ANOTHER PROOF OF THE TRIPLE SUM FORMULA FOR WIGNER 9J-SYMBOLS

HJALMAR ROSENGREN

In Ref. 1 we gave a new simple proof of the triple sum formula for Wigner $9j$-symbols due to Ališauskas and Jucys.[2] In this note we give an even more simple proof. We will use the notation of Ref. 1. In particular we will work with the Lie group $\mathrm{SU}(1,1)$ (and its covering groups), though the same proof works in the physically more important case of $\mathrm{SU}(2)$.

We will denote the (unnormalized) Racah coefficients or $6j$-symbols by

$$R_{qr}^{tu}(\nu_1,\nu_2,\nu_3)=\langle((1,1)_q,1)_r,(1,(1,1)_t)_u\rangle_{\mathcal{A}^{\nu_1}\otimes\mathcal{A}^{\nu_2}\otimes\mathcal{A}^{\nu_3}}$$

(where $q+r=t+u$) and use notation such as

$$x_{12}=x_1+x_2, \qquad |x|=x_1+\cdots+x_n$$

for a vector $x=(x_1,\dots,x_n)$. As in Ref. 1 we denote Wigner $9j$-symbols by

$$W_{pqr}^{tuv}=W_{pqr}^{tuv}(\nu_1,\nu_2,\nu_3,\nu_4),$$

where $p+q+r=t+u+v=s$. It is not hard to derive the expression

$$\begin{aligned}(1)\quad W_{pqr}^{tuv}=&\sum_{j=0}^{\min(p+r,t+v)}\frac{(-1)^{p+r+j+u}}{C_{p+r-j}^{\nu_2,\nu_{34}+2q}C_j^{\nu_1,\nu_{234}+2s-2j}C_{t+v-j}^{\nu_3,\nu_{24}+2u}}\,R_{pr}^{p+r-j,j}(\nu_1,\nu_2,\nu_{34}+2q)\\&\times R_{q,p+r-j}^{u,t+v-j}(\nu_3,\nu_4,\nu_2)\,R_{t+v-j,j}^{t,v}(\nu_{24}+2u,\nu_3,\nu_1);\end{aligned}$$

in fact this (or rather the corresponding formula for $\mathrm{SU}(2)$) is often taken as the definition of Wigner $9j$-symbols. We will prove the triple sum formula by inserting appropriate single sum expressions for the Racah coefficients and changing the order of summation.

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There is a large number of expressions for Racah coefficients as terminating ${}_4F_3$ series, including

$$\begin{aligned}R^{tu}_{qr}(\nu_1,\nu_2,\nu_3) &= (\nu_1)_u(\nu_2)_t(\nu_3)_r(\nu_{12}+q-1)_q(\nu_{23}+r+t)_q(|\nu|+2q+r-1)_r\\ &\quad\times \frac{r!\,u!}{(u-q)!}\,{}_4F_3\left(\begin{matrix}-q,1-\nu_1-q,1+r,\nu_3+r\\ 1+u-q,\nu_{23}+r+t,2-\nu_{12}-2q\end{matrix}\,\middle|\,1\right)\\ &= (\nu_1)_u(\nu_2)_q(\nu_2)_t(\nu_3)_s(\nu_{12}+s)_q(|\nu|+2q+r-1)_r\\ &\quad\times r!\,{}_4F_3\left(\begin{matrix}-q,1-\nu_3-t,\nu_2+t,\nu_{12}+q-1\\ \nu_2,1-\nu_3-s,\nu_{12}+s\end{matrix}\,\middle|\,1\right)\\ &= (\nu_1)_q(\nu_2)_t(\nu_3)_t(\nu_{12}+q-1)_u(\nu_{23}+2t)_u(|\nu|+2q+r-1)_r\\ &\quad\times \frac{(-1)^{q+u}q!\,t!}{(q-u)!}\,{}_4F_3\left(\begin{matrix}-u,1-\nu_1-u,1+t,\nu_3+t\\ 1+q-u,\nu_{23}+2t,2-\nu_{12}-q-u\end{matrix}\,\middle|\,1\right),\end{aligned}$$

where $q+r=t+u=s$. In the first and third expression we use the ${}_4F_3$ notation somewhat formally; for instance the first ${}_4F_3$ should be interpreted as in

$$\frac{1}{(u-q)!}\,{}_4F_3\left(\begin{matrix}-q,a,b,c\\ 1+u-q,d,e\end{matrix}\,\middle|\,1\right)=\sum_{k=\max(0,\,q-u)}^{q}\frac{(-q)_k(a)_k(b)_k(c)_k}{k!(u-q+k)!(d)_k(e)_k},$$

where one may have $u<q$. Inserting these three expressions, in this order, in (1) gives

$$\begin{aligned}W^{tuv}_{pqr} = &\sum_{j=0}^{\min(p+r,t+v)}\frac{(-1)^{q+v}\,r!\,t!(\nu_1)_t(\nu_2)_{s-j}(\nu_3)_t(\nu_4)_q(\nu_4)_u(\nu_{12}+p-1)_p(\nu_{13}+2t)_v}{(\nu_{34})_{s-j}(\nu_{234}+2s-2j)_j(\nu_{234}+2u+t+v-j-1)_{t+v-j}}\\ &\times(\nu_{34})_{2q+r}(\nu_{234}+2q+2r+p-j)_p\\ &\times(\nu_{234}+2u+t+v-j-1)_v(|\nu|+2p+2q+r-1)_r\\ &\times\frac{1}{(j-p)!}\,{}_4F_3\left(\begin{matrix}-p,1-\nu_1-p,1+r,\nu_{34}+2q+r\\ 1+j-p,\nu_{234}+2q+2r+p-j,2-\nu_{12}-2p\end{matrix}\,\middle|\,1\right)\\ &\times{}_4F_3\left(\begin{matrix}-q,1-\nu_2-u,\nu_4+u,\nu_{34}+q-1\\ \nu_4,1-\nu_2-s+j,\nu_{34}+s-j\end{matrix}\,\middle|\,1\right)\\ &\times\frac{1}{(t-j)!}\,{}_4F_3\left(\begin{matrix}-v,1-\nu_{24}-2u-v,1+t,\nu_1+t\\ 1+t-j,\nu_{13}+2t,2-\nu_{234}-2u-t-2v+j\end{matrix}\,\middle|\,1\right).\end{aligned}$$

By Karlsson's summation formula[3] the second ${}_4F_3$ vanishes if $p+r<j\leq t+v$; therefore the condition $j\leq p+r$ is superfluous. Let us now expand the ${}_4F_3$ functions and rearrange the terms according to

$$\sum_{j=0}^{t+v}\sum_{k=\max(0,p-j)}^{p}\sum_{l=0}^{q}\sum_{m=\max(0,j-t)}^{v}C_{jklm}=\sum_{\substack{0\leq k\leq p,\ 0\leq l\leq q,\\ 0\leq m\leq v,\ k+m\geq p-t}}\sum_{j=0}^{t-p+k+m}C_{t+m-j,klm}.$$

This gives

$$\begin{aligned}
W_{pqr}^{tuv} &= (-1)^{q+v} r!\, t!(\nu_1)_t(\nu_2)_{u+v}(\nu_3)_t(\nu_4)_q(\nu_4)_u \\
&\times (\nu_{12}+p-1)_p(\nu_{13}+2t)_v(\nu_{34}+q+r)_q(|\nu|+2p+2q+r-1)_r \\
&\times \sum_{\substack{0\leq k\leq p,\ 0\leq l\leq q,\\ 0\leq m\leq v,\ k+m\geq p-t}} \frac{1}{(\nu_{34}+q+r)_{k+l}(1-\nu_2-u-v)_{l+m}(t-p+k+m)!} \\
&\times \frac{(-p)_k(1-\nu_1-p)_k(1+r)_k(\nu_{34}+2q+r)_k}{k!(2-\nu_{12}-2p)_k} \\
&\times \frac{(-q)_l(1-\nu_2-u)_l(\nu_4+u)_l(\nu_{34}+q-1)_l}{l!(\nu_4)_l} \\
&\times \frac{(-v)_m(1-\nu_{24}-2u-v)_m(1+t)_m(\nu_1+t)_m}{m!(\nu_{13}+2t)_m} \\
&\times \frac{(\nu_{34}+u+v+l-m)_{t-p+k+m}}{(\nu_{234}+2s-2t-2m)_{t-p+k+m}} \sum_{j=0}^{t-p+k+m} \frac{\nu_{234}+2s-2t-2m-1+2j}{\nu_{234}+2s-2t-2m-1} \\
&\times \frac{(p-t-k-m)_j(\nu_{234}+2s-2t-2m-1)_j(\nu_2+u+v-l-m)_j}{j!(\nu_{34}+u+v+l-m)_j(\nu_{234}+2s-p-t+k-m)_j}(-1)^j,
\end{aligned}$$

where, by the summation formula[4]

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{a+2k}{a} \frac{(-n)_k(a)_k(b)_k}{k!(1+a-b)_k(1+a+n)_k}(-1)^k \\
&\qquad = {}_4F_3\left(\begin{matrix} a, 1+\frac{1}{2}a, b, -n \\ \frac{1}{2}a, 1+a-b, 1+a+n \end{matrix} \;\middle|\; -1\right) = \frac{(1+a)_n}{(1+a-b)_n},
\end{aligned}$$

the inner sum equals

$$\frac{(\nu_{234}+2s-2t-2m)_{t-p+k+m}}{(\nu_{34}+u+v+l-m)_{t-p+k+m}}.$$

This leads to the expression

$$\begin{aligned}
W_{pqr}^{tuv} &= (-1)^{q+v} r!\, t!(\nu_1)_t(\nu_2)_{u+v}(\nu_3)_t(\nu_4)_q(\nu_4)_u \\
&\times (\nu_{12}+p-1)_p(\nu_{13}+2t)_v(\nu_{34}+q+r)_q(|\nu|+2p+2q+r-1)_r \\
&\times \sum_{\substack{0\leq k\leq p,\ 0\leq l\leq q,\\ 0\leq m\leq v,\ k+m\geq p-t}} \frac{1}{(\nu_{34}+q+r)_{k+l}(1-\nu_2-u-v)_{l+m}(t-p+k+m)!} \\
&\times \frac{(-p)_k(1-\nu_1-p)_k(1+r)_k(\nu_{34}+2q+r)_k}{k!(2-\nu_{12}-2p)_k} \\
&\times \frac{(-q)_l(1-\nu_2-u)_l(\nu_4+u)_l(\nu_{34}+q-1)_l}{l!(\nu_4)_l} \\
&\times \frac{(-v)_m(1-\nu_{24}-2u-v)_m(1+t)_m(\nu_1+t)_m}{m!(\nu_{13}+2t)_m}
\end{aligned} \tag{2}$$

which is a version of the Ališauskas–Jucys formula.

Finally we remark that Corollary 3.2 of Ref. 1 has previously been obtained (in the case of the group SU(2)) by Masao Nomura.[5] We owe this reference to Sigitas註išauskas.

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