A NEW QUANTUM ALGEBRAIC INTERPRETATION OF THE ASKEY–WILSON POLYNOMIALS

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Dedicated to Richard Askey

Abstract. We introduce a new kind of generalized group elements for the quantum algebra $\mathcal{U}_q(\mathfrak{su}(1,1))$. This leads to new interpretations of Askey–Wilson polynomials and ${}_8W_7$ series as matrix elements for highest weight representations of this algebra. Similarly, the q-Racah polynomials occur as matrix elements for the quantum algebra $\mathcal{U}_q(\mathfrak{su}(2))$. The algebraic interpretation leads to orthogonality relations and addition formulas for such functions.

1. Introduction

The Askey–Wilson polynomials, introduced in [AW2], and the q-Racah polynomials [AW1] generalize a large number of well known orthogonal polynomials. In fact, Askey proposed to define classical orthogonal polynomials to be the Askey–Wilson and q-Racah polynomials together with their limit cases. This leads to the Askey scheme of hypergeometric and basic hypergeometric (or q-) polynomials; cf. [KS], where the fundamental properties of 42 such polynomial systems are listed.

The hypergeometric polynomials in the Askey scheme are connected with representations of three-dimensional Lie groups [K1]. In the late 1980's it was realized that many of the q-polynomials are similarly connected with quantum groups. The first such interpretation of Askey–Wilson polynomials (apart from very degenerate cases) was found by Koornwinder [K2], who obtained a two-parameter family of Askey–Wilson polynomials as zonal spherical functions for the quantum SU(2) group. By considering more general matrix elements one may find a similar interpretation of the full four-parameter family of Askey–Wilson polynomials [NM1, NM2, Ko]. Moreover, a three-parameter family of Askey–Wilson polynomials occur as zonal spherical functions on the quantum SU(n) group [DN, D]. In all these cases Askey–Wilson polynomials appear as q-analogues of Jacobi polynomials.

A different interpretation was given by Koelink and Van der Jeugt [KV], who obtained Askey–Wilson polynomials as Clebsch–Gordan coefficients of the quantum algebra $\mathcal{U}_q(\mathfrak{su}(1,1))$. Here they appear as q-analogues of Jacobi polynomials, Hahn polynomials, and continuous Hahn polynomials.

By contrast, in this paper we obtain the full four-parameter family of Askey–Wilson polynomials as q-analogues of Meixner, Meixner–Pollaczek, and Laguerre polynomials. Actually our interpretation is closely related to the one in [NM1, NM2, Ko], though orthogonality arises in a completely different way. One may compare, for instance, with Jacobi and Meixner polynomials which are both given by Gauss'

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function ${}_2F_1$, though their orthogonality is different. In the quantum case the ${}_2F_1$ becomes a ${}_4\phi_3$ and the orthogonality relations become the same.

Our main tool is a new kind of generalized group elements for the quantum algebra $\mathcal{U}_q(\mathfrak{su}(1,1))$. Thus we work within the "quantum algebra" rather than the (closely related) "quantum group" approach to special functions; cf. [FV1]. The importance of these generalized group elements stems from the fact that they transform so called *twisted primitive* elements of the algebra into *group-like* elements. These two classes play a fundamental role in the work of Koornwinder and others mentioned above.

We compute matrix elements for the change between various bases (continuous and discrete) of a highest weight representation of $\mathcal{U}_q(\mathfrak{su}(1,1))$. These may be expressed in terms of ${}_4\phi_3$ and ${}_8W_7$ series. The interpretation as matrix elements implies addition formulas and orthogonality relations of such series. We obtain many such formulas, though they may all be obtained as special or limit cases of two general addition formulas involving ${}_8W_7$ series. The first of these is in [ARS], while the second one (equation (5.2) below) is a special case of a formula due to Koelink and Van der Jeugt [KV2].

The same method applies to the quantum algebra $\mathcal{U}_q(\mathfrak{su}(2))$. In this case there are two kinds of discrete bases (which diagonalize elements that are twisted primitive in two different senses). The matrix element for a change of base within one of these classes was computed by Koelink [Ko]; they are certain orthogonal ${}_4\phi_3$ functions. We will see that the matrix elements for a change between the two classes are q-Racah polynomials.

It is remarkable that the method of this paper takes us so high up in the Askey scheme. The Racah polynomials (without the q) are Racah coefficients of SU(2), that is, they are connected with a three-fold tensor product of group representations. Similarly, the q-Racah polynomials are Racah coefficients of $\mathcal{U}_q(\mathfrak{su}(2))$ [KR]. Granovskii and Zhedanov [GZ] showed that q-Racah polynomials also occur as Clebsch–Gordan coefficients (connected with a two-fold tensor product). Here we find that it is enough to consider a single representation of $\mathcal{U}_q(\mathfrak{su}(2))$ to obtain the full four-parameter family of q-Racah polynomials. Thus, from this point of view, the q-Racah and Askey–Wilson polynomials appear to be more fundamental objects than their hypergeometric counterparts.

The plan of the paper is as follows. Section 2 contains preliminaries on q-series and quantum algebra. In Section 3 we introduce our generalized group elements and discuss some of their properties. In Section 4 we compute various matrix elements and derive orthogonality relations. In Section 5 we consider addition formulas. In Section 6 we briefly discuss the more elementary case of the quantum algebra $\mathcal{U}_q(\mathfrak{su}(2))$. Finally, in Section 7 we prove some identities involving formal power series over the quantum algebra. These give a better understanding of our approach from an algebraic viewpoint.

We conclude this introduction with a remark on the general philosophy of the paper. Using relations involving formal power series over the quantum algebra we obtain many q-series identities in a simple and natural way. However, since our representation is given by unbounded operators on an infinite-dimensional Hilbert space, it would require a careful analysis to convert these arguments into formal proofs. There are situations when identities satisfied in the algebra are not satisfied on the representation space; cf. [K3, W]. In the case at hand, we prefer to first derive the q-series identities formally and then check by other methods (in the spirit of [GR]) in which sense they are valid. Since these direct proofs are standard and since all our identities are in a sense contained in the two addition formulas (5.1) and (5.2), we usually merely indicate the method of proof.

2. Preliminaries

2.1. **The group case.** It might be instructive to start with the limit case $q \to 1$. The Lie group $\mathrm{SU}(1,1)$ has three conjugacy classes of one-parameter subgroups, called *elliptic*, *hyperbolic*, and *parabolic*. Consider a unitary highest weight representation of $\mathrm{SU}(1,1)$ (or any covering group). Choose two one-parameter subgroups and let e_x, f_y be corresponding joint eigenvectors with eigenvalues x, y. Then $\langle e_x, f_y\rangle$ is called a *matrix element*. These may be identified with classical special functions as follows [BW, K1]:

$$\langle e_x, f_y\rangle = \begin{cases} \text{Meixner polynomial} & \text{case EE (both subgroups elliptic)},\\ \text{Meixner–Pollaczek polynomial} & \text{EH},\\ \text{Laguerre polynomial} & \text{EP},\\ \text{Meixner–Pollaczek function} & \text{HH},\\ \text{Laguerre function} & \text{HP},\\ \text{Bessel function} & \text{PP}. \end{cases}$$

The unitarity and the group law imply orthogonality relations and addition formulas for such functions; cf. [VK].

2.2. **Basic hypergeometric series.** Throughout the paper q will be a fixed number with $0 < q < 1$. We will use the notation for basic hypergeometric series (or q-series) of [GR], though since q is fixed we will often suppress it from the notation. Thus we write

$${}_r\phi_s\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(q)_k (b_1)_k \cdots (b_s)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k,$$

where

$$(a)_k = (a;q)_k = \prod_{i=0}^{k-1} (1 - aq^i).$$

The very-well-poised series is denoted by

$$\begin{aligned} {}_{r+1}W_r(a; b_1, \dots, b_{r-2}; q, z) &= {}_{r+1}\phi_r\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b_1, \dots, b_{r-2} \\ a^{\frac12}, -a^{\frac12}, qa/b_1, \dots, qa/b_{r-2} \end{matrix}; q, z\right] \\ &= \sum_{k=0}^{\infty} \frac{1 - aq^{2k}}{1 - a} \frac{(a)_k (b_1)_k \cdots (b_{r-2})_k}{(q)_k (qa/b_1)_k \cdots (qa/b_{r-2})_k} z^k. \end{aligned}$$

We will also write

$$(a)_\infty = (a;q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i),$$

$$(ab^{\pm})_\infty = (ab)_\infty (ab^{-1})_\infty,$$

$$(ab^{\pm}c^{\pm})_\infty = (abc)_\infty (abc^{-1})_\infty (ab^{-1}c)_\infty (ab^{-1}c^{-1})_\infty,$$

and similarly for finite products $(ab^{\pm})_n$. Among the facts about q-series used here we mention the summation formula [GR]

$$(2.1)\qquad {}_6W_5(a;b,c,d;q,aq/bcd)=\frac{(aq)_\infty(aq/bc)_\infty(aq/bd)_\infty(aq/cd)_\infty}{(aq/b)_\infty(aq/c)_\infty(aq/d)_\infty(aq/bcd)_\infty}$$

(where $|aq/bcd|<1$), which will play a central role. We also recall the definition of Askey–Wilson polynomials:

$$(2.2)\quad p_n(\cos\theta;a,b,c,d|q)=(ab)_n(ac)_n(ad)_na^{-n}\,{}_4\phi_3\left[\begin{matrix}q^{-n},abcdq^{n-1},ae^{i\theta},ae^{-i\theta}\\ ab,ac,ad\end{matrix};q,q\right];$$

this is a polynomial in $\cos\theta$ of degree n which is symmetric in the four parameters a, b, c, and d.

2.3. **The quantum algebra.** The quantum algebra (or quantized universal enveloping algebra) $\mathcal{U}_q(\mathfrak{su}(1,1))$ $(0<q<1)$ is an associative involutive algebra over the complex numbers, defined by generators

$$X_+,\qquad X_-,\qquad K^+,\qquad K^-,$$

relations

$$(2.3)\qquad \begin{cases} K^+K^-=K^-K^+=1,\\[2ex] K^+X_\pm K^-=q^{\pm\frac12}X_\pm,\\[2ex] X_+X_--X_-X_+=\dfrac{K^2-K^{-2}}{q^{\frac12}-q^{-\frac12}},\end{cases}$$

and involution

$$(2.4)\qquad (K^\pm)^*=K^\pm,\quad X_\pm^*=-X_\mp.$$

It has more algebraic structure (coproduct, counit, antipode) which we do not need to consider. We refer to [CP] for an introduction to quantum group theory.

2.4. **Holomorphic realization.** We will work with the holomorphic realization of highest weight representations of $\mathcal{U}_q(\mathfrak{su}(1,1))$. Equivalent versions of this realization occur in [J, KL, Kor].

For $\nu>0$ and $0<q<1$, let $\mathcal{A}_q^\nu$ be the Hilbert space of analytic functions with the norm

$$\|f\|^2=\langle f,f\rangle=\sum_{k=0}^\infty\frac{(q)_k}{(q^\nu)_k}\,q^{\frac12k(\nu-1)}\,|\hat f(k)|^2,$$

where

$$f(z)=\sum_{k=0}^\infty\hat f(k)z^k.$$

(Note that the norm is invariant if one formally replaces q by q^{-1}.) By the q-binomial formula [GR], $\mathcal{A}_q^\nu$ has the reproducing kernel

$$k_w(z)=\sum_{k=0}^\infty\frac{(q^\nu)_k}{(q)_k}\,q^{\frac12k(1-\nu)}(z\bar w)^k=\frac{(q^{\frac12+\frac\nu2}z\bar w)_\infty}{(q^{\frac12-\frac\nu2}z\bar w)_\infty},$$

and thus the natural domain of definition is the disc

$$\{z \in \mathbb{C};\ |z| < q^{\frac{\nu}{4}-\frac{1}{4}}\}.$$

We will use the notation

$$\langle f, g\rangle = \sum_{k=0}^{\infty} \frac{(q)_k}{(q^\nu)_k}\, q^{\frac{1}{2}k(\nu-1)} \hat{f}(k)\overline{\hat{g}(k)}$$

also when f or g is not in $\mathcal{A}_q^\nu$, provided that the sum on the right-hand side converges.

A unitary representation of $\mathcal{U}_q(\mathfrak{su}(1,1))$ on $\mathcal{A}_q^\nu$ is given by

$$\begin{cases} X_+ f(z) = -\dfrac{f(q^{\frac{1}{2}}z) - f(q^{-\frac{1}{2}}z)}{q^{\frac{1}{2}}z - q^{-\frac{1}{2}}z}, \\[2ex] X_- f(z) = z\,\dfrac{q^{\frac{\nu}{2}} f(q^{\frac{1}{2}}z) - q^{-\frac{\nu}{2}} f(q^{-\frac{1}{2}}z)}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \\[2ex] K^{\pm} f(z) = q^{\mp\frac{\nu}{4}} f(q^{\mp\frac{1}{2}}z). \end{cases}$$

Thus these operators satisfy the structure equations (2.3) and the unitarity conditions (2.4), when $*$ is the Hilbert space adjoint of densely defined operators.

In the limit $q \to 1$ one recovers the Lie algebra operators

$$\begin{cases} X_+ \to -\dfrac{d}{dz} = E, \\[2ex] X_- \to z^2\dfrac{d}{dz} + \nu z = F, \\[2ex] 2\log_q K^{\pm} \to \mp\left(\dfrac{\nu}{2} + z\dfrac{d}{dz}\right) = H, \end{cases} \tag{2.5}$$

which correspond to the group action

$$f(z) \mapsto f\left(\frac{az+b}{cz+d}\right)\frac{1}{(cz+d)^\nu} = f(\phi(z))(\phi'(z))^{\frac{\nu}{2}},$$

with

$$\phi(z) = \frac{az+b}{cz+d}, \qquad ad - bc = 1$$

an analytic automorphism of the unit disc. For $\nu > 1$,

$$\lim_{q\to 1} \|f\|^2_{\mathcal{A}_q^\nu} = \frac{\nu-1}{\pi}\int_{|z|<1} |f(z)|^2(1-|z|^2)^{\nu-2}\,dxdy \qquad (z = x+iy).$$

There is an analogous integral formula for the norm of $\mathcal{A}_q^\nu$ for $\nu > 1$, though we will not use it.

2.5. **Bases.** We will write

$$e_n(z) = z^n.$$

These elements form a discrete basis for $\mathcal{A}_q^\nu$, with Plancherel formula

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \frac{(q^\nu)_n}{(q)_n}\, q^{\frac{1}{2}n(1-\nu)} \langle f, e_n\rangle\langle e_n, g\rangle. \tag{2.6}$$

Continuous bases were considered by Koelink and Van der Jeugt [KV]. Let

$$Y_t = \left(q^{\frac{1}{4}} X_- - q^{-\frac{1}{4}} X_+ + t\, \frac{K - K^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \right) K^{-1},$$

where t is real. Then Y_t is essentially self-adjoint. In our realization,

$$\begin{aligned} Y_t f(z) &= \frac{1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \left(q^{\frac{3\nu}{4}+\frac{1}{4}} z + q^{\frac{\nu}{4}-\frac{1}{4}} z^{-1} - q^{\frac{\nu}{2}} t \right) f(qz) \\ &\quad - \frac{1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \left(q^{\frac{1}{4}-\frac{\nu}{4}} z + q^{\frac{\nu}{4}-\frac{1}{4}} z^{-1} - t \right) f(z). \end{aligned}$$

Since this only involves the value of f at two points, the formal eigenvectors of Y_t may be expressed as infinite products. For x another real number we write

$$v_{xt}(z) = \prod_{j=0}^{\infty} \frac{1 - z q^{\frac{1}{4}+\frac{\nu}{4}+j} t + z^2 q^{\frac{1}{2}+\frac{\nu}{2}+2j}}{1 - z q^{\frac{1}{4}-\frac{\nu}{4}+j} x + z^2 q^{\frac{1}{2}-\frac{\nu}{2}+2j}};$$

then

$$Y_t\, v_{xt} = \frac{t - x}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\, v_{xt}.$$

One may also write

$$v_{xt}(z) = \frac{(z q^{\frac{1}{4}+\frac{\nu}{4}} \tau)_\infty (z q^{\frac{1}{4}+\frac{\nu}{4}} \tau^{-1})_\infty}{(z q^{\frac{1}{4}-\frac{\nu}{4}} \xi)_\infty (z q^{\frac{1}{4}-\frac{\nu}{4}} \xi^{-1})_\infty},$$

where

$$t = \tau + \tau^{-1}, \qquad x = \xi + \xi^{-1}.$$

Then τ and ξ are either non-zero real or unimodular numbers,

The matrix elements $\langle e_n, v_{xt}\rangle$ are Al-Salam–Chihara polynomials, that is, Askey–Wilson polynomials with two parameters equal to 0. In fact, v_{xt} is a well known generating function for these polynomials. We note the explicit expressions

$$\begin{aligned} \langle e_n, v_{xt}\rangle &= (q^{\frac{\nu}{4}+\frac{1}{4}}\tau)^{-n}\, {}_3\phi_2\left[\begin{matrix} q^{-n},\ q^{\frac{\nu}{2}}\tau\xi,\ q^{\frac{\nu}{2}}\tau\xi^{-1} \\ q^\nu,\ 0 \end{matrix}; q, q\right] \\ &= \frac{(q^{\frac{\nu}{2}}\xi\tau^{-1})_n}{(q^\nu)_n} (q^{\frac{\nu}{4}-\frac{1}{4}}\xi^{-1})^n\, {}_2\phi_1\left[\begin{matrix} q^{-n},\ q^{\frac{\nu}{2}}\tau\xi^{-1} \\ q^{1-n-\frac{\nu}{2}}\tau\xi^{-1} \end{matrix}; q, q^{1-\frac{\nu}{2}}\tau\xi\right] \\ &= \frac{1}{(q^\nu)_n}\, q^{\frac{1}{4}n(\nu-1)}\, p_n(\tfrac{x}{2}; q^{\frac{\nu}{2}}\tau, q^{\frac{\nu}{2}}\tau^{-1}, 0, 0|q). \end{aligned} \tag{2.7}$$

The orthogonality measure for these polynomials (cf. [GR] or [KS] for the explicit formula) is then the spectral measure of Y_t. We let dm_t be this measure, normalized so that one has a Plancherel formula

$$\langle f, g\rangle = \int \langle f, v_{xt}\rangle\langle v_{xt}, g\rangle\, dm_t(x)$$

(for f, g in a dense subset of $\mathcal{A}_q^\nu$). For $|t| < q^{\frac{\nu}{2}} + q^{-\frac{\nu}{2}}$, the measure is absolutely continuous with support $-2 \leq x \leq 2$. For larger $|t|$ one must add a finite number of point-masses; if $\tau > q^{-\frac{\nu}{2}}$ they are located at the points

$$q^{\frac{\nu}{2}+k}\tau + q^{-\frac{\nu}{2}-k}\tau^{-1},$$

where k are the non-negative integers such that $q^{\frac{\nu}{2}+k}\tau > 1$.

The matrix element

$$\langle \delta_c v_{xt}, v_{yu} \rangle = \sum_{n=0}^{\infty} \frac{(q^\nu)_n}{(q)_n}\, q^{\frac{1}{2}n(1-\nu)} c^n \langle v_{xt}, e_n \rangle \langle e_n, v_{yu} \rangle, \tag{2.8}$$

where δ_c is the dilatation operator $\delta_c f(z) = f(cz)$, is a Poisson kernel for Al-Salam–Chihara polynomials. It has recently been shown [ARS, IS] that this kernel is expressible as an ${}_8W_7$ series. This fact will be recovered below, with an independent proof. One may compare with [VJ], where a different interpretation of (2.8) (involving a tensor product $\mathcal{A}_q^{\nu_1} \otimes \mathcal{A}_q^{\nu_2}$) was found.

If one formally replaces q by q^{-1} in the definition of Y_t, one obtains the operator

$$\tilde{Y}_t = \left(q^{-\frac{1}{4}} X_- - q^{\frac{1}{4}} X_+ + t\, \frac{K - K^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \right) K. \tag{2.9}$$

This is an unbounded symmetric operator. It has formal eigenvectors

$$\tilde{v}_{xt}(z) = \frac{(zq^{\frac{3}{4}+\frac{\nu}{4}}\xi)_\infty (zq^{\frac{3}{4}+\frac{\nu}{4}}\xi^{-1})_\infty}{(zq^{\frac{3}{4}-\frac{\nu}{4}}\tau)_\infty (zq^{\frac{3}{4}-\frac{\nu}{4}}\tau^{-1})_\infty},$$

satisfying

$$\tilde{Y}_t\, \tilde{v}_{xt} = \frac{x - t}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\, \tilde{v}_{xt}. \tag{2.10}$$

Note that v and $\tilde{v}$ are connected by

$$\tilde{v}_{xt}(z) = v_{tx}(q^{\frac{1}{2}} z). \tag{2.11}$$

The matrix elements $\langle e_n, \tilde{v}_{xt} \rangle$ are Al-Salam–Chihara polynomials with q replaced by q^{-1}. The corresponding moment problem is determinate if and only if $|t| \geq q^{\frac{1}{2}} + q^{-\frac{1}{2}}$ [AI]. Then the orthogonality measure is infinite discrete. This discrete orthogonality of Al-Salam–Chihara polynomials will also be recovered below.

2.6. **The "classical" quantum algebra approach.** Recall the two q-exponential functions [GR]

$$E_q(z) = (-z)_\infty = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} z^n}{(q)_n},$$

$$e_q(z) = \frac{1}{(z)_\infty} = \sum_{n=0}^{\infty} \frac{z^n}{(q)_n}.$$

In the quantum algebra approach to q-series one considers generalized group elements such as $e_q(A)E_q(B)$, where A and B are in the quantum algebra. Though we will use a more complicated construction, we will give an example of this "classical" quantum algebra approach for comparison and motivation.

Consider, for real λ, the following two q-analogues of translation operators acting as densely defined operators on $\mathcal{A}_q^\nu$:

$$T_\lambda = E_q(\lambda q^{-\frac14}(1-q)X_+K^{-1}),$$
$$\tilde{T}_\lambda = e_q(\lambda q^{\frac14}(1-q)X_+K).$$

(Formally $\tilde{T}$ is obtained from T by replacing q by q^{-1}.) Let

$$T_{\lambda\mu} = T_\lambda^* \tilde{T}_\mu.$$

One may compute the matrix elements

$$\langle T_{\lambda\mu}\, e_k, e_l\rangle = (-1)^{k+l}\lambda^l\mu^k\, q^{\frac14(\nu+1)(l-k)+\binom{l}{2}-\binom{k}{2}}\, {}_2\phi_1\left[\begin{matrix} q^{-k}, q^{-l} \\ q^\nu \end{matrix}; q, \frac{q^{\frac{\nu}{2}+\frac12+k}}{\lambda\mu}\right],$$

$$\langle (T_{\lambda\mu}^*)^{-1} e_k, e_l\rangle = q^{\frac14(\nu+1)(k-l)}\lambda^k\mu^l\, \frac{(q^{\frac{\nu}{2}+\frac12+l}\lambda\mu)_\infty}{(q^{\frac12-\frac{\nu}{2}-k}\lambda\mu)_\infty}\, {}_2\phi_1\left[\begin{matrix} q^{-k}, q^{-l} \\ q^\nu \end{matrix}; q, \frac{q^{\frac{\nu}{2}+\frac12+k}}{\lambda\mu}\right].$$

The relation

$$\langle e_k, e_l\rangle = \langle T_{\lambda\mu}\, e_k, (T_{\lambda\mu}^*)^{-1}e_l\rangle = \sum_{n=0}^\infty \frac{(q^\nu)_n}{(q)_n}\, q^{\frac12 n(1-\nu)}\langle T_{\lambda\mu}\, e_k, e_n\rangle\langle e_n, (T_{\lambda\mu}^*)^{-1}e_l\rangle \tag{2.12}$$

is then, for $\lambda\mu < 0$, equivalent to the orthogonality relation for q-Meixner polynomials; cf. [FV2, KMM, Z]. The fact that the matrix elements of $T_{\lambda\mu}$ have nice properties is connected with the "master relation"

$$T_\lambda^*\, \tilde{T}_\mu\, e_q(q^{\frac12}\lambda\mu K^2) = e_q(q^{\frac12}\lambda\mu K^{-2})\, \tilde{T}_\mu\, T_\lambda^*. \tag{2.13}$$

If one wants to do something similar involving continuous bases one might find the relation

$$\langle e_k, e_l\rangle = \int \langle T_\lambda e_k, v_{xt}\rangle\langle v_{xt}, (T_\lambda^*)^{-1}e_l\rangle\, dm_t(x),$$

which gives the orthogonality of continuous dual q-Hahn polynomials, that is, Askey–Wilson polynomials with one parameter equal to zero. This is in itself already a new fact. To generalize it, it is natural to seek an operator $U_{\lambda\mu}$ which interpolates between $(T_\lambda^*)^{-1}$ and T_μ, just as $T_{\lambda\mu}$ interpolates between T_λ^* and $\tilde{T}_\mu$. We will construct such an operator $U_{\lambda\mu}$ for which

$$\langle e_k, e_l\rangle = \int \langle U_{\lambda\mu} e_k, v_{xt}\rangle\langle v_{xt}, (U_{\lambda\mu}^*)^{-1}e_l\rangle\, dm_t(x) \tag{2.14}$$

gives the orthogonality of Askey–Wilson polynomials.

3. New generalized group elements

Let us consider the operator defined by

$$U_{\lambda\mu}e_n(z) = \frac{z^n(q^{\frac14+\frac{\nu}{4}}\mu/z)_n(q^{\frac34+\frac{3\nu}{4}+n}\lambda z)_\infty}{(q\lambda\mu)_n(q^{\frac34-\frac{\nu}{4}}\lambda z)_\infty},$$

where λ and μ are complex numbers. Initially we think of it as defined on the space of polynomials. For $|\lambda| < q^{-\frac12}$ this gives a densely defined operator on $\mathcal{A}_q^\nu$, since the poles of the right-hand side are then larger than $q^{\frac{\nu}{4}-\frac{1}{4}}$ in modulus. Let us mention already the following fact, which will be proved below.

Proposition 3.1. *For $|\lambda| < q^{-\frac{1}{2}}$, the densely defined operator*

$$U_{\lambda\bar{\lambda}}\Delta_{|\lambda|^2}^{1/2},$$

where

$$\Delta_{|\lambda|^2}^{1/2} e_n = \left(\frac{(q|\lambda|^2)_\infty (q|\lambda|^2)_n}{(q^{\nu+n}|\lambda|^2)_\infty} (1 - q^{\nu+2n}|\lambda|^2) \right)^{\frac{1}{2}} e_n,$$

extends to a unitary operator on $\mathcal{A}_q^\nu$.

Remark 3.2. Because of the relation

$$\delta_c U_{\lambda\mu} = U_{c\lambda, c^{-1}\mu}\delta_c \tag{3.1}$$

(where as above $\delta_c f(z) = f(cz)$), it would be no substantial restriction to consider only the operators $U_{\lambda\bar{\lambda}}$. The advantage with considering the general case is that we incorporate also $U_{\lambda 0}$ and $U_{0\mu}$, which makes certain limit transitions in the Askey scheme more transparent.

In the limit

$$\lim_{q\to 1} (U_{\lambda\mu}e_n)(z) = \frac{(z-\mu)^n}{(1-\lambda\mu)^n(1-\lambda z)^{\nu+n}} = e_n(\phi(z))\phi'(z)^{\frac{\nu}{2}},$$

where ϕ is the (complexified) group element

$$\phi(z) = \frac{1}{1-\lambda\mu}\frac{z-\mu}{1-\lambda z}.$$

In terms of the Lie algebra generators (2.5) one may write

$$\lim_{q\to 1} U_{\lambda\mu} = \exp(\lambda F)\exp(\tfrac{\mu}{1-\lambda\mu}E) = \exp(\mu E)\exp(\tfrac{\lambda}{1-\lambda\mu}F)(1-\lambda\mu)^H.$$

We will now give analogous expressions for $U_{\lambda\mu}$ in terms of the quantum algebra generators.

Proposition 3.3. *The operator $U_{\lambda\mu}$ may be expressed as*

$$\begin{aligned} U_{\lambda\mu} &= \sum_{k,l=0}^\infty \frac{\lambda^k\mu^l q^{kl+\frac{1}{4}k-\frac{1}{4}l+\binom{l}{2}}}{(q)_k(q)_l(q\lambda\mu)_l}(1-q)^{k+l}(X_-K^{-1})^k(X_+K^{-1})^l \\ &= \sum_{l=0}^\infty \frac{\mu^l q^{-\frac{1}{4}l+\binom{l}{2}}}{(q)_l}(1-q)^l(X_+K^{-1})^l \sum_{k=0}^\infty \frac{\lambda^k q^{\frac{1}{4}k}}{(q)_k}(1-q)^k(X_-K^{-1})^k \frac{(q^{1+k}\lambda\mu K^{-4})_\infty}{(q\lambda\mu)_\infty}. \end{aligned}$$

Note that these expressions are independent of the parameter ν, so that $U_{\lambda\mu}$ also makes sense as a formal power series over the quantum algebra. The equality of the two expressions (a new "master relation") may be proved in this context; cf. Section 7. Note that in particular

$$U_{\lambda 0} = (T_\lambda^*)^{-1}, \qquad U_{0\mu} = T_\mu$$

as indicated at the end of the previous section.

Proof. First one checks that

$$(X_+K^{-1})^k e_n = q^{-\binom{k}{2}+k(\frac{\nu}{4}+n+\frac12)}\frac{(q^{-n})_k}{(1-q)^k}\, e_{n-k},$$
$$(X_-K^{-1})^k e_n = q^{k(\frac12-\frac{\nu}{4})}\frac{(q^{\nu+n})_k}{(1-q)^k}\, e_{n+k}.$$

To prove the first expression we write

$$\begin{aligned}\sum_{k,l=0}^{\infty}&\frac{\lambda^k\mu^l q^{kl+\frac14 k-\frac14 l+\binom{l}{2}}}{(q)_k(q)_l(q\lambda\mu)_l}(1-q)^{k+l}(X_-K^{-1})^k(X_+K^{-1})^l e_n(z)\\
&=\sum_{l=0}^{n}\frac{(q^{-n})_l}{(q)_l(q\lambda\mu)_l}\,q^{l(\frac{\nu}{4}+n+\frac14)}\mu^l z^{n-l}\sum_{k=0}^{\infty}\frac{(q^{\nu+n-l})_k}{(q)_k}\,q^{k(\frac34+l-\frac{\nu}{4})}\lambda^k z^k\\
&=\sum_{l=0}^{n}\frac{(q^{-n})_l}{(q)_l(q\lambda\mu)_l}\,q^{l(\frac{\nu}{4}+n+\frac14)}\mu^l z^{n-l}\frac{(q^{\frac34+\frac{3\nu}{4}+n}\lambda z)_\infty}{(q^{\frac34-\frac{\nu}{4}+l}\lambda z)_\infty}\\
&=\frac{(q^{\frac34+\frac{3\nu}{4}+n}\lambda z)_\infty}{(q^{\frac34-\frac{\nu}{4}}\lambda z)_\infty}\sum_{l=0}^{n}\frac{(q^{-n})_l(q^{\frac34-\frac{\nu}{4}}\lambda z)_l}{(q)_l(q\lambda\mu)_l}\,q^{l(\frac{\nu}{4}+n+\frac14)}\mu^l z^{n-l}\\
&=\frac{z^n(q^{\frac14+\frac{\nu}{4}}\mu/z)_n(q^{\frac34+\frac{3\nu}{4}+n}\lambda z)_\infty}{(q\lambda\mu)_n(q^{\frac34-\frac{\nu}{4}}\lambda z)_\infty}=U_{\lambda\mu}e_n(z),\end{aligned}$$

where we used the q-binomial theorem and the q-Vandermonde formula [GR]. The second expression is proved similarly using the q-Gauss formula. In both cases, the resulting series converge for $|\lambda z|<q^{\frac{\nu}{4}-\frac34}$ and generic μ. □

The second expression of the proposition gives the factorization

$$U_{\lambda\mu}=T_\mu AD, \tag{3.2}$$

where T_μ is as above,

$$A=\sum_{k=0}^{\infty}\frac{1}{(q)_k}\,y^k\frac{1}{(x)_k},$$

with

$$x=q\lambda\mu K^{-4},\qquad y=q^{\frac14}(1-q)\lambda X_-K^{-1} \tag{3.3}$$

and where

$$D=\frac{(q\lambda\mu K^{-4})_\infty}{(q\lambda\mu)_\infty}.$$

This may be used to invert $U_{\lambda\mu}$ as a formal power series. The factor T_μ is of the form $E_q(X)$, having the inverse $e_q(-X)$. To invert the second factor, we use the following lemma.

Lemma 3.4. *Let x and y be variables such that $xy=q^2yx$. Then the formal power series*

$$A=\sum_{k=0}^{\infty}\frac{1}{(q)_k}\,y^k\frac{1}{(x)_k}$$

is invertible with the inverse

$$B = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q)_k} \frac{1}{(q^{-k-1}x)_k}\, y^k.$$

The factors $1/(x)_k$ and $1/(q^{-k-1}x)_k$ should be interpreted as formal power series in x (given by the q-binomial formula). Many identities of a similar nature are given in [K3].

Proof. The power series A is invertible since the constant term is non-zero. (The usual proof for commuting variables works, since the relation $xy = qyx$ preserves homogeneity.) Thus it suffices to prove that $BA = 1$. One has that

$$\begin{aligned} BA &= \sum_{k,l=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q)_k (q)_l} \frac{1}{(q^{-k-1}x)_k}\, y^{k+l} \frac{1}{(x)_l} \\ &= \sum_{k,l=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q)_k (q)_l}\, y^{k+l} \frac{1}{(q^{k+2l-1}x)_k (x)_l} \\ &= \sum_{j=0}^{\infty} y^j \sum_{k+l=j} \frac{(-1)^k q^{\binom{k}{2}}}{(q)_k (q)_l} \frac{1}{(q^{j+l-1}x)_k (x)_l}. \end{aligned}$$

The inner series may be written as a terminating ${}_2\phi_1$, which equals δ_{j0} by the q-Vandermonde formula.

Alternatively, one may compute AB, which leads to the sum

$${}_4W_3(a; q^{-j}; q, q^j) = \delta_{j0}$$

(this is the special case $b = q^{-j}$, $cd = aq$ of (2.1)).

□

Thus we obtain the following expression for $U_{\lambda\mu}^{-1}$.

Proposition 3.5. *As a formal power series, $U_{\lambda\mu}$ is invertible with the inverse*

$$\begin{aligned} U_{\lambda\mu}^{-1} = \frac{(q\lambda\mu)_\infty}{(q\lambda\mu K^{-4})_\infty} &\sum_{k=0}^{\infty} \frac{\lambda^k q^{\frac14 k + \binom{k}{2}}}{(q)_k} (-1)^k (1-q)^k \frac{1}{(q^{-k}\lambda\mu K^{-4})_k} (X_- K^{-1})^k \\ \times &\sum_{l=0}^{\infty} \frac{\mu^l q^{-\frac14 l}}{(q)_l} (-1)^l (1-q)^l (X_+ K^{-1})^l \end{aligned}$$

and thus (with the involution $*$ extended to power series in the natural way)

$$\begin{aligned} (U_{\bar\lambda\bar\mu}^*)^{-1} &= \sum_{l=0}^{\infty} \frac{\mu^l q^{\frac14 l}}{(q)_l} (1-q)^l (X_- K^{-1})^l \\ &\times \sum_{k=0}^{\infty} \frac{\lambda^k q^{-\frac14 k + \binom{k}{2}}}{(q)_k} (1-q)^k (X_+ K^{-1})^k \frac{1}{(q^{-k}\lambda\mu K^{-4})_k} \frac{(q\lambda\mu)_\infty}{(q\lambda\mu K^{-4})_\infty}. \end{aligned} \tag{3.4}$$

It is tempting to conclude that the expression (3.4) also holds at the level of densely defined operators on $\mathcal{A}_q^\nu$, which would be useful when making equation (2.14) explicit. This would, however, require a careful analysis of the convergence of the series involved; cf. the remark at the end of the introduction. We will instead use (3.4) to formally derive some identities satisfied by matrix elements. We will check those identities by other means, and then reverse the argument, showing that this implies that (3.4) holds in a certain sense.

4. Matrix elements

4.1. **Discrete matrix elements.** To compute the matrix element $\langle U_{\lambda\mu}e_k, e_l\rangle$ is equivalent to expanding $U_{\lambda\mu}e_k$ as a power series. One may use Proposition 3.3 or compare with the generating function for Al-Salam–Chihara polynomials. Using transformation formulas from [GR] one may derive many expressions such as

$$\begin{aligned}
\langle U_{\lambda\mu}e_k, e_l\rangle &= (-1)^k q^{\binom{k}{2}+\frac14(\nu+1)(k+l)+kl}\frac{\mu^k\lambda^l}{(q\lambda\mu)_k}\,{}_3\phi_2\left[\begin{matrix}q^{-k},q^{-l},q^{-k}/\lambda\mu\\ q^{\nu},0\end{matrix};q,q\right]\\
&= \lambda^{l-k}q^{\frac14 k(\nu-3)+\frac14 l(\nu+1)}\frac{1}{(q^{\nu})_k}\,{}_2\phi_1\left[\begin{matrix}q^{-k},q^{\nu+k}\lambda\mu\\ q\lambda\mu\end{matrix};q,q^{1+l}\right] &(4.1)\\
&= (-1)^k q^{\binom{k}{2}+\frac14(\nu+1)(k+l)}\frac{\mu^k\lambda^l}{(q\lambda\mu)_k}\,{}_3\phi_1\left[\begin{matrix}q^{-k},q^{-l},q^{\nu+k}\lambda\mu\\ q^{\nu}\end{matrix};q,\frac{q^l}{\lambda\mu}\right] &(4.2)\\
&= (-1)^k q^{\binom{k}{2}+\frac14(\nu+1)(k+l)}\frac{\mu^{k-l}}{(q\lambda\mu)_k}\,{}_3\phi_1\left[\begin{matrix}q^{-l},q^{-k}/\lambda\mu,q^{\nu+k}\\ q^{\nu}\end{matrix};q,q^l\lambda\mu\right]. &(4.3)
\end{aligned}$$

If one formally uses (3.4), a computation similar to the proof of Proposition 3.3 gives

$$(U^*_{\bar\mu\bar\lambda})^{-1}e_n=\frac{(q\lambda\mu)_\infty(q\lambda\mu)_n}{(q^{\nu+n}\lambda\mu)_\infty}(1-q^{\nu+2n}\lambda\mu)\,U_{\lambda\mu}e_n,$$

that is,

$$(U^*_{\bar\mu\bar\lambda})^{-1}=U_{\lambda\mu}\,\Delta_{\lambda\mu}, \tag{4.4}$$

where $\Delta_{\lambda\mu}$ is the diagonal operator

$$\Delta_{\lambda\mu}e_n=\frac{(q\lambda\mu)_\infty(q\lambda\mu)_n}{(q^{\nu+n}\lambda\mu)_\infty}(1-q^{\nu+2n}\lambda\mu)e_n.$$

This suggests that the matrix elements satisfy

$$\langle e_k,e_l\rangle=\sum_{n=0}^{\infty}\frac{(q^{\nu})_n}{(q)_n}\,q^{\frac12 n(1-\nu)}\langle U_{\lambda\mu}\Delta_{\lambda\mu}e_k,e_n\rangle\langle e_n,U_{\bar\mu\bar\lambda}e_l\rangle, \tag{4.5}$$

$$\langle e_k,e_l\rangle=\sum_{n=0}^{\infty}\frac{(q^{\nu})_n}{(q)_n}\,q^{\frac12 n(1-\nu)}\langle e_k,U_{\bar\mu\bar\lambda}e_n\rangle\langle U_{\lambda\mu}\Delta_{\lambda\mu}e_n,e_l\rangle. \tag{4.6}$$

One may check that (4.5) indeed holds for $|\lambda\mu|<q^{-1}$, while (4.6) holds for all generic values of λ and μ. For $0<\lambda\mu<q^{-1}$, (4.5) gives the orthogonality relation for little q-Jacobi polynomials. For $0<\lambda\mu\le q$, (4.6) gives the discrete orthogonality relation for Al-Salam–Chihara polynomials found by Askey and Ismail [AI]. It is also an orthogonality relation for $q<\lambda\mu<\min(q^{-1},q^{-\nu})$; this gives a special solution to an indeterminate moment problem.

To prove (4.5) one may use (4.1); the same calculation is in [AA1]. To prove (4.6) one may use expressions (4.2) and (4.3). After changing the order of summation one has to apply the formula

$$ {}_4\phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b \\ \sqrt{a}, -\sqrt{a}, aq/b, 0, 0 \end{matrix}; q, \frac{aq}{b}\right] = \frac{(aq)_\infty}{(aq/b)_\infty}, $$

which is the limit case $c,\ d \to \infty$ of (2.1).

The expression (4.4) also suggests that the reproducing kernels satisfy

$$ \langle k_w, k_z\rangle = \sum_{n=0}^{\infty} \frac{(q^\nu)_n}{(q)_n}\, q^{\frac{1}{2}n(1-\nu)} \langle k_w, U_{\bar\mu\bar\lambda} e_n\rangle \langle U_{\lambda\mu}\Delta_{\lambda\mu} e_n, k_z\rangle, $$

that is,

$$ k_w(z) = \sum_{n=0}^{\infty} \frac{(q^\nu)_n}{(q)_n}\, q^{\frac{1}{2}n(1-\nu)} \overline{U_{\bar\mu\bar\lambda} e_n(w)}\, U_{\lambda\mu}\Delta_{\lambda\mu} e_n(z), $$

for $|w|,\ |z| < q^{\frac{\nu}{4}-\frac{1}{4}}$. Indeed, the right-hand side is given by

$$ \begin{aligned} &\sum_{n=0}^{\infty} \frac{(q^\nu)_n}{(q)_n}\, q^{\frac{1}{2}n(1-\nu)} \frac{(q\lambda\mu)_\infty (q\lambda\mu)_n}{(q^{\nu+n}\lambda\mu)_\infty}(1-q^{\nu+2n}\lambda\mu) \\ &\qquad\qquad \times \frac{\bar w^n (q^{\frac{1}{4}+\frac{\nu}{4}}\lambda/\bar w)_n (q^{\frac{3}{4}+\frac{3\nu}{4}+n}\mu\bar w)_\infty}{(q\lambda\mu)_n (q^{\frac{3}{4}-\frac{\nu}{4}}\mu\bar w)_\infty} \frac{z^n (q^{\frac{1}{4}+\frac{\nu}{4}}\mu/z)_n (q^{\frac{3}{4}+\frac{3\nu}{4}+n}\lambda z)_\infty}{(q\lambda\mu)_n (q^{\frac{3}{4}-\frac{\nu}{4}}\lambda z)_\infty} \\ &\qquad = \frac{(q^{\frac{3}{4}+\frac{3\nu}{4}}\mu\bar w)_\infty (q^{\frac{3}{4}+\frac{3\nu}{4}}\lambda z)_\infty (q\lambda\mu)_\infty}{(q^{\frac{3}{4}-\frac{\nu}{4}}\mu\bar w)_\infty (q^{\frac{3}{4}-\frac{\nu}{4}}\lambda z)_\infty (q^{\nu+1}\lambda\mu)_\infty} \\ &\qquad\quad \times {}_6W_5(q^\nu\lambda\mu; q^{\frac{1}{4}+\frac{\nu}{4}}\lambda/\bar w, q^{\frac{1}{4}+\frac{\nu}{4}}\mu/z, q^\nu; q, q^{\frac{1}{2}-\frac{\nu}{2}} z\bar w) \\ &\qquad = \frac{(q^{\frac{1}{2}+\frac{\nu}{2}} z\bar w)_\infty}{(q^{\frac{1}{2}-\frac{\nu}{2}} z\bar w)_\infty} = k_w(z) \end{aligned} $$

by (2.1). In particular, for $|\lambda| < q^{-\frac{1}{2}}$ and for fixed $|w| < q^{\frac{\nu}{4}-\frac{1}{4}}$, each term in the sum is in $\mathcal{A}_q^\nu$ as a function of z, and the sum converges uniformly on a disc of radius greater than $q^{\frac{\nu}{4}-\frac{1}{4}}$. This implies convergence in $\mathcal{A}_q^\nu$. Since the span of all reproducing kernels is dense, it follows that the image of $U_{\lambda\mu}$ is dense in $\mathcal{A}_q^\nu$. On the other hand, it follows from (4.5) that for $\max(|\lambda|, |\mu|) < q^{-\frac{1}{2}}$ the range of $U_{\lambda\mu}$ is contained in the domain of $U^*_{\bar\mu\bar\lambda}$, and that

$$ U^*_{\bar\mu\bar\lambda} U_{\lambda\mu}\Delta_{\lambda\mu} e_n = e_n. $$

Thus, under this condition, $U^*_{\bar\mu\bar\lambda}$ and $U_{\lambda\mu}\Delta_{\lambda\mu}$ are both densely defined and mutually inverse. This gives an interpretation of (3.4) at the level of operators. In particular, Proposition 3.1 follows.

4.2. **The connection with twisted primitive elements.** Recall that (4.6) gives the discrete orthogonality of Al-Salam–Chihara polynomials, connected with the operator $\tilde Y_t$ defined by (2.9). The reason is that

$$ U_{\lambda\lambda} e_n = (-1)^n q^{\binom{n}{2}+\frac{1}{4}n(\nu+1)} \lambda^n \frac{1}{(q\lambda^2)_n}\, \tilde v_{xt}, \tag{4.7} $$

where

$$x = q^{\frac{\nu}{2}+n}\lambda + q^{-\frac{\nu}{2}-n}\lambda^{-1}, \qquad t = \lambda + \lambda^{-1}.$$

(More generally, in view of (3.1), $U_{\lambda\mu}e_n$ may be expressed as a $\delta_c\tilde{v}_{xt}$ if $\lambda\mu \neq 0$.) Thus $U_{\lambda\lambda}e_n$ is an eigenvector of $\tilde{Y}_{\lambda+\lambda^{-1}}$. The eigenvalue equation (2.10) gives

$$\tilde{Y}_{\lambda+\lambda^{-1}}\, U_{\lambda\lambda}e_n = U_{\lambda\lambda}\, \frac{\lambda(K^{-2}-1)+\lambda^{-1}(K^2-1)}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\, e_n,$$

so that formally

$$U_{\lambda\lambda}^{-1}\, \tilde{Y}_{\lambda+\lambda^{-1}}\, U_{\lambda\lambda} = \frac{\lambda(K^{-2}-1)+\lambda^{-1}(K^2-1)}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}.$$

Since this identity is independent of ν it makes sense at the level of formal power series, and it will be proved in that context in Section 7. Thus conjugation with $U_{\lambda\lambda}$ transforms the twisted primitive elements $\tilde{Y}$ into group-like elements. More generally, we will prove that

$$U_{\lambda\mu}^{-1}\left(\lambda q^{-\frac{1}{4}}X_- - \mu q^{\frac{1}{4}}X_+ + (\lambda\mu+1)\frac{K-K^{-1}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\right) K\, U_{\lambda\mu} = \frac{\lambda\mu(K^{-2}-1)+(K^2-1)}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}. \tag{4.8}$$

Though we will not use this identity, it illuminates our approach, and also gives the connection with the work of Koornwinder and others mentioned in the introduction.

Remark 4.1. We have used the operators $U_{\lambda\mu}$ to find the spectrum of $\tilde{Y}_t$. Since the operator Y_t has a partly continuous spectrum, we cannot expect to find it in a similar way. However, formally replacing q by q^{-1} in $U_{\lambda\mu}$ gives the operator

$$\tilde{U}_{\lambda\mu}e_n(z) = \frac{z^n(q^{\frac{3}{4}-\frac{\nu}{4}-n}\mu/z)_n(q^{\frac{1}{4}+\frac{\nu}{4}}\lambda z)_\infty}{(q^{-n}\lambda\mu)_n(q^{\frac{1}{4}-\frac{3\nu}{4}-n}\lambda z)_\infty}. \tag{4.9}$$

The right-hand side belongs to $\mathcal{A}_q^\nu$ for $|\lambda| < q^{\frac{\nu}{2}+n}$. If $\lambda = \mu \neq 0$, it is a multiple of a v_{xt}, where x is a discrete mass-point of dm_t, and conversely every such v_{xt} may be obtained in this way. Similar formal arguments as for $U_{\lambda\mu}$ suggest the equality

$$\langle \tilde{U}_{\lambda\mu}\tilde{\Delta}_{\lambda\mu}e_k, \tilde{U}_{\bar{\mu}\bar{\lambda}}e_l\rangle = \langle e_k, e_l\rangle,$$

where

$$\tilde{\Delta}_{\lambda\mu}e_n = \frac{(q^{1-\nu-n}\lambda\mu)_\infty(q^{-n}\lambda\mu)_n}{(\lambda\mu)_\infty}(1-q^{-\nu-2n}\lambda\mu).$$

This gives the correct expression for the weight of these point-masses. Thus one obtains precisely the discrete part of the spectrum of Y_t in this way. Note also the equality

$$\tilde{U}_{\lambda\mu}e_n = \frac{(q^{1-\nu-2n}\lambda\mu)_n}{(q^{-n}\lambda\mu)_n}\, U_{\lambda q^{-\frac{1}{2}-\frac{\nu}{2}-n},\, \mu q^{\frac{1}{2}-\frac{\nu}{2}-n}}e_n, \tag{4.10}$$

which is essentially a special case of (2.11).

4.3. **Discrete/continuous matrix elements: Askey–Wilson polynomials.** We will now consider matrix elements of the form $\langle U_{\lambda\mu}e_k, v_{xt}\rangle$.

Proposition 4.2. *For* $|\lambda| < q^{-\frac{1}{2}}$ *and* $x \in \mathrm{supp}(dm_t)$*, the matrix element* $\langle U_{\lambda\mu}e_k, v_{xt}\rangle$ *is given by the Askey–Wilson polynomial*

$$\langle U_{\lambda\mu}e_k, v_{xt}\rangle = q^{\frac{1}{4}k(\nu-1)} \frac{(q^{\frac{\nu}{2}+\frac{1}{2}+k}\lambda\tau^{\pm})_\infty}{(q\lambda\mu)_k(q^\nu)_k(q^{\frac{1}{2}}\lambda\xi^{\pm})_\infty}\, p_k(\tfrac{x}{2}; q^{\frac{\nu}{2}}\tau, q^{\frac{\nu}{2}}\tau^{-1}, q^{\frac{1}{2}}\lambda, q^{\frac{1}{2}}\mu|q).$$

Proof. Insert expressions (2.7) and (4.1) in the discrete Plancherel formula (2.6). This gives

$$\begin{aligned}\langle U_{\lambda\mu}e_k, v_{xt}\rangle &= \lambda^{-k}q^{\frac{1}{4}k(\nu-3)}\frac{1}{(q^\nu)_k}\sum_{n=0}^{\infty}\frac{(q^{\frac{\nu}{2}}\xi\tau^{-1})_n}{(q)_n}(q^{\frac{1}{2}}\lambda\xi^{-1})^n \\ &\times\ {}_2\phi_1\left[\begin{matrix}q^{-n},\ q^{\frac{\nu}{2}}\tau\xi^{-1}\\ q^{1-n-\frac{\nu}{2}}\tau\xi^{-1}\end{matrix};q,q^{1-\frac{\nu}{2}}\tau\xi\right]\ {}_2\phi_1\left[\begin{matrix}q^{-k}, q^{\nu+k}\lambda\mu\\ q\lambda\mu\end{matrix};q,q^{1+n}\right].\end{aligned}$$

Expanding the series and changing the order of summation gives

$$\begin{aligned}\langle U_{\lambda\mu}e_k, v_{xt}\rangle &= \lambda^{-k}q^{\frac{1}{4}k(\nu-3)}\frac{1}{(q^\nu)_k}\sum_{i=0}^{k}\frac{(q^{-k})_i(q^{\nu+k}\lambda\mu)_i}{(q)_i(q\lambda\mu)_i}\,q^i\sum_{j=0}^{\infty}\frac{(q^{\frac{\nu}{2}}\tau\xi^{-1})_j}{(q)_j}(q^{1-\frac{\nu}{2}}\tau\xi)^j \\ &\times\sum_{n=j}^{\infty}\frac{(q^{\frac{\nu}{2}}\xi\tau^{-1})_n(q^{-n})_j}{(q)_n(q^{1-n-\frac{\nu}{2}}\tau\xi^{-1})_j}(q^{\frac{1}{2}+i}\lambda\xi^{-1})^n.\end{aligned}$$

Replacing n by $n+j$ in the inner sum gives

$$\begin{aligned}\langle U_{\lambda\mu}e_k, v_{xt}\rangle &= \lambda^{-k}q^{\frac{1}{4}k(\nu-3)}\frac{1}{(q^\nu)_k}\sum_{i=0}^{k}\frac{(q^{-k})_i(q^{\nu+k}\lambda\mu)_i}{(q)_i(q\lambda\mu)_i}\,q^i \\ &\times\ {}_1\phi_0\left[\begin{matrix}q^{\frac{\nu}{2}}\tau\xi^{-1}\\ -\end{matrix};q,q^{\frac{1}{2}+i}\lambda\xi\right]\ {}_1\phi_0\left[\begin{matrix}q^{\frac{\nu}{2}}\xi\tau^{-1}\\ -\end{matrix};q,q^{\frac{1}{2}+i}\lambda\xi^{-1}\right].\end{aligned}$$

If x is in the continuous part of $\mathrm{supp}(dm_t)$, then both the inner series converge and are summable by the q-binomial theorem. If x is in the discrete part, then one can choose ξ so that one of the series terminates and the other converges. In both cases it follows that

$$\langle U_{\lambda\mu}e_k, v_{xt}\rangle = \lambda^{-k}q^{(\frac{\nu}{4}-\frac{3}{4})k}\frac{(q^{\frac{1}{2}+\frac{\nu}{2}}\lambda\tau^{\pm})_\infty}{(q^\nu)_k(q^{\frac{1}{2}}\lambda\xi^{\pm})_\infty}\,{}_4\phi_3\left[\begin{matrix}q^{-n},\ q^{\nu+k}\lambda\mu,\ q^{\frac{1}{2}}\lambda\xi,\ q^{\frac{1}{2}}\lambda\xi^{-1}\\ q\lambda\mu,\ q^{\frac{1}{2}+\frac{\nu}{2}}\lambda\tau,\ q^{\frac{1}{2}+\frac{\nu}{2}}\lambda\tau^{-1}\end{matrix};q,q\right],$$

which proves the proposition. □

The relation

$$\langle e_k, e_l\rangle = \int \langle U_{\lambda\mu}\Delta_{\lambda\mu}e_k, v_{xt}\rangle\langle v_{xt}, U_{\bar\mu\bar\lambda}e_l\rangle\, dm_t(x) \tag{4.11}$$

holds for $|\lambda|,\ |\mu| < q^{-\frac{1}{2}}$. When λ and μ are real or complex conjugates this is the orthogonality relation for Askey–Wilson polynomials.

4.4. **Continuous matrix elements: ${}_8W_7$ series.** We will now prove that the matrix element $\langle \delta_c v_{x_1t_1}, v_{x_2t_2}\rangle$ is given by an ${}_8W_7$ series. As was observed above, expanding in the discrete basis e_k leads to the Poisson kernel (2.8) for Al-Salam–Chihara polynomials. However, one may more generally write

$$\langle \delta_c v_{x_1t_1}, v_{x_2t_2}\rangle = \sum_{n=0}^{\infty} \frac{(q^\nu)_n}{(q)_n}\, q^{\frac12 n(1-\nu)} \langle \delta_{c_1} v_{x_1t_1}, U_{\bar\mu\bar\lambda} e_n\rangle \langle \delta_{c_2} U_{\lambda\mu}\Delta_{\lambda\mu} e_n, v_{x_2t_2}\rangle, \tag{4.12}$$

where $c = c_1c_2$, $\max(|c_1|,|c_2|) < 1$ and $\max(|\lambda|,|\mu|) < q^{-\frac12}$. Now choose

$$\lambda = q^{-\frac12} c_1\xi_1, \qquad \mu = q^{-\frac12} c_2\xi_2,$$

where

$$x_1 = \xi_1 + \xi_1^{-1}, \quad |\xi_1| \le 1$$

and similarly for ξ_2. In view of (3.1) and Proposition 4.2, this leads to Askey–Wilson polynomials of the form

$$p_n(\tfrac12(\xi+\xi^{-1}); \xi, b, c, d|q) = (\xi b)_n (\xi c)_n (\xi d)_n \xi^{-n},$$

where the last equality follows from (2.2). Thus we get a single sum, which may be written as

$$\begin{aligned}\langle \delta_c v_{x_1t_1}, v_{x_2t_2}\rangle &= \frac{(q^{\frac{\nu}{2}} c\xi_2\tau_1^{\pm})_\infty (q^{\frac{\nu}{2}} c\xi_1\tau_2^{\pm})_\infty}{(q^\nu c\xi_1\xi_2)_\infty (c\xi_1\xi_2)_\infty (c\xi_1\xi_2^{-1})_\infty (c\xi_1^{-1}\xi_2)_\infty} \\ &\quad \times {}_8W_7(q^{\nu-1}c\xi_1\xi_2; c\xi_1\xi_2, q^{\frac{\nu}{2}}\xi_1\tau_1^{\pm}, q^{\frac{\nu}{2}}\xi_2\tau_2^{\pm}; q, c/\xi_1\xi_2).\end{aligned}$$

Note that if x_1 and x_2 are both in the continuous part of the spectrum, then ξ_1 and ξ_2 are unimodular and the series converges; if one of them is in the discrete part the series terminates. By the transformation formulas for ${}_8W_7$ series [GR], one also has that

$$\begin{aligned}\langle \delta_c v_{x_1t_1}, v_{x_2t_2}\rangle &= \frac{(c/\tau_1\tau_2)_\infty (q^{\frac{\nu}{2}} c\tau_2\xi_1^{\pm})_\infty (q^{\frac{\nu}{2}} c\tau_1\xi_2^{\pm})_\infty}{(q^\nu c\tau_1\tau_2)_\infty (c\xi_1^{\pm}\xi_2^{\pm})_\infty} \\ &\quad \times {}_8W_7(q^{\nu-1}c\tau_1\tau_2; c\tau_1\tau_2, q^{\frac{\nu}{2}}\tau_1\xi_1^{\pm}, q^{\frac{\nu}{2}}\tau_2\xi_2^{\pm}; q, c/\tau_1\tau_2),\end{aligned}$$

provided that $|c| < |\tau_1\tau_2|$ or that this series terminates.

4.5. **More discrete matrix elements.** Let us consider matrix elements of the form $\langle U_{\lambda_1\mu_1} e_k, U_{\bar\lambda_2\bar\mu_2} e_l\rangle$.

Proposition 4.3. *For* $|q\lambda_1\lambda_2| < 1$,

$$\begin{aligned}\langle U_{\lambda_1\mu_1} e_k, U_{\bar\lambda_2\bar\mu_2} e_l\rangle &= \lambda_1^{l-k} q^{\frac14 k(\nu-3)+\frac14 l(\nu+1)} \frac{(\mu_2/\lambda_1)_l (q^{1+\nu+l}\lambda_1\lambda_2)_\infty}{(q\lambda_2\mu_2)_l (q^\nu)_k (q\lambda_1\lambda_2)_\infty} \\ &\quad \times {}_4\phi_3\left[\begin{matrix} q^{-k}, q^{\nu+k}\lambda_1\mu_1, q\lambda_1\lambda_2, q\lambda_1/\mu_2 \\ q\lambda_1\mu_1, q^{1+\nu+l}\lambda_1\lambda_2, q^{1-l}\lambda_1/\mu_2 \end{matrix}; q, q\right].\end{aligned}$$

This is as a degenerate case of Proposition 4.2, since it follows from (3.1), (4.7), and (2.11) that $U_{\lambda\mu} e_n$ may be expressed as a $\delta_c v_{xt}$ with $\tau = q^{\frac{\nu}{2}+n}\xi$. It may be proved similarly by inserting (4.1) in (2.6); this requires knowledge of the q-binomial and q-Saalschütz formulas [GR]. Another proof is indicated in Section 7.

The equality

$$\langle e_k, e_l\rangle = \sum_{n=0}^{\infty} \frac{(q^\nu)_n}{(q)_n}\, q^{\frac{1}{2}n(1-\nu)} \langle U_{\lambda_1\mu_1}\Delta_{\lambda_1\mu_1} e_k, U_{\bar\mu_2\bar\lambda_2} e_n\rangle \langle U_{\lambda_2\mu_2}\Delta_{\lambda_2\mu_2} e_n, U_{\bar\mu_1\bar\lambda_1} e_l\rangle \tag{4.13}$$

generalizes both (4.5) and (4.6). It holds for $|q\lambda_1\mu_1| < 1$ and generic λ_2 and μ_2. Since this is not quite clear from the above, let us give a direct proof.

After applying the symmetry

$$\langle U_{\lambda_1\mu_1} e_k, U_{\bar\lambda_2\bar\mu_2} e_l\rangle = \langle U_{\lambda_2\mu_2} e_l, U_{\bar\lambda_1\bar\mu_1} e_k\rangle$$

to the second factor, insert the expression given in Proposition 4.3. The right-hand side of (4.13) is then

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(q^\nu)_n}{(q)_n}\, q^{\frac{1}{2}n(1-\nu)} \frac{(q\lambda_1\mu_1)_\infty (q\lambda_1\mu_1)_k}{(q^{\nu+k}\lambda_1\mu_1)_\infty}(1-q^{\nu+2k}\lambda_1\mu_1)\\
&\times \frac{(q\lambda_2\mu_2)_\infty (q\lambda_2\mu_2)_n}{(q^{\nu+n}\lambda_2\mu_2)_\infty}(1-q^{\nu+2n}\lambda_2\mu_2)\\
&\times \lambda_1^{n-k} q^{\frac{1}{4}k(\nu-3)+\frac{1}{4}n(\nu+1)} \frac{(\lambda_2/\lambda_1)_n (q^{1+\nu+n}\lambda_1\mu_2)_\infty}{(q\lambda_2\mu_2)_n (q^\nu)_k (q\lambda_1\mu_2)_\infty}\\
&\times {}_4\phi_3\left[\begin{matrix} q^{-k}, q^{\nu+k}\lambda_1\mu_1, q\lambda_1\mu_2, q\lambda_1/\lambda_2 \\ q\lambda_1\mu_1, q^{1+\nu+n}\lambda_1\mu_2, q^{1-n}\lambda_1/\lambda_2\end{matrix}; q, q\right]\\
&\times \mu_1^{n-l} q^{\frac{1}{4}l(\nu-3)+\frac{1}{4}n(\nu+1)} \frac{(\mu_2/\mu_1)_n (q^{1+\nu+n}\mu_1\lambda_2)_\infty}{(q\lambda_2\mu_2)_n (q^\nu)_l (q\mu_1\lambda_2)_\infty}\\
&\times {}_4\phi_3\left[\begin{matrix} q^{-l}, q^{\nu+l}\lambda_1\mu_1, q\mu_1\lambda_2, q\mu_1/\mu_2 \\ q\lambda_1\mu_1, q^{1+\nu+n}\mu_1\lambda_2, q^{1-n}\mu_1/\mu_2\end{matrix}; q, q\right].
\end{aligned}$$

Expanding the ${}_4\phi_3$ series and changing the order of summation, this may be written as

$$\begin{aligned}
&\lambda_1^{-k}\mu_1^{-l}q^{\frac14(\nu-3)(k+l)}\frac{(q\lambda_1\mu_1)_\infty(q\lambda_1\mu_1)_k(q\lambda_2\mu_2)_\infty(q^{\nu+1}\lambda_1\mu_2)_\infty(q^{\nu+1}\mu_1\lambda_2)_\infty}{(q^\nu)_k(q^\nu)_l(q^{\nu+k}\lambda_1\mu_1)_\infty(q^{\nu+1}\lambda_2\mu_2)_\infty(q\lambda_1\mu_2)_\infty(q\mu_1\lambda_2)_\infty}\\
&\qquad\times(1-q^{\nu+2k}\lambda_1\mu_1)\sum_{i=0}^k\frac{(q^{-k})_i(q^{\nu+k}\lambda_1\mu_1)_i(q\lambda_1\mu_2)_i}{(q)_i(q\lambda_1\mu_1)_i(q^{1+\nu}\lambda_1\mu_2)_i}\,q^i\\
&\qquad\times\sum_{j=0}^l\frac{(q^{-l})_j(q^{\nu+l}\lambda_1\mu_1)_j(q\mu_1\lambda_2)_j}{(q)_j(q\lambda_1\mu_1)_j(q^{1+\nu}\mu_1\lambda_2)_j}\,q^j\\
&\qquad\times\,{}_6W_5(q^\nu\lambda_2\mu_2;q^\nu,q^{-i}\lambda_2/\lambda_1,q^{-j}\mu_2/\mu_1;q,q^{1+i+j}\lambda_1\mu_1)\\
&\quad=\lambda_1^{-k}\mu_1^{-l}q^{\frac14(\nu-3)(k+l)}\frac{(q\lambda_1\mu_1)_\infty(q\lambda_1\mu_1)_k}{(q^\nu)_k(q^\nu)_l(q^{\nu+k}\lambda_1\mu_1)_\infty}(1-q^{\nu+2k}\lambda_1\mu_1)\\
&\qquad\times\sum_{i=0}^k\frac{(q^{-k})_i(q^{\nu+k}\lambda_1\mu_1)_i}{(q)_i(q\lambda_1\mu_1)_i}\,q^i\sum_{j=0}^l\frac{(q^{-l})_j(q^{\nu+l}\lambda_1\mu_1)_j}{(q)_j(q\lambda_1\mu_1)_j}\,q^j\frac{(q^{\nu+1+i+j}\lambda_1\mu_1)_\infty}{(q^{1+i+j}\lambda_1\mu_1)_\infty}\\
&\quad=\lambda_1^{-k}\mu_1^{-l}q^{\frac14(\nu-3)(k+l)}\frac{(q\lambda_1\mu_1)_k(q^{\nu+1}\lambda_1\mu_1)_\infty}{(q^\nu)_k(q^\nu)_l(q^{\nu+k}\lambda_1\mu_1)_\infty}(1-q^{\nu+2k}\lambda_1\mu_1)\\
&\qquad\times\sum_{i=0}^k\frac{(q^{-k})_i(q^{\nu+k}\lambda_1\mu_1)_i}{(q)_i(q^{\nu+1}\lambda_1\mu_1)_i}\,q^i\,{}_3\phi_2\left[\begin{matrix}q^{-l},q^{\nu+l}\lambda_1\mu_1,q^{1+i}\lambda_1\mu_1\\ q\lambda_1\mu_1,q^{1+i+\nu}\lambda_1\mu_1\end{matrix};q,q\right]\\
&\quad=\lambda_1^{l-k}q^{\frac14(\nu-3)(k+l)}\frac{(q\lambda_1\mu_1)_k(q^{\nu+l+1}\lambda_1\mu_1)_\infty}{(q^\nu)_k(q\lambda_1\mu_1)_l(q^{\nu+k}\lambda_1\mu_1)_\infty}(1-q^{\nu+2k}\lambda_1\mu_1)\\
&\qquad\times\sum_{i=0}^k\frac{(q^{-k})_i(q^{\nu+k}\lambda_1\mu_1)_i}{(q)_i(q^{\nu+l+1}\lambda_1\mu_1)_i}\,q^i(q^{-i})_l\,q^{l(i+1)},
\end{aligned}$$

where we used (2.1) and the q-Saalschütz formula. In the final sum, the terms with $i<l$ vanish. By symmetry, it vanishes unless $k=l$, when it reduces to the term with $i=k=l$. In that case the expression simplifies to

$$\frac{(q)_k}{(q^\nu)_k}\,q^{\frac12k(\nu-1)}=\|e_k\|^2.$$

In general the equality (4.13) is a biorthogonality relation. It gives an orthogonal system only if

$$\bar\lambda_1\mu_2=\bar\lambda_2\mu_1. \tag{4.14}$$

This is, however, not a polynomial system. If one formally replaces ν by a non-positive integer (which corresponds to the quantum algebra $\mathcal{U}_q(\mathfrak{su}(2))$; cf. Section 6), one recovers orthogonal functions discovered by Koelink [Ko]. To make this identification one may use Sears' transformation formula [GR] to write

$$\tag{4.15} \begin{aligned} \langle U_{\lambda_1\mu_1}e_k, U_{\bar\mu_2\bar\lambda_2}e_l\rangle &= \lambda_1^l\mu_2^k q^{\frac14(\nu+1)(k+l)+kl}\frac{(\mu_1/\mu_2)_k(q^{-k}\lambda_2/\lambda_1)_l(q^{1+\nu+k+l}\lambda_1\mu_2)_\infty}{(q\lambda_1\mu_1)_k(q\lambda_2\mu_2)_l(q\lambda_1\mu_2)_\infty} \\ &\quad\times {}_4\phi_3\left[\begin{matrix} q^{-k}, q^{-l}, q^{-k}/\lambda_1\mu_1, q^{\nu+l}\lambda_2\mu_2 \\ q^\nu, q^{1-k}\mu_2/\mu_1, q^{-k}\lambda_2/\lambda_1\end{matrix}; q, q\right] \end{aligned}$$

(note that (4.14) means that one denominator parameter equals q times the conjugate of another).

4.6. **Two more orthogonality relations.** So far, we have obtained two orthogonality relations for ${}_4\phi_3$ series, (4.11) and (4.13), the latter generalizing (4.5) and (4.6). Two more such relations may be found by incorporating the operator $\tilde U_{\lambda\mu}$ defined by (4.9), namely,

$$\tag{4.16} \langle e_k, e_l\rangle = \sum_{n=0}^\infty \frac{(q^\nu)_n}{(q)_n}\, q^{\frac12 n(1-\nu)}\langle \tilde U_{\lambda_1\mu_1}\tilde\Delta_{\lambda_1\mu_1}e_k, U_{\bar\mu_2\bar\lambda_2}e_n\rangle\langle U_{\lambda_2\mu_2}\Delta_{\lambda_2\mu_2}e_n, \tilde U_{\bar\mu_1\bar\lambda_1}e_l\rangle$$

and

$$\tag{4.17} \langle e_k, e_l\rangle = \int \langle \tilde U_{\lambda\mu}\tilde\Delta_{\lambda\mu}e_k, v_{xt}\rangle\langle v_{xt}, \tilde U_{\bar\mu\bar\lambda}e_l\rangle\, dm_t(x).$$

Expressions for the matrix elements as ${}_4\phi_3$-series follow from Propositions 4.2 and 4.3 using (4.10).

It turns out that (4.16) leads to a polynomial orthogonality relation. Formally replacing ν by a non-positive integer in this equality gives the orthogonality of q-Racah polynomials; cf. Section 6. The direct proof given in [AW1] generalizes to show that (4.16) holds for

$$|\lambda_1\mu_1| < q^{\nu+k+l}.$$

Thus for $\lambda_1\mu_1 \neq 0$ we have only a finite family of orthogonal polynomials. However, if $\lambda_1 = 0$ or $\mu_1 = 0$ we obtain an orthogonality relation for continuous dual q^{-1}-Hahn polynomials on an infinite discrete set. There are suggestions about such an orthogonality relation in the literature [AA2, AW2], but we have not found it written down explicitly.

The equation (4.17) may be written in terms of Askey–Wilson polynomials as

$$\begin{aligned} \int \frac{1}{(\lambda q^{-\frac{\nu}{2}-k}\xi^\pm)_\infty(\mu q^{-\frac{\nu}{2}-l}\xi^\pm)_\infty}\, & p_k(\tfrac{x}{2}; q^{\frac{\nu}{2}}\tau, q^{\frac{\nu}{2}}\tau^{-1}, q^{-\frac{\nu}{2}-k}\lambda, q^{1-\frac{\nu}{2}-k}\mu; q) \\ &\times p_l(\tfrac{x}{2}; q^{\frac{\nu}{2}}\tau, q^{\frac{\nu}{2}}\tau^{-1}, q^{-\frac{\nu}{2}-l}\mu, q^{1-\frac{\nu}{2}-l}\lambda; q)\, dm_t(x) \\ &= \frac{(q)_k(q^\nu)_k(q^{-k}\lambda\mu)_\infty}{(q^{1-\nu-k}\lambda\mu)_\infty(\lambda\tau^\pm)_\infty(\mu\tau^\pm)_\infty(1-q^{-\nu-2k}\lambda\mu)}\,\delta_{kl}, \end{aligned}$$

valid for $|\lambda| < q^{\frac{\nu}{2}+k}$, $|\mu| < q^{\frac{\nu}{2}+l}$. For $\lambda = \bar\mu$, this is an orthogonality relation for a finite family of ${}_4\phi_3$ functions.

5. Addition formulas

The interpretation of ${}_8W_7$ series as matrix elements yields two addition formulas. The first one is

$$\langle \delta_{c_1c_2} v_{x_1t_1}, v_{x_2t_2}\rangle = \int \langle \delta_{c_1} v_{x_1t_1}, v_{x_3t_3}\rangle \langle \delta_{c_2} v_{x_3t_3}, v_{x_2t_2}\rangle \, dm_{t_3}(x_3), \tag{5.1}$$

which is in [ARS]. The second one is equation (4.12).

The matrix elements $\langle U_{\lambda_1\mu_1}e_k, U_{\bar{\lambda}_2\bar{\mu}_2}e_l\rangle$ and $\langle U_{\lambda\mu}e_k, v_{xt}\rangle$ also satisfy discrete and continuous addition formulas. These may formally be obtained from (4.12) and (5.1) as the cases when $\tau_1 = q^{\frac{\nu}{2}+k}\xi_1$ and/or $\tau_2 = q^{\frac{\nu}{2}+l}\xi_2$. Then the left-hand side may be expressed as a terminating ${}_8W_7$ series, and thus in view of Watson's transformation formula [GR] as a ${}_4\phi_3$. Further specialization of the parameters yields the orthogonality relations (4.11), (4.13), (4.16), and (4.17) of the previous section.

Writing (4.12) in terms of Askey–Wilson polynomials and changing the names of the variables gives the identities

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(ABq^{-1})_n}{(q)_n(A)_n(B)_n(ad)_n(bd)_n(\alpha\delta)_n(\beta\delta)_n} \frac{1-ABq^{-1+2n}}{1-ABq^{-1}} t^n \\
&\quad \times p_n(\cos\theta; a,b,c,d|q)\, p_n(\cos\psi; \alpha,\beta,\gamma,\delta|q) \\
&= \frac{(AB)_\infty (de^{\pm i\theta})_\infty (\delta e^{\pm i\psi})_\infty (\alpha te^{i\theta})_\infty (\beta te^{i\theta})_\infty (ate^{i\psi})_\infty (bte^{i\psi})_\infty}{(B)_\infty (ad)_\infty (bd)_\infty (\alpha\delta)_\infty (\beta\delta)_\infty (Ate^{i\theta}e^{i\psi})_\infty (te^{i\theta}e^{i\psi})_\infty (te^{i\theta}e^{-i\psi})_\infty (te^{-i\theta}e^{i\psi})_\infty} \\
&\quad \times {}_8W_7(q^{-1}Ate^{i\theta}e^{i\psi}; te^{i\theta}e^{i\psi}, ae^{i\theta}, be^{i\theta}, \alpha e^{i\psi}, \beta e^{i\psi}; q, te^{-i\theta}e^{-i\psi}) \\
&= \frac{(AB)_\infty (tA/a\alpha)_\infty (t\alpha e^{\pm i\theta})_\infty (tae^{\pm i\psi})_\infty (de^{\pm i\theta})_\infty (\delta e^{\pm i\psi})_\infty}{(B)_\infty (ad)_\infty (bd)_\infty (\alpha\delta)_\infty (\beta\delta)_\infty (ta\alpha)_\infty (te^{\pm i\theta}e^{\pm i\psi})_\infty} \\
&\quad \times {}_8W_7(q^{-1}ta\alpha; ta\alpha/A, ae^{\pm i\theta}, \alpha e^{\pm i\psi}; q, tA/a\alpha),
\end{aligned} \tag{5.2}$$

where

$$ab = \alpha\beta = A, \qquad cd = \gamma\delta = B \tag{5.3}$$

and

$$t = \frac{B}{c\gamma} = \frac{d}{\gamma} = \frac{\delta}{c}. \tag{5.4}$$

The same kernel, with the condition (5.3) but in general without (5.4), is studied thoroughly in [ARS]. The explicit expression (5.2) is obtained in a special case when the ${}_8W_7$ series reduces to a summable ${}_6W_5$. In fact, the general case may be proved by similar methods. The authors of [ARS] express the kernel as a double Jackson q-integral, the integrand being a ${}_6W_5$ series. In the case (5.4) (which corresponds to $t = 1$ in the notation of [ARS]) this series is summable. The q-integration may then be performed using formulas (2.10.18) and (2.10.19) of [GR]. This direct proof shows that the first expression in (5.2) is valid for

$$|te^{\pm i\theta}e^{\pm i\psi}| < 1$$

and otherwise generic values of the parameters. The second expression is valid when in addition

$$|tA| < |a\alpha|.$$

As was remarked above, the two expressions are also valid in certain cases when the series terminate.

A formula even more general than (5.2) is proved in [KV2], using a different quantum algebraic interpretation of Askey–Wilson polynomials (as Clebsch–Gordan coefficients).

6. The compact case

The quantum algebra $\mathcal{U}_q(\mathfrak{su}(2))$ is defined by the same relations as $\mathcal{U}_q(\mathfrak{su}(1,1))$, though with a different involution. It has a series of finite-dimensional unitary representations, which have natural realizations on spaces of polynomials. Our approach may be adapted to treat this algebra, the main difference being that in this case all operators are trivially bounded and have finite spectrum; in particular the technical difficulties discussed at the end of the introduction do not arise. Formally, this case corresponds to replacing the parameter ν by $-N$, where N is a non-negative integer. Then the infinite sums connected with expansions in eigenfunctions of $\tilde{Y}_t$ will terminate. Expansions involving the compact analogue of Y_t follow from these by formally replacing q by q^{-1}.

For instance, replacing ν by $-N$ in (2.12) gives the orthogonality relation for quantum q-Krawtchouk polynomials. Similarly (4.5) gives the orthogonality of q^{-1}-Krawtchouk polynomials and (4.6) the orthogonality of dual q^{-1}-Krawtchouk polynomials. As was remarked above, (4.13) gives a family of orthogonal functions discovered by Koelink [Ko] and (4.16) the orthogonality of q-Racah polynomials.

Koelink's function is a Poisson kernel for dual q-Krawtchouk polynomials. Similarly, by expanding in the discrete basis (e_k), one obtains an expression for q-Racah polynomials as a kernel involving products of dual q- and dual q^{-1}-Krawtchouk polynomials. Explicitly, in the notation of [KS],

$$\sum_{n=0}^{N} \frac{(q^{-N})_n}{(q)_n} t^n K_n(q^{-k}+aq^{k-N};a,N|q) K_n(q^l+bq^{N-l};b,N|q^{-1})$$
$$= \frac{(aq^{-N}t)_k(bq^{-l}t)_l(q^{-N}t)_N}{(q^{-k}t)_k(q^{-N}t)_l} {}_4\phi_3\left[\begin{matrix} q^{-k}, aq^{k-N}, q^{-l}, q^{l-N}/b \\ q^{-N}, aq^{-N}t, q/bt \end{matrix}; q, q\right],$$

where the ${}_4\phi_3$ may be expressed as a q-Racah polynomial in several ways, such as

$$R_l(q^{-k}+aq^{k-N}; q^{-N-1}, b^{-1}, b^{-1}t^{-1}, abq^{-N-1}t|q).$$

It is also interesting to rewrite this identity using [S]

$$K_n(q^l+bq^{N-l};b,N|q^{-1}) = (bq^{N-l})^l K_{N-n}(q^{-l}+b^{-1}q^{l-N};b^{-1},N|q).$$

7. Formal power series

In this section we will do some computations on formal power series over the quantum algebra. In particular we will prove equation (4.8), which, though not used explicitly, is in a sense the key identity of this paper.

The following lemma may be proved by induction or by generating functions (i.e. essentially by the master relation (2.13)).

Lemma 7.1. *In the algebra* $\mathcal{U}_q(\mathfrak{su}(1,1))$,

$$q^{kl}(X_-K^{-1})^k(X_+K^{-1})^l = \sum_{m=0}^{\min(k,l)} \frac{(q)_k(q)_l\, q^{\frac{1}{2}(m^2+m)}}{(q)_m(q)_{k-m}(q)_{l-m}(1-q)^{2m}} \times (X_+K^{-1})^{l-m}(X_-K^{-1})^{k-m}(q^{1-m+k-l}K^{-4})_m.$$

This may be used to prove that the two expressions of Proposition 3.3 are equal as formal power series. In fact, the first of these expressions takes the form

$$\sum_{k,l=0}^{\infty}\sum_{m=0}^{\min(k,l)} \frac{\lambda^k\mu^l q^{\frac{1}{4}k-\frac{1}{4}l+\binom{l}{2}+\frac{1}{2}(m^2+m)}}{(q)_{k-m}(q)_{l-m}(q)_m(q\lambda\mu)_l}(1-q)^{k+l-2m} \times (X_+K^{-1})^{l-m}(X_-K^{-1})^{k-m}(q^{1-m+k-l}K^{-4})_m.$$

Replacing k, l by $k+m$, $l+m$ gives

$$\sum_{k,l=0}^{\infty} \frac{\lambda^k\mu^l q^{\frac{1}{4}k-\frac{1}{4}l+\binom{l}{2}}}{(q)_k(q)_l(q\lambda\mu)_l}(1-q)^{k+l}(X_+K^{-1})^l(X_-K^{-1})^k\, {}_1\phi_1\left[\begin{matrix} q^{l-k}K^4 \\ q^{1+l}\lambda\mu\end{matrix}; q, q^{1+k}\lambda\mu K^{-4}\right].$$

By the ${}_1\phi_1$ summation formula [GR] (as a power series identity), this equals the second expression in Proposition 3.3.

Another interesting formal power series identity, which may be obtained in a similar way is

$$U_{\lambda_1\mu_1}^{-1}U_{\lambda_2\mu_2} = \frac{(q\lambda_1\mu_1)_\infty}{(q\lambda_2\mu_1)_\infty}\sum_{j,m=0}^{\infty} \frac{\lambda_1^j(q^{1+m-j}\lambda_2/\lambda_1)_j\, \mu_1^m(\mu_2/\mu_1)_m\, q^{\binom{j}{2}+\frac{1}{4}j-\frac{1}{4}m}}{(q)_j(q)_m(q\lambda_2\mu_2)_m} \times (-1)^{j+m}(1-q)^{j+m}\frac{1}{(q\lambda_1\mu_1K^{-4})_\infty}(X_-K^{-1})^j\frac{(q^{1+m+j}\mu_1\lambda_2K^{-4})_\infty}{(q^j\lambda_1\mu_1K^{-4})_j}(X_+K^{-1})^m.$$

Note that for $\lambda_1=\lambda_2$ and $\mu_1=\mu_2$, this identity reduces to $UU^{-1}=1$. We do not write out the proof. Note, however, that this may be used to give another derivation of Proposition 4.3. If one uses this formula to compute the matrix element

$$\langle U_{\lambda_1\mu_1}^{-1}U_{\lambda_2\mu_2}e_k, e_l\rangle,$$

which, in view of (4.4), is equivalent to computing

$$\langle U_{\lambda_2\mu_2}e_k, U_{\bar\mu_1\bar\lambda_1}e_l\rangle,$$

one obtains a sum over j, m satisfying $k-m+j=l$. This may be written as a single sum over $k-m=l-j$, which turns out to be a ${}_4\phi_3$-series. Actually this does not lead to the expression given in Proposition 4.3, but rather to (4.15).

Finally we prove that (4.8) holds at the level of formal power series. We use the factorization $U_{\lambda\mu}=T_\mu AD$ given in (3.2). Since the operator D commutes with the right-hand side of (4.8), and since one may subtract the constant term $-(\lambda\mu+1)/(q^{\frac{1}{2}}-q^{-\frac{1}{2}})$ from both sides, it suffices to prove that

$$T_\mu^{-1}\left(\lambda q^{-\frac{1}{4}}X_-K - \mu q^{\frac{1}{4}}X_+K + \frac{\lambda\mu+1}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}K^2\right)T_\mu = A\,\frac{\lambda\mu K^{-2}+K^2}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}A^{-1}. \tag{7.1}$$

To compute the left-hand side of (7.1) one may use the two identities

$$T_\mu^{-1}\left(q^{-\frac14}X_-K+\frac{\mu}{q^{\frac12}-q^{-\frac12}}\,K^2\right)T_\mu = q^{-\frac14}X_-K+\frac{\mu}{q^{\frac12}-q^{-\frac12}}\,K^{-2},$$
$$T_\mu^{-1}\left(-\mu q^{\frac14}X_+K+\frac{1}{q^{\frac12}-q^{-\frac12}}\,K^2\right)T_\mu = \frac{1}{q^{\frac12}-q^{-\frac12}}\,K^{2}.$$

Identities of this kind are discussed in [FV2, KMM, Z]. They may be obtained using a q-analogue of the Baker–Campbell–Hausdorff formula; writing $L_xy=xy$, $R_xy=yx$, the q-binomial formula (as a formal power series identity) gives

$$e_q(x)yE_q(-x)=e_q(L_x)E_q(-R_x)y=\sum_{n=0}^{\infty}\frac{1}{(q)_n}\prod_{i=0}^{n-1}(L_x-q^iR_x)y.$$

It follows that the left-hand side of (7.1) equals

$$q^{-\frac14}\lambda X_-K+\frac{\lambda\mu K^{-2}+K^2}{q^{\frac12}-q^{-\frac12}}.$$

The right-hand side of (7.1) may be written as

$$\frac{(\lambda\mu)^{\frac12}}{q^{\frac12}-q^{-\frac12}}\,A(q^{-\frac12}x^{\frac12}+q^{\frac12}x^{-\frac12})A^{-1},$$

where, as in (3.3), $x=q\lambda\mu K^{-4}$. Then (7.1) follows once we have proved the following lemma.

Lemma 7.2. *Let $x^{\frac12}$, $x^{-\frac12}$, and y be variables such that*

$$x^{\frac12}x^{-\frac12}=x^{-\frac12}x^{\frac12}=1,\qquad x^{\frac12}y=qyx^{\frac12},$$

and let

$$A=\sum_{k=0}^{\infty}\frac{1}{(q)_k}\,y^k\frac{1}{(x)_k}$$

(where $x=(x^{\frac12})^2$). Then the formal power series identity

$$A(q^{-\frac12}x^{\frac12}+q^{\frac12}x^{-\frac12})A^{-1}=q^{-\frac12}x^{\frac12}+q^{\frac12}x^{-\frac12}-q^{-\frac12}yx^{-\frac12}\tag{7.2}$$

holds.

Proof. By Lemma 3.4, the left-hand side of (7.2) is given by

$$\sum_{j,k=0}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}}{(q)_j(q)_k}\,y^j\frac{q^{-\frac12}x^{\frac12}+q^{\frac12}x^{-\frac12}}{(x)_j(q^{-k-1}x)_k}\,y^k=\sum_{j,k=0}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}}{(q)_j(q)_k}\,y^{j+k}\frac{q^{k-\frac12}x^{\frac12}+q^{\frac12-k}x^{-\frac12}}{(q^{2k}x)_j(q^{k-1}x)_k}.$$

Rewriting the sum using $l=j+k$ and k as summation variables gives

$$\sum_{l=0}^{\infty}\frac{1}{(q)_l}\,y^l\frac{1}{(q^{-1}x)_{l+1}}\sum_{k=0}^{l}\frac{(q^{-l})_k(q^{-1}x)_k(q^{k-\frac12}x^{\frac12}+q^{\frac12-k}x^{-\frac12})(1-q^{2k-1}x)}{(q)_k(q^lx)_k}\,q^{lk}.$$

Since

$$\begin{aligned}(q^{k-\frac12}x^{\frac12}+q^{\frac12-k}x^{-\frac12})(1-q^{2k-1}x)&=q^{\frac12-k}x^{-\frac12}(1+q^{2k-1}x)(1-q^{2k-1}x)\\&=q^{\frac12-k}x^{-\frac12}(1-q^{-2}x^2)\frac{(q^{\frac12}x^{\frac12})_k(iq^{\frac12}x^{\frac12})_k(-q^{\frac12}x^{\frac12})_k(-iq^{\frac12}x^{\frac12})_k}{(q^{-\frac12}x^{\frac12})_k(iq^{-\frac12}x^{\frac12})_k(-q^{-\frac12}x^{\frac12})_k(-iq^{-\frac12}x^{\frac12})_k},\end{aligned}$$

one may also write the sum as

$$\sum_{l=0}^{\infty}\frac{1}{(q)_l}\,y^l\frac{1}{(q^{-1}x)_{l+1}}\,q^{\frac12}x^{-\frac12}(1-q^{-2}x^2)\,{}_6W_5(q^{-1}x;q^{-l},iq^{\frac12}x^{\frac12},-iq^{\frac12}x^{\frac12};q,q^{l-1}).$$

The terminating version of (2.1) gives

$$\begin{aligned}{}_6W_5(q^{-1}x;q^{-l},iq^{\frac12}x^{\frac12},-iq^{\frac12}x^{\frac12};q,q^{l-1})&=\frac{(x)_l(q^{-1})_l}{(iq^{-\frac12}x^{\frac12})_l(-iq^{-\frac12}x^{\frac12})_l}\\&=\begin{cases}1 & l=0,\\(1-x)(1-q^{-1})/(1+q^{-1}x) & l=1,\\0 & l=2,3,\dots.\end{cases}\end{aligned}$$

This proves the lemma. □

References

[AA1] G. E. Andrews and R. Askey, *Enumeration of partitions: the role of Eulerian series and q-orthogonal polynomials*, in M. Aigner (ed.), *Higher Combinatorics*, pp. 3–26, Reidel, Boston, 1977.

[AA2] G. E. Andrews and R. Askey, *Classical orthogonal polynomials*, Lecture Notes Math. 1171, pp. 36–62, Springer-Verlag, 1985.

[AI] R. Askey and M. E. H. Ismail, *Recurrence relations, continued fractions and orthogonal polynomials*, Mem. Amer. Math. Soc. 300 (1984).

[ARS] R. Askey, M. Rahman, and S. K. Suslov, *On a general q-Fourier transformation with non-symmetric kernels*, J. Comput. Appl. Math. 68 (1996), 25–55.

[AW1] R. Askey and J. Wilson, *A set of orthogonal polynomials that generalize the Racah coefficients or* 6-j *symbols*, SIAM J. Math. Anal. 10 (1979), 1008–1016.

[AW2] R. Askey and J. A. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. 319 (1985).

[BW] D. Basu and K. B. Wolf, *The unitary irreducible representations of* $\mathrm{SL}(2,\mathbb{R})$ *in all subgroup reductions*, J. Math. Phys. 23 (1982), 189–205.

[CP] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, Cambridge 1994.

[D] M. S. Dijkhuizen, *Some remarks on the construction of quantum symmetric spaces*, Acta Appl. Math. 44 (1996), 59–80.

[DN] M. S. Dijkhuizen and M. Noumi, *A family of quantum projective spaces and related q-hypergeometric orthogonal polynomials*, Trans. Amer. Math. Soc. 350 (1998), 3269–3296.

[FV1] R. Floreanini and L. Vinet, *On the quantum group and quantum algebra approach to q-special functions*, Lett. Math. Phys. 27 (1993), 179–190.

[FV2] R. Floreanini and L. Vinet, $U_q(\mathrm{sl}(2))$ *and q-special functions*, Contemp. Math. 160 (1994), 85–100.

[GR] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.

[GZ] Y. I. Granovskii and A. S. Zhedanov, *'Twisted' Clebsch–Gordan coefficients for* $\mathrm{SU}_q(2)$, J. Phys. A 25 (1992), L1029–L1032.

[IS] M. E. H. Ismail and D. Stanton, *Classical orthogonal polynomials as moments*, Canad. J. Math. 49 (1997), 520–542.

[J] B. Jurčo, *On coherent states for the simplest quantum groups*, Lett. Math. Phys. 21 (1991), 51–58.

[KMM] E. G. Kalnins, W. Miller, Jr., and S. Mukherjee, *Models of q-algebra representations: matrix elements of* $U_q(\mathrm{su}_2)$, Contemp. Math. 160 (1994), 191–208.

[KR] A. N. Kirillov and N. Y. Reshetikhin, *Representations of the algebra* $U_q(\mathrm{sl}(2))$, *q-orthogonal polynomials and invariants of links*, in V. G. Kac (ed.), *Infinite-dimensional Lie Algebras and Groups*, pp. 285–339, World Scientific, 1989.

[KL] S. Klimek and A. Lesniewski, *A two-parameter quantum deformation of the unit disc*, J. Funct. Anal. 115 (1993), 1–23.

[KS] R. Koekoek and R. F. Swarttouw, *The Askey scheme of hypergeometric orthogonal polynomials and its q-analogue*, report 98-17, Delft University of Technology, 1998, available at `http://aw.twi.tudelft.nl/~koekoek/research.html`.

[Ko] H. T. Koelink, *Askey-Wilson polynomials and the quantum* SU(2) *group: survey and applications*, Acta Appl. Math. 44 (1996), 295–352.

[KV] H. T. Koelink and J. Van der Jeugt, *Convolutions for orthogonal polynomials from Lie and quantum algebra representations*, SIAM J. Math. Anal. 29 (1998), 794–822.

[KV2] H. T. Koelink and J. Van der Jeugt, *Bilinear generating functions for orthogonal polynomials*, Constr. Approx., to appear.

[K1] T. H. Koornwinder, *Group theoretic interpretation of Askey's scheme of hypergeometric orthogonal polynomials*, Lecture Notes Math. 1329, pp. 46–72, Springer-Verlag, 1988.

[K2] T. H. Koornwinder, *Askey–Wilson polynomials as zonal spherical functions on the* SU(2) *quantum group*, SIAM J. Math. Anal. 24 (1993), 795–813.

[K3] T. H. Koornwinder, *Special functions and q-commuting variables*, in *Special Functions, q-series and Related Topics*, pp. 131–166, Fields Inst. Commun. 14, Amer. Math. Soc., Providence, 1997.

[Kor] L. I. Korogodsky, *Quantum G-spaces and q-oscillators*, preprint, Institute for Theoretical Physics, Kiev, 1991.

[NM1] M. Noumi and K. Mimachi, *Askey–Wilson polynomials and the quantum group* $\mathrm{SU}_q(2)$, Proc. Japan Acad. Ser. A Math. Sci. 66 (1990), 146-149.

[NM2] M. Noumi and K. Mimachi, *Askey–Wilson polynomials as spherical functions on* $\mathrm{SU}_q(2)$, Lecture Notes Math. 1510, pp. 98–103, Springer-Verlag, 1992.

[S] D. Stanton, *Some q-Krawtchouk polynomials on Chevalley groups*, Amer. J. Math. 102 (1980), 625–662.

[VJ] J. Van der Jeugt and R. Jagannathan, *Realizations of* $\mathrm{su}(1,1)$ *and* $U_q(\mathrm{su}(1,1))$ *and generating functions for orthogonal polynomials*, J. Math. Phys., to appear.

[VK] N. Ya. Vilenkin and A. U. Klimyk, *Representation of Lie Groups and Special Functions*, Vol. 1, Dordrecht, Kluwer Academic Publishers, 1991.

[W] S. L. Woronowicz, *Operator equalities related to quantum* $E(2)$ *group*, Comm. Math. Phys. 144 (1992), 417–428.

[Z] A. S. Zhedanov, *Q rotations and other Q transformations as unitary nonlinear automorphisms of quantum algebras*, J. Math. Phys. 34 (1993), 2631–2647.

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