)\, G_m(1/p) \ge 0,$$

$$\left(\textit{resp.}, \limsup_{p\mapsto 0,\ m\to\infty} f'(|p|)\, G_m(-1/p) \le 0\right).$$

Let u_m be a subsolution (resp., supersolution) of

$$\frac{\partial u_m}{\partial t} = \left|Du_m\right| G_m\left(\operatorname{div}\frac{Du_m}{\left|Du_m\right|}\right) \text{ in } \mathcal{O}.$$

Then

$$u^+(z) = \limsup_{r\mapsto 0}\left\{u_m(y), |y-z| \le r,\ m > 1/m\right\}, \tag{2.2}$$

$$\left(\textit{resp.}, u_+(z) = \liminf_{r\to 0}\left\{u_m(y), |y-z| \le r,\ m > 1/m\right\}\right) \tag{2.3}$$

is a subsolution (resp., supersolution) of (1.2) *in $\mathcal{O}$ provided that $u^+ < \infty$ (resp., $u_+ > -\infty$).*

3. A convolution-thresholding method for a generalized curvature flow.

3.1. Convergence of approximation schemes. Here we make use of a theorem by Barles and Souganidis proved in [5]. In order to base the proof of our main result on this theorem, we follow Pasquignon [28] and restate it in terms of (1.2).

Let $H(h)$ be the approximation operator, i.e.,

$$u_h(x,(n+1)h) = H(h)\,u_h(x,nh) = H(h)^{n+1}u_0(x),$$
$$u_h(x,0) = u_0(x).$$

Definition 3.1.

1. *Consistency.*
   An approximation operator $H(h)$, $h>0$, *is consistent with* (1.2) *if for any* $\phi \in C^\infty(\bar{\Omega})$ *and for any* $x \in \bar{\Omega}$, *the following holds:*

$$\frac{(H(h)\,\phi)(x) - \phi(x)}{h} = |D\phi|\, G\left(\operatorname{div}\frac{D\phi}{|D\phi|}\right) + o_x(1) \text{ for } D\phi \neq 0. \tag{3.1}$$

   If the convergence of $o_x(1)$ *is locally uniform on sets, where* $D\phi \neq 0$, *then* $H(h)$ *is said to be uniformly consistent with the PDE.*
2. *Monotonicity.*
   An operator $H(h)$, $h>0$, *is locally monotone if there exists* $r>0$ *such that for any functions* $u(y), v(y) \in \mathbb{B}(\bar{\Omega})$ *with* $u \geq v$ *on* $B(x,r)\setminus\{x\}$, *the following holds:*

$$H(h)\,u(y) \geq H(h)\,v(y) + o(h),$$

   where the convergence of $o(h)$ *is uniform on* $B(x,r)\setminus\{x\}$.
3. *Stability.*
   An approximation scheme $H(h)$ *is stable if* $H(h)^n u \in \mathbb{B}(\bar{\Omega})$ *for every* $u \in B(\bar{\Omega})$, $n \in \mathbb{N}, h>0$, *with a bound independent of* h *and* n.

In this setting the result of Barles and Souganidis reads as follows.

Theorem 3.2. *Consider a monotone, stable approximation operator* $H(h)$ *that commutes with additions of constants* (*i.e.*, $H(h)(u+C) = H(h)\,u + C$ *for all* $C \in \mathbb{R}$) *and is uniformly consistent with* (1.2). *Suppose also that*

$$\lim_{h\to 0}\frac{H(h)\,(f(|x-x_0|))\,(x_0)}{h} = 0 \tag{3.2}$$

for any $f \in \mathcal{F}(G)$. *Then* $u_h(x,nh)$ *converges locally uniformly to the unique viscosity solution* $u(x,t)$ *of* (1.2) *as* $nh \mapsto t$.

3.2. Properties of $\mathcal{H}$. We consider a convolution generated motion of a hypersurface in $\mathbb{R}^n$ defined by (1.4) and the corresponding evolution of an initially bounded function $g: \mathbb{R}^n \mapsto \mathbb{R}$ defined by (1.5). Consider also the initial value problem (1.3) with given G and g. We are looking for such a thresholding function F in (1.4) so that $H^m_{t/m}g(x)$ would converge (in some sense) to the unique viscosity solution of (1.3).

For example, set $F(M_1,M_2) = M_1 - \frac{1}{2}$ and $\tilde{\rho}_1(x) = \frac{1}{(4\pi)^{n/2}}e^{-x^2/4}$ to get corresponding operators $\mathcal{H}_h$ and $H(h)$ by (1.4) and (1.5). Then we get the Bence–Merriman–Osher procedure to which the main result of [15] applies, and $H(h)^n u_0$ converges locally uniformly to the unique viscosity solution of (1.3) with $G(k)=k$.

We will see that it is necessary to compute two convolutions M_1 and M_2 and use the thresholding function depending on both these values to resolve the problem when G is not linear.

Let us now consider an operator $H(h)$ defined by (1.5) with the help of an operator $\mathcal{H}_h$ with an arbitrary thresholding function (1.4). We look for requirements on F sufficient to fulfill the conditions of Theorem 3.2.

Stability. Suppose $u(x) \in \mathbb{B}(\mathbb{R}^n)$. We show that $H(h)u \in \mathbb{B}(\mathbb{R}^n)$. Intuitively, we require

$$\mathcal{H}_h\mathbb{R}^n = \mathbb{R}^n, \tag{3.3}$$

$$\mathcal{H}_h\emptyset = \emptyset, \tag{3.4}$$

and denote $A = \max|u|$. With these settings, we have $[u \le A] = \mathbb{R}^n$ and

$$-A \le H(h)u(x) = \inf\{\lambda \in \mathbb{R} : x \in \mathcal{H}_h[u \le \lambda]\} \le A.$$

It remains to find out for which F the conditions (3.3) and (3.4) are satisfied. To do this, we substitute the corresponding sets into the definition of $\mathcal{H}$:

$$\begin{aligned}
\mathcal{H}_h\mathbb{R}^n &= \{x \in \mathbb{R}^n : F(M_1\mathbb{R}^n(x,h), M_2\mathbb{R}^n(x,h)) \ge 0\}\\
&= \left\{x \in \mathbb{R}^n : F\left(\int_{\mathbb{R}^n}\rho_1 dx, \int_{\mathbb{R}^n}\rho_2 dx\right) \ge 0\right\} = \mathbb{R}^n\\
\mathcal{H}_h\emptyset &= \{x \in \mathbb{R}^n : F(M_1\emptyset(x,h), M_2\emptyset(x,h)) \ge 0\}\\
&= \{x \in \mathbb{R}^n : F(0,0) \ge 0\} = \emptyset.
\end{aligned}$$

Thus, the requirements on F become

$$\begin{aligned}
F\left(\int_{\mathbb{R}^n}\rho_1 dx, \int_{\mathbb{R}^n}\rho_2 dx\right) &\ge 0,\\
F(0,0) &< 0.
\end{aligned}$$

Monotonicity. Let us now show that if $\mathcal{H}_h$ satisfies the so-called inclusion principle, then H_h is monotonous.

LEMMA 3.3. *Assume, that $\mathcal{H}_h$ satisfies the inclusion principle, i.e.,*

$$\forall C_1, C_2 \subseteq \mathbb{R}^n : C_1 \subseteq C_2 \text{ we have } \mathcal{H}_hC_1 \subseteq \mathcal{H}_hC_2; \tag{3.5}$$

then H_h is monotone, that is,

$$\forall u, v \in \mathbb{C}(\mathbb{R}^n) : v \le u \text{ we have } H_h(v) \le H_h(u).$$

Proof. Suppose, there exists x_0 s.t. $H(h)u(x_0) < H(h)v(x_0)$. We denote $\lambda_1 = H(h)u(x_0)$, $\lambda_2 = H(h)v(x_0)$, and $\epsilon = \frac{\lambda_2-\lambda_1}{2} > 0$. Since

$$\lambda_1 + \epsilon < \inf\{\lambda \in \mathbb{R} : x_0 \in \mathcal{H}_h[v \le \lambda]\},$$

we have $x_0 \notin \mathcal{H}_h[v \le \lambda_1 + \epsilon]$, but

$$\mathcal{H}_h[v \le \lambda_1 + \epsilon] \supseteq \mathcal{H}_h[u \le \lambda_1 + \epsilon].$$

Therefore $x_0 \notin \mathcal{H}_h[u \le \lambda_1 + \epsilon]$, which contradicts the definition of λ_1. ∎

Consistency. We sum up some calculations in the following lemma.

LEMMA 3.4. *Let $\phi \in C^\infty(\mathbb{R}^n)$ $\phi(0) = 0$ and $D\phi(0) = (0, 0, \ldots, \beta)$. Then the consistency of an operator $H(h)$ with (1.3) is equivalent to*

$$\gamma(h, 0) = hG(-\Delta\gamma(h, 0)) + o(h), \tag{3.6}$$

where $\Delta\gamma(h, 0) = \sum_{i=1}^{n-1} \partial^2\gamma/\partial x_i^2(0)$ and $x_n = \gamma(h, \acute{x})$ is a parameterization of the surface

$$\{x \in \mathbb{R}^n : \phi(x) = H(h)\phi(0)\}$$

near $\acute{x} = 0$.

We observe that in these settings $-\Delta\gamma(h, 0) \equiv k$ is the mean curvature of the graph of γ at the point $(0, \gamma(h, 0))$.

Proof. Without loss of generality, one can consider the consistency condition (3.1) only for ϕ as in the statement. We rewrite (3.1) in a more convenient form:

$$(H(h)\phi)(0) = h|D\phi(0)|G\left(\operatorname{div}\frac{D\phi}{|D\phi|}(0)\right) + o(h). \tag{3.7}$$

We use the equality

$$\operatorname{div}\left(\frac{D\phi}{|D\phi|}\right) = \frac{1}{|D\phi|}\sum_{i,j=1}^{n}\left(\delta_{i,j} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right)\phi_{x_ix_j}.$$

Since $\phi(0) = 0$ and $\phi_{x_i}(0) = \delta_{ni}\beta$,

$$\begin{aligned}\operatorname{div}\frac{D\phi}{|D\phi|}\bigg|_{x=0} &= \frac{1}{\beta}\left[\sum_{i=1}^{n}\phi_{x_ix_i}(0) - \frac{\phi_{x_n}(0)\,\phi_{x_n}(0)}{\beta^2}\phi_{x_nx_n}(0)\right]\\ &= \frac{1}{\beta}\Delta'\phi(0).\end{aligned} \tag{3.8}$$

Here $\Delta'\phi = \sum_{i=1}^{n-1}\phi_{x_ix_i}$. Our next step is to take small $\acute{x}$, namely $|\acute{x}| < Rh$. For such $\acute{x}$ we apply the inverse function theorem to ϕ,

$$H(h)\phi(0) = \phi(\acute{x}, \gamma(h, \acute{x})) = \phi(0) + \beta\gamma(h, 0) + O\left(h^2\right). \tag{3.9}$$

Putting (3.9) and (3.8) into (3.7) we get

$$\gamma(h, 0) = hG\left(\frac{1}{\beta}\Delta'\phi(0)\right) + o(h). \tag{3.10}$$

Furthermore, differentiating both sides of $H(h)\phi(0) = \phi(\acute{x}, \gamma(h, \acute{x}))$ gives

$$\phi_{x_i} + \phi_{x_n}\gamma_{x_i} = 0,$$
$$\phi_{x_ix_j} + \phi_{x_ix_n}\gamma_{x_j} + \phi_{x_nx_j}\gamma_{x_i} + \phi_{x_nx_n}\gamma_{x_j}\gamma_{x_i} + \phi_{x_n}\gamma_{x_ix_j} = 0$$

for $j, i = 1, \ldots, n-1$. We deduce $\gamma_{x_i}(h, 0) = 0$ from the first equality and rewrite the second one for $i = j$,

$$\phi_{x_jx_j}(0) + \phi_{x_n}(0)\gamma_{x_jx_j}(h, 0) = 0.$$

After a summation over j this becomes

$$\frac{1}{\beta}\Delta'\phi(0) = -\Delta\gamma(h, 0).$$

It remains to put this relation into (3.10) to get the desired equality (3.6). ∎

3.3. The convergence result for general G. In this subsection we construct the thresholding function $F(M_1, M_2)$ and show that the corresponding convolution thresholding scheme (1.4), (1.5) converges to the viscosity solution $u(x,t)$ of (1.3),

$$H^m_{\frac{t}{m}} g(x) \to u(x,t) \text{ as } m \to \infty.$$

We start with $F(M_1 C(x,h), M_2 C(x,h))$, where

$$M_i C(x,h) = \int_C \rho_i(x-y)\, dy.$$

For each ρ_i we expand this integral into the power series in h (see (3.19)), i.e.,

$$M_i[\phi \le H(h)\phi(0)](0,h) = A_i + \sqrt{h} v C_i + \sqrt{h} \Delta\gamma(h,0) B_i + O\left(h^{3/2}\right), \tag{3.11}$$

where

$$A_i = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{0} \rho_i(|y|)\, dy_n d\acute{y}, \tag{3.12}$$

$$B_i = \frac{1}{2} \int_{\mathbb{R}^{n-1}} y_k^2 \rho_i(\acute{y}, 0)\, d\acute{y}, \tag{3.13}$$

$$C_i = \int_{\mathbb{R}^{n-1}} \rho_i(\acute{y}, 0)\, d\acute{y}, \tag{3.14}$$

and $i = 1, 2$. This is a system of linear algebraic equations for $\Delta\gamma(h,0)$ and v. We choose the kernels so that the determinant of this system is positive,

$$D = C_1 B_2 - C_2 B_1 > 0,$$

denote $N_i = M_i[\phi \le H(h)\phi(0)](0,h) - A_i$, and write the solution

$$v = \frac{\gamma(h,0)}{h} = \frac{1}{\sqrt{h}} \frac{N_1 B_2 - N_2 B_1}{C_1 B_2 - C_2 B_1} + O(h),$$

$$\Delta\gamma(h,0) = \frac{1}{\sqrt{h}} \frac{N_2 C_1 - N_1 C_2}{C_1 B_2 - C_2 B_1} + O(h).$$

Lemma 3.4 implies that the operator H is consistent with the PDE in (1.3) if we take

$$\begin{aligned} F(N_1, N_2) &= v - G(-\Delta\gamma(h,0)) \\ &= \frac{1}{\sqrt{h}} \frac{N_1 B_2 - N_2 B_1}{D} - G\left(\frac{1}{\sqrt{h}} \frac{N_1 C_2 - N_2 C_1}{D}\right). \end{aligned} \tag{3.15}$$

In the case of the thresholding function of one variable, the inclusion principle (3.5) holds for $\mathcal{H}$ when F is nondecreasing. In the case of two variables we require

$$\frac{\partial F}{\partial N_1} = \frac{B_2}{D} - \frac{C_2}{D} G' \ge 0, \tag{3.16}$$

$$\frac{\partial F}{\partial N_2} = -\frac{B_1}{D} + \frac{C_1}{D} G' \ge 0. \tag{3.17}$$

This implies

$$\frac{B_1}{C_1} \le G^{'} \le \frac{B_2}{C_2}. \tag{3.18}$$

Therefore, for awhile we restrict ourselves with G having a bounded and positive derivative. Comparing (3.14) with (3.13) one sees that it is possible to make the lower bound in (3.18) small by choosing ρ_1 with mass concentration close to the origin. The upper bound will be large if the mass of ρ_2 is concentrated relatively far from the origin.

Next, we state some auxiliary results.

LEMMA 3.5. *Suppose* (3.16) *and* (3.17) *hold and* $\mathcal{H}$ *is defined by* (1.4)*; then for all* $h \in \mathbb{R}_+$,

1. $\mathcal{H}(h)(\mathbb{R}^n) = \mathbb{R}^n$, $\mathcal{H}(h)(\emptyset) = \emptyset$,
2. *for all* $a, b \in \mathbb{X} : a \subseteq b \Rightarrow \mathcal{H}(h)\, a \subseteq \mathcal{H}(h)\, b$.

Proof.

1. It is enough to show that $F\big(M_1(\mathbb{R}^n)(x,h), M_2(\mathbb{R}^n)\big(x,h\big)\big) \ge 0$, and $F\big(M_1(\emptyset)\big(x,h\big), M_2(\emptyset)(x,h)\big) < 0$. First we observe that $F(A_1, A_2) = 0$, $M_i(\mathbb{R}^n)(x,h) \ge A_i$, and $M_i(\emptyset)(x,h) = 0 < A_i$. This, together with $\frac{\partial F}{\partial N_i} > 0$, gives the desired inequalities.
2. Since $M_i(b) \ge M_i(a)$, $F(M_1(b), M_2(b)) \ge F(M_1(a), M_2(a))$, therefore $[F(M_1(a), M_2(a)) \ge 0] \subseteq [F(M_1(b), M_2(b)) \ge 0]$, which is equivalent to $\mathcal{H}(h)\, a \subseteq \mathcal{H}(h)\, b$. □

PROPOSITION 3.6. *Define* H *by* (1.5) *and* $\mathcal{H}$ *by* (1.4)*; then for each* $h > 0$ *and* $u \in \mathbb{B}(\mathbb{R}^n)$ *one has* $H(h)\, u \in \mathbb{B}(\mathbb{R}^n)$.

Proof. Without loss of generality we assume that $S_1 \le u(x) \le S_2$ for some $S_1, S_2 \in \mathbb{R}$. From

$$\forall h \in \mathbb{R}_+ \quad \mathcal{H}(h)(\mathbb{R}^n) = \mathbb{R}^n \text{and } \mathcal{H}(h)(\emptyset) = \emptyset$$

it follows that $x \in \mathcal{H}(h)[u \le S_2]$ and $x \notin \mathcal{H}(h)[u \le S_1]$. Therefore, we see that

$$S_1 \le H(h)\, u(x) = \inf\{\lambda \in \mathbb{R} : x \in \mathcal{H}(h)[u \le \lambda]\} \le S_2. \qquad \square$$

With the results above, we are ready to state the convergence of the approximations $H(t/m)^m g$ to the unique viscosity solution of (1.3).

THEOREM 3.7. *Let* $H(h)$ *be defined by*

$$[H(h)\, u](x) = \sup\{\lambda \in \mathbb{R} : x \in \mathcal{H}_h[u \ge \lambda]\}$$

with

$$\mathcal{H}_h C = \{x \in \mathbb{R}^n : F(M_1(C)(x,h), M_2(C)(x,h)) \ge 0\},$$

where

$$F(N_1, N_2) = \frac{1}{\sqrt{h}} \frac{N_1 B_2 - N_2 B_1}{D} - G\left(\frac{1}{\sqrt{h}} \frac{N_2 C_1 - N_1 C_2}{D}\right),$$

and where $\tilde{\rho}_1, \tilde{\rho}_2$ *have compact support and* G *is continuous nondecreasing satisfying* (3.18). *Then*

$$H^m_{t/m} g(x) \to u(x,t)$$

locally uniformly when $m \to \infty$. *Here* $u(x,t)$ *is the unique viscosity solution of* (1.3) *with* G *satisfying* (3.18).

Proof. Our aim is to show here that the operator $H(h)$ satisfies the conditions of Theorem 3.2. The *monotonicity* of H_h is ensured by Lemmas 3.3 and 3.5.

The *stability* of H is exactly the result of Proposition 3.6: $H(h)\,u \in \mathbb{B}\left(\bar{\Omega}\right)$.

Another requirement in Theorem 3.2 is that $H(h)$ must commute with the addition of constants, i.e.,

$$\forall a \in \mathbb{R} \quad H(h)(u(x) + a) = H(h)\,u(x) + a.$$

This follows from the very definition of $H(h)$:

$$\begin{aligned} H(h)(u(x) + a) &= \inf\{\lambda \in \mathbb{R} : x \in \mathcal{H}(h)[u(x) + a \le \lambda]\} \\ &= \inf\{\beta + a \in \mathbb{R} : x \in \mathcal{H}(h)[u(x) \le \beta]\} = H(h)\,u(x) + a. \end{aligned}$$

The operator $H(h)$ has to fulfill (3.2) as well. The limit we are interested in is

$$\lim_{h \to 0} \frac{H(h)\,u(x_0)}{h} = 0$$

for u of the form $u(x) = f(|x - x_0|)$, where $f \in C^2([0,\infty))$ with $f(0) = f'(0) = f''(0) = 0$ and $f''(r) > 0$ for $r > 0$.

It is enough to show that this is true for $x_0 = 0$. First, we observe that $\mathcal{H}_h^{-1}[\{0\}] = \{u \le \lambda_1\}$, where $\lambda_1 = H(h)\,u(0)$. Since both ρ_1 and ρ_2 have compact support, we can be sure that there exists R s.t. $\left\{ |x| \le R\sqrt{h} \right\} \supseteq \mathcal{H}_h^{-1}[\{0\}]$. Now we observe, that $\left\{ |x| \le R\sqrt{h} \right\} = \{u \le \lambda_2\}$ for some $\lambda_2 > \lambda_1$. From the latter equality we deduce $\lambda_2 = O\left(h^{3/2}\right)$ and conclude with

$$\lim_{h \to 0} \frac{H(h)\,u(x_0)}{h} \le \lim_{h \to 0} \frac{O\left(h^{3/2}\right)}{h} = 0.$$

To show that our approximation operator is *consistent* with the PDE, we use Lemma 3.4. It is enough to prove the following:

$$\gamma(h, 0) = hG(-\Delta\gamma(h,0)) + o(h),$$

where $x_n = \gamma(h, \acute{x})$ is a parameterization of the surface

$$\{x \in \mathbb{R}^n : u(x) = H(h)\,u(0)\}$$

near $\acute{x} = 0$. To show this, we use the fact that

$$F(M_1[u \le \mu], M_2[u \le \mu])|_{x=0} = 0.$$

We begin by writing the expressions for M_i in detail:

$$M_i = \left(\chi_{[u\le\mu]} \star \frac{1}{h^{n/2}}\rho_i\left(\frac{|\cdot|}{\sqrt{h}}\right)\right)(0) = \int_{\mathbb{R}^n} \chi_{[u\le\mu]}(y)\,\frac{1}{h^{n/2}}\rho_i\left(\frac{|y|}{\sqrt{h}}\right)dy$$

$$= \int_{\mathbb{R}^{n-1}}\int_{-\infty}^{\gamma(h,\acute{y})} \frac{1}{h^{n/2}}\rho_i\left(\frac{|y|}{\sqrt{h}}\right)dy_n d\acute{y} = A_i + \int_{\mathbb{R}^{n-1}}\int_0^{(1/\sqrt{h})\gamma\left(h,\sqrt{h}\acute{y}\right)} \rho_i(|y|)\,dy_n d\acute{y}.$$

Here A_i is given by (3.12). Expanding $\gamma\big(h,\sqrt{h}\acute{y}\big)$ in the Taylor series with respect to the spatial variables (keeping h as a parameter) we get

$$\frac{1}{\sqrt{h}}\gamma\left(h,\sqrt{h}\acute{y}\right) = \sqrt{h}\frac{\gamma(h,0)}{h} + \frac{\sqrt{h}}{2}\sum_{i,j=1}^{n-1}\gamma_{y_iy_j}(h,0)\,y_iy_j$$
$$+\frac{h}{6}\sum_{i,j,l=1}^{n-1}\gamma_{y_iy_jy_l}(h,0)\,y_iy_jy_l + O\left(h^{3/2}\acute{y}^4\right).$$

Observing that $\gamma(h,0) = O\big(\sqrt{h}\big)$, we denote $\frac{\gamma(h,0)}{h} = v$. The expression for M_i becomes

$$M_i = A_i + \int_{\mathbb{R}^{n-1}}\rho_i(\acute{y},0)\left[\sqrt{h}v + \frac{\sqrt{h}}{2}\sum_{i,j=1}^{n-1}\gamma_{y_iy_j}(h,0)\,y_iy_j + O\left(h^{3/2}\acute{y}^4\right)\right]dy_n d\acute{y}$$
$$= A_i + \sqrt{h}vC_i + \sqrt{h}\Delta\gamma(h,0)\,B_i + O\left(h^{3/2}\right), \tag{3.19}$$

where we have used the fact that $\rho_i(\acute{x},x_n)$ is smooth and radially symmetric, in particular,

$$\frac{\partial\rho_i}{\partial x_n}(\acute{x},0) = 0.$$

The constants B_i, C_i depend only on ρ_i and are given by (3.13) and (3.14).

Remark 1. At this point it is easy to see that a scheme with a thresholding depending only on one variable can be consistent with the PDE (1.2) only in the case of linear G. The thresholding condition becomes

$$F\left(A + \sqrt{h}vC + \sqrt{h}\Delta\gamma(h,0)\,B + O\left(h^{3/2}\right)\right) = 0.$$

As was required by the inclusion principle, the function F is nondecreasing. This implies

$$A + \sqrt{h}vC + \sqrt{h}\Delta\gamma(h,0)\,B + O\left(h^{3/2}\right) = a,$$

where a is the unique solution of $F(a) = 0$. Thus

$$v = \frac{\gamma(h,0)}{h} = -\frac{B}{C}\Delta\gamma(h,0) - \frac{a-A}{\sqrt{h}C} + o\left(\sqrt{h}\right).$$

Comparing this relationship with the one in Lemma 3.4, we see that the only G's we can resolve by thresholding depending on one variable are the linear ones: $G(k) = \mathrm{const}\cdot k + \mathrm{const}$.

Let us denote here $k = \Delta\gamma(h, 0)$.

Now we can express v and k in terms of M_i and constants A_i, B_i, and C_i :

$$v = \frac{1}{\sqrt{h}} \frac{N_1 B_2 - N_2 B_1}{C_1 B_2 - C_2 B_1} + O(h),$$

$$k = \frac{1}{\sqrt{h}} \frac{N_2 C_1 - N_1 C_2}{C_1 B_2 - C_2 B_1} + O(h).$$

Since $F(M_1, M_2) = v - G(-k) = 0$, we have

$$\gamma(h, 0) = hG(-\Delta\gamma(h, 0)) + o(h).$$

Remark 2. As was already mentioned above, convolution kernels $\tilde{\rho}_i$ can also be taken with unbounded support. For example, the exponential decay for large arguments is sufficient in order for Theorem 3.7 to hold.

The requirement (3.18) is quite restrictive. Our next result shows that it is enough to take G_ϵ satisfying (3.18) and uniformly close to G in order to approximate the solutions of (1.3).

PROPOSITION 3.8. *Suppose G_ϵ, G are continuous and $G_\epsilon \to G$ uniformly on $\mathbb{R}$ as $\epsilon \to 0$. Then $\mathcal{F}(G) = \mathcal{F}(G_\epsilon)$.*

Proof. Suppose $f \in \mathcal{F}(G)$. It means that $f(0) = f'(0) = f''(0)$, $f(r) > 0$ for $r > 0$, and

$$\lim_{p \to 0} f'(p)\, G\left(\frac{1}{p}\right) = \lim_{p \to 0} f'(p)\, G\left(\frac{-1}{p}\right) = 0.$$

Since $G_\epsilon \to G$ uniformly, $G(k) = G_\epsilon(k) + o_\epsilon(1)\,\alpha(k)$, where $\alpha \in \mathbb{B}(\mathbb{R})$. We write

$$\begin{aligned} 0 &= \lim_{p \mapsto 0} f'(p)\, G\left(\frac{1}{p}\right) \\ &= \lim_{p \to 0} f'(p) \left(G_\epsilon\left(\frac{1}{p}\right) + o_\epsilon(1)\,\alpha\left(\frac{1}{p}\right)\right) = \lim_{p \to 0} f'(p)\, G_\epsilon\left(\frac{1}{p}\right) \end{aligned}$$

to see that $f \in \mathcal{F}(G_\epsilon)$.

The proof of the reverse inclusion is analogous. ∎

LEMMA 3.9. *Suppose G_ϵ, G are nondecreasing continuous and $G_\epsilon \to G$ uniformly on $\mathbb{R}$ as $\epsilon \to 0$. Suppose also that for each $\epsilon > 0$ the operator H_ϵ is monotone, stable, commuting with additions of constants, and consistent with*

$$\frac{\partial u_\epsilon}{\partial t} = |Du_\epsilon|\, G_\epsilon\left(\operatorname{div} \frac{\mathrm{D}u_\epsilon}{|\mathrm{D}u_\epsilon|}\right). \tag{3.20}$$

Additionally, let the following limit hold:

$$\lim_{h \to} \frac{H_h(h)\left(f(|x - x_0|)\right)(x_0)}{h} = 0 \tag{3.21}$$

for each $f \in \mathcal{F}(G)$. Then

$$H^m_{t/m}(t/m)\, u_0(x) \to u(x, t)$$

locally uniformly as $m \to \infty$, where $u(x, t)$ is the unique viscosity solution of (1.3).

Proof. We show here that the operator $H_h(h)$ satisfies the conditions of Theorem 3.2. This operator commutes with additions of constants and satisfies limit (3.21) by the assumption. Since the operator H_ϵ is stable for all $\epsilon > 0$, it is particularly stable for $\epsilon = h$ for each $h > 0$.

Since the operator H_ϵ is monotonic for all $\epsilon > 0$, it is particularly monotonic for $\epsilon = h$ for each $h > 0$.

We have to show consistency; i.e., for each $\phi \in C^\infty(\mathbb{R}^n)$ at each point where $|D\phi| \neq 0$,

$$H_h(h)\phi(x) - \phi(x) = h|D\phi(x)|\,G\left(\operatorname{div}\frac{D\phi(x)}{|D\phi(x)|}\right) + o(h) \tag{3.22}$$

has to hold. Since the operator H_ϵ is consistent with (3.20) and $G_h(k) = G(k) + o_h(1)\alpha(k)$ for some $\alpha \in \mathbb{B}(\mathbb{R})$, we write

$$\begin{aligned} H_h(h)\phi(x) - \phi(x) &= h|D\phi(x)|\,G_h\left(\operatorname{div}\frac{D\phi(x)}{|D\phi(x)|}\right) + o(h) \\ = h|D\phi(x)|\left(G\left(\operatorname{div}\frac{D\phi(x)}{|D\phi(x)|}\right) + o_h(1)\,\alpha\left(\operatorname{div}\frac{D\phi(x)}{|D\phi(x)|}\right)\right) &+ o(h) \\ &= h|D\phi(x)|\,G\left(\operatorname{div}\frac{D\phi(x)}{|D\phi(x)|}\right) + o(h); \end{aligned}$$

here $o_h(1) \to 0$ as $h \to 0$. ∎

THEOREM 3.10. *Consider a convolution-thresholding scheme*

$$\begin{aligned} H_\epsilon(h)u(x) &= \inf\{\lambda \in \mathbb{R} : x \in \mathcal{H}_\epsilon(h)[u \le \lambda]\}, \\ \mathcal{H}_\epsilon(h)C &= \{x \in \mathbb{R}^n : F_\epsilon(M_1(C)(x,h), M_2(C)(x,h)) \ge 0\}, \end{aligned}$$

where the thresholding function $F_\epsilon(M_1, M_2)$ is chosen so that the scheme is monotone and consistent with (3.20) *and the convolution kernels have compact support. If $G_\epsilon \to G$ uniformly, then*

$$H^m_{t/m}(t/m)\,u_0(x) \to u(x,t)$$

locally uniformly as $m \to \infty$, where $u(x,t)$ is the unique viscosity solution of (1.3).

Proof. The convergence follows form Lemma 3.9 if we show that the limit (3.21) holds. Let us set $x_0 = 0$; then the set $[f(|x|) \le \lambda]$ is a ball centered at the origin with radius $O(\lambda^{1/3})$. We denote $H_h(h)f(0) = \lambda_1$. Observe that λ_1 can be characterized as a number for which $\mathcal{H}_h(h)[f \le \lambda_1] = \{0\}$. Since we know that $F_h(A_1, A_2) > 0$, the radius of $[f \le \lambda_1]$ must be less than or equal to the radius of the greatest support of the kernel: $O(\lambda_1^{1/3}) \le R\sqrt{h}$. From this inequality we deduce $H_h(h)f(0) = \lambda_1 \le O(h^{3/2})$. This establishes the desired limit (3.21). ∎

Let us now consider the particular interesting case with $G(k) = k|k|^{\alpha-1}$ with $\alpha > 1$. We set

$$G_m(k) = \begin{cases} (1-\alpha)m^\alpha + \alpha m^{\alpha-1}k & \text{for } k < -n, \\ m^{1-\alpha}k & \text{for } |k| < 1/n, \\ -(1-\alpha)m^\alpha + \alpha m^{\alpha-1}k & \text{for } k > n, \\ k|k|^{\alpha-1} & \text{elsewhere.} \end{cases}$$

G_m is continuous, increasing, and its derivative is bounded from below and above: $m^{1-\alpha} \leq G_m' \leq \alpha m^{\alpha-1}$. Moreover, $G_m \to k\,|k|^{\alpha-1}$ locally uniformly as $m \to \infty$. Using Theorem 2.4 it is easy to show the following.

THEOREM 3.11. *Let u_m be the viscosity solution of*

$$\frac{\partial u_m}{\partial t} = |Du_m|\, G_m \left(\operatorname{div} \frac{Du_m}{|Du_m|} \right) \text{ in } \mathcal{O},$$

where G_m is defined above. Then $u_m \to u$ locally uniformly as $m \to \infty$, where u is the viscosity solution of (1.2) *in $\mathcal{O}$, with $G(k) = k\,|k|^{\alpha-1}$, $\alpha > 1$.*

Proof. First we establish the inclusion $\mathcal{F}(G) \subset \mathcal{F}(G_m)$. Take $f \in \mathcal{F}(G)$. By the definition of $\mathcal{F}(G)$, $f'(x) = o(x^\alpha)$. This immediately gives

$$\lim_{p \to 0} f'(p)\, G_m(1/p) = \lim_{p \to 0} f'(p)/p = 0,$$

since $\alpha > 1$. We observe also that the remaining conditions of Theorem 2.4 are satisfied. Hence a subsolution and a supersolution u^+ and u_+ can be constructed by means of (2.2) and (2.3). Since the equation has the strong comparison property (see [12]), $u^+ = u_+$ and the result follows. □

Remark 3. In a more general case when $G(k) = O(k^\alpha)$, $\alpha > 1$, one can pick a sequence of increasing functions with derivative bounded below and above and apply Theorem 2.4 to get a result similar to Theorem 3.11.

4. Numerical implementation. This section is devoted to a description of our numerical implementations of the convolution-thresholding scheme developed in section 3.

Given a compact set $C \subset \mathbb{R}^n$, we fix convolution kernels ρ_1, ρ_2 and the time step h and approximate C_t at a time moment $t = mh$ by $(\mathcal{H}(h))^m C$. The algorithm of computations consists of the following steps:

1. Compute convolutions and the thresholding function

$$M_i C(x,h) = \int_{\mathbb{R}^n} \chi_C(y)\, \rho_i(x-y)\, dy, \qquad i = 1,2, \tag{4.1}$$

$$F(x,h) = F(M_1 C(x,h), M_2 C(x,h)). \tag{4.2}$$

2. Find the evolved set $\mathcal{H}(h)\, C = \{x \in \mathbb{R}^n : F(x,h) \geq 0\}$.
3. Repeat the procedure with the evolved set to get $\mathcal{H}^2(h)C$ and so on.

We used two different algorithms for the calculation of the convolution step, which constitutes the main computational part of the algorithm.

4.1. Spatial discretization. We assume that initially the surface is closed and contained in a unit cube. The surface under consideration is always an isosurface of some function. In our implementation we use a modification of the so-called marching cubes algorithm for extracting an isosurface. The algorithm was originally proposed in [24] and was first applied for the mean curvature flow calculations in [30]. The algorithm creates an adaptive spatial discretization of C (see Figure 4.1).

By our implementation, we significantly reduce the number of grid points. In addition, the accurate piecewise polynomial approximation of the ∂C can be arranged.

4.2. Spectral method. One can use a Fourier series to calculate the convolutions (4.1). Numerical aspects of this approach have been presented by Ruuth in [30].

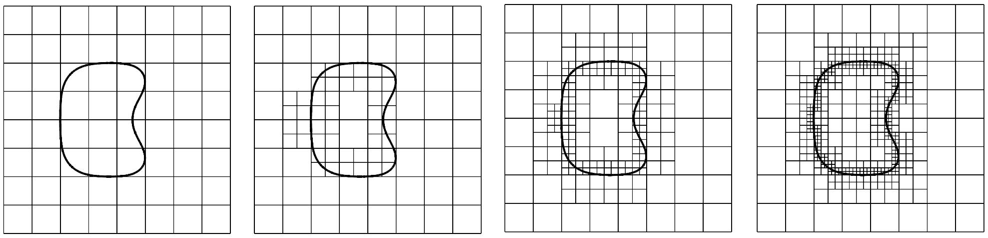

FIG. 4.1. *On the spatial discretization.*

In order to compute Fourier coefficients of χ_C given on a nonuniform grid, the unequally spaced approximate fast Fourier transform algorithm [6] is used. The numerical cost of this transform algorithm combined with the marching cubes procedure is (see [30]) $O\big(m^n N_p + N_f^n \log(N_f)\big)$, where m is a constant depending on a desired accuracy in the calculation of the Fourier coefficients (in case $m = 23$, the accuracy is comparable with the machine truncation error), N_f is a number of the Fourier modes along each axis, and N_p is the number of nodes in the grid.

4.3. Direct method. If ρ_1 and ρ_2 are simple enough and have compact support, their convolutions with χ_C can be calculated explicitly. Let us choose

$$\tilde{\rho}_1(x) = \begin{cases} \frac{1}{|\mathcal{B}_1|} & \text{if } x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{\rho}_2(x) = \frac{1}{\alpha^n} \tilde{\rho}_1\left(\frac{x}{\alpha}\right),$$

where $|\mathcal{B}_1|$ is the Lebesgue measure of a unit ball in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}_+,\ \alpha < 1$. In this case, convolution values (4.1) are proportional to the measure of the intersection of C with a ball of radius proportional to $\sqrt{h}$ centered at the point x.

We present expressions for the thresholding function $F(M_1, M_2)$ in the case $n = 2$:

$$F(M_1, M_2) = v - G(k), \text{ where}$$

$$v = \frac{\pi\alpha(2\alpha M_1 - 2M_2 - \alpha + 1)}{4\sqrt{h}(\alpha^2 - 1)},$$

$$k = \frac{-3\pi(2M_1 - 2\alpha M_2 + \alpha - 1)}{2\sqrt{h}(\alpha^2 - 1)}.$$

In this case convolutions M_1 and M_2 can be calculated as follows. We represent C as a disjoint union of squares and triangles (or cubes in tetrahedron in case $n = 3$) using the marching cubes method and calculate the area (volume) of intersection of the ball (supp ρ) with each square and triangle. The numerical cost of each step of the evolution can be estimated by $O(N_p * N_i + N_p)$, where N_i is the number of points inside the ball of radius $\sqrt{h}$ with the center at some grid point. When h is large, the accuracy of the method is low; therefore one can take less grid points. Thus, N_i is entirely determined by the desired accuracy.

4.4. Computed examples. In the case of the mean curvature curve evolution in $\mathbb{R}^2$, the accuracy of calculations can be monitored with the help of the Von Neumann–Mullins parabolic law. It asserts that $dS/dt = -2\pi$, where S is the area enclosed by the curve.

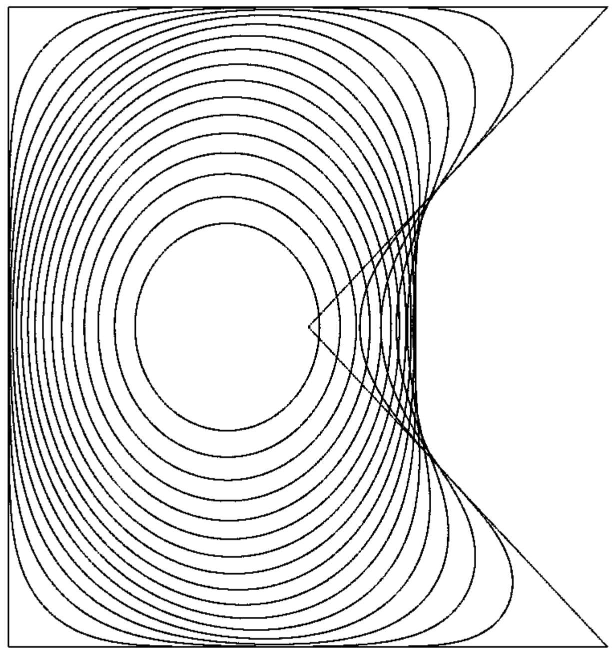

FIG. 4.2. *The mean curvature evolution of a nonsmooth, nonconvex curve.*

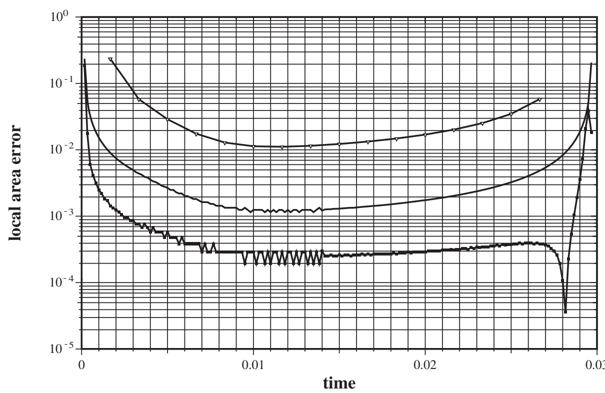

FIG. 4.3. *Local area error dependence on time. The first order method with time step* 1/600*—the line with triangle markers; the first order method with time step* 1/6000*—the thin line; the second order method with time step* 1/6000*—the line with square markers.*

Consider a nonconvex, nonsmooth initial curve, depicted in Figure 4.2. The mean curvature evolution of this curve was calculated using the direct method with time step values $dt = 1/600$ and $1/6000$. The shape of the curve is plotted in Figure 4.2 for times $t = 1/600,\ 2/600, \dots$ when calculated with the fine time step. The comparison between local relative errors

$$e_i = \frac{|S_i - S_{i+1} - 2\pi dt|}{2\pi dt} \tag{4.3}$$

for calculations with different time steps is seen in Figure 4.3. One can observe that the error indeed depends linearly on the time step.

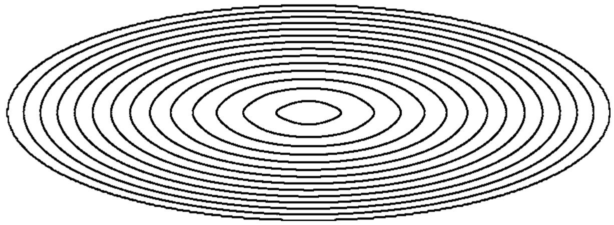

FIG. 4.4. *The evolution $v = k^{1/3}$ of an ellipse.*

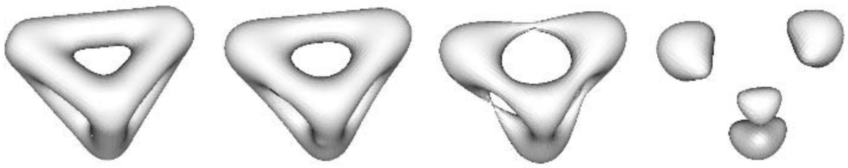

FIG. 4.5. *Computed mean curvature evolution.*

The evolution with the velocity $v = k^{1/3}$ is depicted in Figure 4.4. In this case the flow is affine invariant [1]; hence the eccentricity e of the evolving ellipse remains constant. In this particular example, the curvature is bounded from above and below by some positive constants for some evolution time. This means that we never use the parts of $G(k) = k^{1/3}$, where its derivative is too large or too small. This allows us to apply the thresholding procedure without any approximation of G.

In Figures 4.5 and 4.6 computed three-dimensional evolution of a nonconvex surface is represented for curvature flow and for a flow with velocity $v = G(k)$, as in Figure 4.7 with ~ 200000 triangles approximating the surface.

4.5. On the higher order schemes for the mean curvature motion. Let us now look at approximations to the mean curvature evolution. It is easy to see that if the surface is smooth, the Bence–Merriman–Osher method gives the first order approximation in time for a curvature flow. A higher order scheme by an extrapolation argument in time was proposed by Ruuth in [30]. We propose here higher order approximations to the mean curvature evolution using some properties of functions M_i.

We rewrite the equations (3.11) and keep an additional term of order $h^{3/2}$ in each equation with a kernel-dependent multiplier E_i to get the error term of order $h^{5/2}$. Considering two equations we get the relation

$$E_2 N_1 - E_1 N_2 = \sqrt{h}[(E_2 C_1 - E_1 C_2)v + (E_2 B_1 - E_1 B_2)\gamma''(h,0)] + O\big(h^{5/2}\big). \tag{4.4}$$

This relationship motivates us to take the thresholding function $F(N_1, N_2) = E_2 N_1 - E_1 N_2$ to approximate the mean curvature evolution with the second order accuracy for smooth curves. However, this thresholding function does not simultaneously satisfy (3.16) and (3.17) and, therefore, the stability of the numerical scheme is not guaranteed by the previous argument.

The calculations with the above thresholding function were performed. No sign of instability was observed in the numerical experiments and, as one can see in Figure 4.3, the accuracy was increased by approximately one order. This increase agrees with the construction (4.4).

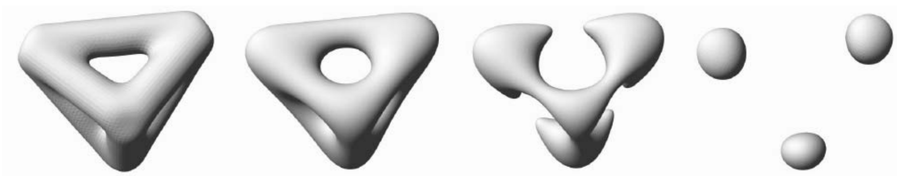

FIG. 4.6. *Computed generalized mean curvature evolution.*

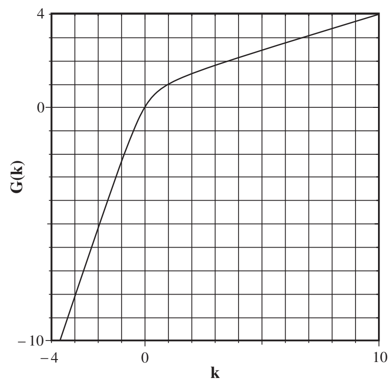

FIG. 4.7. *Function $G(k)$ used in the computation.*

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REFERENCES

[1] L. ALVAREZ, F. GUICHARD, P.-L. LIONS, AND J.-M. MOREL, *Axioms and fundamental equations of image processing*, Arch. Rational Mech. Anal., 123 (1993), pp. 199–257.

[2] S. ANGENENT, T. ILMANEN, AND D. L. CHOPP, *A computed example of nonuniqueness of mean curvature flow in* $\mathbf{R}^3$, Comm. Partial Differential Equations, 20 (1995), pp. 1937–1958.

[3] G. BARLES AND C. GEORGELIN, *A simple proof of convergence for an approximation scheme for computing motions by mean curvature*, SIAM J. Numer. Anal., 32 (1995), pp. 484–500.

[4] G. BARLES, H. M. SONER, AND P. E. SOUGANIDIS, *Front propagation and phase field theory*, SIAM J. Control Optim., 31 (1993), pp. 439–469.

[5] G. BARLES AND P. E. SOUGANIDIS, *Convergence of approximation schemes for fully nonlinear second order equations*, Asymptotic Anal., 4 (1991), pp. 271–283.

[6] G. BEYLKIN, *On the fast Fourier transform of functions with singularities*, Appl. Comput. Harmon. Anal., 2 (1995), pp. 363–381.

[7] K. A. BRAKKE, *The Motion of a Surface by Its Mean Curvature*, Princeton University Press, Princeton, NJ, 1978.

[8] L. BRONSARD AND R. V. KOHN, *Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics*, J. Differential Equations, 90 (1991), pp. 211–237.

[9] F. CAO, *Partial differential equations and mathematical morphology*, J. Math. Pures Appl. (9), 77 (1998), pp. 909–941.

[10] F. CATTÉ, F. DIBOS, AND G. KOEPFLER, *A morphological scheme for mean curvature motion and applications to anisotropic diffusion and motion of level sets*, SIAM J. Numer. Anal., 32 (1995), pp. 1895–1909.

[11] Y. G. Chen, Y. Giga, and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geom., 33 (1991), pp. 749–786.
[12] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. AMS (N.S.), 27 (1992), pp. 1–67.
[13] M. G. Crandall and P.-L. Lions, *Convergent difference schemes for nonlinear parabolic equations and mean curvature motion*, Numer. Math., 75 (1996), pp. 17–41.
[14] E. De Giorgi, *Conjectures on limits of some quasilinear parabolic equations and flow by mean curvature*, in Partial Differential Equations and Related Subjects (Trento, 1990), Longman Scientific and Technical, Harlow, 1992, pp. 85–95.
[15] L. C. Evans, *Convergence of an algorithm for mean curvature motion*, Indiana Univ. Math. J., 42 (1993), pp. 533–557.
[16] L. C. Evans, H. M. Soner, and P. E. Souganidis, *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math., 45 (1992), pp. 1097–1123.
[17] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature.* I, J. Differential Geom., 33 (1991), pp. 635–681.
[18] F. Guichard and J.-M. Morel, *Partial differential equations and image iterative filtering*, in The State of the Art in Numerical Analysis (York, 1996), Oxford University Press, New York, 1997, pp. 525–562.
[19] T. Ilmanen, *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Differential Geom., 38 (1993), pp. 417–461.
[20] H. Ishii, *A generalization of the Bence, Merriman and Osher algorithm for motion by mean curvature*, in Curvature Flows and Related Topics (Levico, 1994), Gakkōtosho, Tokyo, 1995, pp. 111–127.
[21] H. Ishii, G. E. Pires, and P. E. Souganidis, *Threshold dynamics type approximation schemes for propagating fronts*, J. Math. Soc. Japan, 51 (1999), pp. 267–308.
[22] H. Ishii and P. Souganidis, *Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor*, Tohoku Math. J. (2), 47 (1995), pp. 227–250.
[23] F. Leoni, *Convergence of an approximation scheme for curvature-dependent motions of sets*, SIAM J. Numer. Anal., 39 (2001), pp. 1115–1131.
[24] W. Lorensen and H. Cline, *Marching cubes: A high resolution* 3d *surface construction algorithm*, ACM Comput. Graphics, 21 (1987), pp. 163–170.
[25] B. Merriman, J. K. Bence, and S. J. Osher, *Motion of multiple junctions: A level set approach*, J. Comput. Phys., 112 (1994), pp. 334–363.
[26] T. Ohta, D. Jasnow, and K. Kawasaki, *Universal scaling in the motion of random interfaces*, Phys. Rev. Lett., 47 (1982), pp. 1223–1226.
[27] S. Osher and J. A. Sethian, *Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulations*, J. Comput. Phys., 79 (1988), pp. 12–49.
[28] D. Pasquignon, *Approximation of viscosity solution by morphological filters*, ESAIM Control Optim. Calc. Var., 4 (1999), pp. 335–359 (electronic).
[29] J. Rubinstein, P. Sternberg, and J. B. Keller, *Fast reaction, slow diffusion, and curve shortening*, SIAM J. Appl. Math., 49 (1989), pp. 116–133.
[30] S. J. Ruuth, *Efficient algorithms for diffusion-generated motion by mean curvature*, J. Comput. Phys., 144 (1998), pp. 603–625.
[31] J. A. Sethian, *Level Set Methods and Fast Marching Methods*, 2nd ed., Cambridge University Press, Cambridge, UK, 1999.