A NON-COMMUTATIVE BINOMIAL FORMULA

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Abstract. We prove a binomial formula for variables x and y satisfying a quadratic relation $xy = ax^2 + qyx + by^2$. Such relations are important in quantum group theory and non-commutative geometry.

1. Introduction

The classical binomial formula is given by

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k x^{n-k}.$$

Since the binomial coefficients are integers, this identity makes sense in an arbitrary ring with 1, provided that x and y satisfy the commutation relation $xy = yx$.

An important generalization is obtained by considering polynomials in two non-commuting variables x and y satisfying

$$xy = qyx, \tag{1}$$

where q is a scalar. In that case one has

$$(x+y)^n = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}\, y^k x^{n-k}, \tag{2}$$

where we use the standard notation [GR]

$$(a;q)_k = \prod_{i=0}^{k-1} (1 - aq^i).$$

Since the q-binomial coefficients occurring in (2) are polynomials in q (the Gaussian polynomials), this identity makes sense for x, y, and q in an arbitrary ring, assuming also that q commutes with x and y. The q-binomial formula (2) was first given explicitly by Schützenberger [S].

Recently, H. B. Benaoum [B2] proved the more general formula

$$(x+y)^n = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \prod_{j=0}^{k-1} (1 + [j]_q h)\, y^k x^{n-k},$$

where

$$[j]_q = 1 + q + \cdots + q^{j-1}$$

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and where x and y satisfy

$$xy = qyx + hy^2 \tag{3}$$

with q and h scalars. Assuming that $q \neq 1$ and $h \neq q-1$, one may rewrite the formula using

$$\prod_{j=0}^{k-1}(1+[j]_q h) = \left(\frac{1-q+h}{1-q}\right)^k (h/1-q+h;q)_k.$$

For $q=1$ one has the formula [B1]

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \prod_{j=0}^{k-1}(1+jh)\, y^k x^{n-k},$$

where

$$xy = yx + hy^2, \tag{4}$$

while the case when $h = q-1$ is

$$(x+y)^n = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}\, q^{\binom{k}{2}}\, y^k x^{n-k},$$

where

$$xy = qyx + (q-1)y^2.$$

Relations such as (1) and (4) play a fundamental role in quantum group theory. Loosely speaking, quantum groups may be constructed as symmetry "groups" of algebras defined by quadratic relations [M2]. In the case of the q-deformed quantum plane (1) this construction gives the standard quantum deformation of GL(2) [M1], while the h-deformed quantum plane (4) gives the non-standard, or Jordanian, deformation [D, Ku, Z]. From the viewpoint of non-commutative geometry, the h-deformed plane is in some ways better behaved than the q-deformed plane [CMP].

The purpose of this paper is to obtain a binomial formula for variables x and y satisfying the more general quadratic relation

$$xy = ax^2 + qyx + by^2. \tag{5}$$

Here x and y are elements of some ring with 1, and a, q, and b are central elements. For instance one may consider the algebra of polynomials in x and y over the complex field, and let a, q, and b be scalars. Actually these algebras are all isomorphic to (1) or to (4), though this observation does not simplify our problem.

Note that the elements $y^i x^j$ need not span the space of polynomials in x and y, and thus a binomial formula need not exist in general. As an illustration we try to express $x^2 y$ as a sum of such elements:

$$\begin{aligned} x^2 y &= x(ax^2 + qyx + by^2) = ax^3 + qxyx + bxy^2 \\ &= ax^3 + qxyx + b(ax^2 + qyx + by^2)y = ax^3 + qxyx + abx^2 y + qbyxy + b^2 y^3. \end{aligned}$$

If, for instance, $ab = 1$, the x^2y-terms will cancel, and there seems to be no such expansion. However, if $1 - ab$ is invertible we get

$$x^2 y = (1-ab)^{-1}(ax^3 + qxyx + qbyxy + b^2 y^3).$$

Using again (5) and simplifying gives

$$x^2y = (1-ab)^{-1}\left(a(1+q)x^3 + q(q+ab)yx^2 + bq(1+q)y^2x + b^2(1+q)y^3\right).$$

In fact, some further computations show that if $1-ab$ is invertible, the elements $(y^ix^j)_{i+j=3}$ span the space of cubic polynomials in x and y. Similarly, if both $1-ab$ and $1-2ab-abq$ are invertible, the elements $(y^ix^j)_{i+j=4}$ span the space of quartic polynomials.

To formulate our binomial formula, we define recursively $\alpha_0 = 0$, $\beta_0 = 1$,

$$\begin{cases} \alpha_{n+1} = q\alpha_n + \beta_n \\ \beta_{n+1} = -ab\alpha_n + \beta_n. \end{cases} \tag{6}$$

The first few cases are

$$\alpha_1 = 1, \quad \alpha_2 = 1+q, \quad \alpha_3 = 1-ab+q+q^2,$$

$$\beta_1 = 1, \quad \beta_2 = 1-ab, \quad \beta_3 = 1-2ab-abq,$$

where we recognise β_2 and β_3 from the previous paragraph. We will prove a binomial formula

$$(x+y)^n = \sum_{k=0}^{n} C_{kn}\, y^k x^{n-k}$$

under the additional condition that $\beta_2, \dots, \beta_{n-1}$ are invertible. Assuming that this is the case we define

$$\gamma_k = \frac{\alpha_k}{\beta_k}$$

and introduce the polynomials

$$h_k(x) = \prod_{i=0}^{k-1}(1+\gamma_i x),$$

the first few cases being

$$h_0(x) = h_1(x) = 1, \quad h_2(x) = 1+x, \quad h_3(x) = (1+x)\left(1+\frac{1+q}{1-ab}\,x\right).$$

We are now ready to formulate our main result.

Theorem 1.1. *In the notation above, assuming that the elements $\beta_2, \dots, \beta_{n-1}$ are invertible, the formula*

$$(x+y)^n = \sum_{k=0}^{n} \frac{h_n(q)}{h_k(q)h_{n-k}(q)}\, h_k(b)h_{n-k}(a)\, y^k x^{n-k}$$

holds.

Working in a polynomial algebra, so that a, q, and b are complex numbers, one may solve the recursion (6). This gives a more explicit version of Theorem 1.1.

Corollary 1.2. *For generic values of a, q, and b,*

$$(x+y)^n=\sum_{k=0}^n\frac{(Q;Q)_n}{(Q;Q)_k(Q;Q)_{n-k}}\,\frac{\left(\frac{Q-q+b(1+Q)}{1-qQ+b(1+Q)};Q\right)_k\left(\frac{Q-q+a(1+Q)}{1-qQ+a(1+Q)};Q\right)_{n-k}}{\left(\frac{Q-q}{1-qQ};Q\right)_n}$$
$$\times\frac{(1-qQ+b(1+Q))^k(1-qQ+a(1+Q))^{n-k}}{(1-qQ)^n}\,y^kx^{n-k},$$

where Q satisfies

$$Q+Q^{-1}=\frac{q^2+1-2ab}{q+ab}. \tag{7}$$

Using the identity

$$(a;Q^{-1})_k=(a^{-1};Q)_k(-a)^kQ^{-\binom{k}{2}}$$

one may check that the coefficients in Corollary 1.2 are invariant if replacing Q by Q^{-1}, and thus are indeed rational functions of a, q, and b.

An interesting degenerate case of Corollary 1.2 is when $Q=1$, which happens for $(q-1)^2=4ab$. This yields a two-parameter version of the h-deformed quantum plane (4), with the binomial formula

$$\begin{aligned}(x+y)^n \\ =\sum_{k=0}^n\binom{n}{k}\frac{\prod_{j=0}^{k-1}(1+q+(1-q+2b)j)\prod_{j=0}^{n-k-1}(1+q+(1-q+2a)j)}{\prod_{j=0}^{n-1}(1+q+(1-q)j)}\,y^kx^{n-k},\end{aligned} \tag{8}$$

or, if $q\neq 1,\ 1+2a,\ 1+2b$,

$$(x+y)^n=\sum_{k=0}^n\binom{n}{k}\frac{(1-q+2b)^k(1-q+2a)^{n-k}}{(1-q)^n}\,\frac{\left(\frac{1+q}{1-q+2b}\right)_k\left(\frac{1+q}{1-q+2a}\right)_{n-k}}{\left(\frac{1+q}{1-q}\right)_n}\,y^kx^{n-k},$$

where

$$(a)_k=a(a+1)\cdots(a+k-1).$$

The plan of the paper is as follows. In Section 2 we prove Theorem 1.1. Our proof is different from the one given for the case $a=0$ in [B2]. In Section 3 we apply Theorem 1.1 to compute the coefficients in the expansion

$$x^my^n=\sum_{l=0}^{m+n}C_{lmn}\,y^lx^{m+n-l},$$

which exists if $\beta_2,\dots,\beta_{m+n-1}$ are invertible. For generic complex values of a, q, and b, the coefficients C_{lmn} may be expressed as terminating balanced ${}_4\phi_3$ series [GR] of a special type. In Section 4 we derive Corollary 1.2.

2. Proof of the binomial formula

In this section we will prove Theorem 1.1. First we observe that if x and y satisfy relation (5), and if c and d are central invertible elements, then cx and dy satisfy a similar relation with (a, q, b) replaced by $(ad/c, q, bc/d)$. Since this does not change q or ab, the polynomials h_j are the same for the two rings. Thus Theorem 1.1 implies

$$(cx + dy)^n = \sum_{k=0}^{n} \frac{h_n(q)}{h_k(q)h_{n-k}(q)} h_k(bc/d)h_{n-k}(ad/c)d^k c^{n-k}\, y^k x^{n-k}. \tag{9}$$

With the interpretation

$$d^k h_k(bc/d) = \prod_{i=0}^{k-1}(d + \gamma_i bc) \tag{10}$$

and similarly for h_{n-k}, this equality makes sense also if c or d is not invertible. We will prove Theorem 1.1 by induction on n, the induction hypothesis being that (9) holds for all central elements c and d.

Lemma 2.1. *Let A_0, B_0, C_0, and D_0 be arbitrary central elements, and define recursively*

$$\begin{cases} A_{n+1} = A_n + aB_n \\ B_{n+1} = -bA_n + qB_n, \end{cases}$$

$$\begin{cases} C_{n+1} = C_n + aD_n \\ D_{n+1} = -bC_n + qD_n. \end{cases}$$

Then, for all n, $m \geq 0$,

$$(C_n x + D_n y)^m (A_0 x + B_0 y)^n = (A_m x + B_m y)^n (C_0 x + D_0 y)^m.$$

Proof. Let L_{mn} be the statement of the lemma for fixed m and n. Then L_{m0} and L_{0n} are trivial, and it is also clear that L_{mn} and L_{11} together imply $L_{m+1,n}$ and $L_{m,n+1}$. Thus it suffices to prove L_{11}, that is, the identity

$$(C_1 x + D_1 y)(A_0 x + B_0 y) = (A_1 x + B_1 y)(C_0 x + D_0 y),$$

or equivalently

$$[C_0(x - by) + D_0(ax + qy)](A_0 x + B_0 y) = [A_0(x - by) + B_0(ax + qy)](C_0 x + D_0 y).$$

This follows from the commutation relation (5), written in the form

$$(x - by)y = (ax + qy)x.$$

$\square$

Let us put $m = 1$, $A_0 = B_0 = C_0 = 1$, $D_0 = 0$ in Lemma 2.1. One may check that in this case

$$C_n = \beta_n, \quad D_n = -b\alpha_n,$$

so that the lemma gives

$$(\beta_n x - b\alpha_n y)(x + y)^n = [(a + 1)x + (q - b)y]^n\, x.$$

Thus, if β_n is invertible,

$$
\begin{aligned}
(x+y)^{n+1} &= \left(\frac{1}{\beta_n}(\beta_n x - b\alpha_n y) + (1+b\gamma_n)y\right)(x+y)^n \\
&= \frac{1}{\beta_n}\left[(a+1)x+(q-b)y\right]^n x + (1+b\gamma_n)y(x+y)^n,
\end{aligned}
$$

and hence, by the induction hypothesis (9),

(11)

$$
\begin{aligned}
(x+y)^{n+1} &= \frac{1}{\beta_n}\sum_{k=0}^{n} \frac{h_n(q)h_k\left(\frac{b(a+1)}{q-b}\right)h_{n-k}\left(\frac{a(q-b)}{a+1}\right)}{h_k(q)h_{n-k}(q)}(q-b)^k(a+1)^{n-k}\, y^k x^{n-k+1} \\
&\quad + (1+b\gamma_n)\sum_{k=0}^{n}\frac{h_n(q)}{h_k(q)h_{n-k}(q)}\, h_k(b)h_{n-k}(a)\, y^{k+1}x^{n-k}.
\end{aligned}
$$

The terms in the first sum may be simplified using the following lemma.

Lemma 2.2. *In the notation above,*

$$
\beta_k h_{k+1}(x) = (1+x)^k h_k\left(\frac{qx-ab}{1+x}\right).
$$

Proof. By induction on k this reduces to

$$
\beta_{k+1} + x\alpha_{k+1} = (1+x)\beta_k + (qx-ab)\alpha_k,
$$

which is equivalent to (6). □

In particular, choosing $x=a$ and replacing k by $n-k$ gives

$$
(a+1)^{n-k}h_{n-k}\left(\frac{a(q-b)}{a+1}\right) = \beta_{n-k}h_{n-k+1}(a),
$$

while choosing $x = b(a+1)/(q-b)$ and replacing k by $k-1$ gives

$$
(q-b)^k h_k\left(\frac{b(a+1)}{q-b}\right) = (q-b)\frac{(q+ab)^{k-1}}{\beta_{k-1}}\, h_{k-1}(b).
$$

Plugging these expressions into (11) we obtain

$$
\begin{aligned}
(x+y)^{n+1} &= h_{n+1}(a)x^{n+1} + h_{n+1}(b)y^{n+1} \\
&\quad + \sum_{k=1}^{n}\left(\frac{1}{\beta_n}\frac{h_n(q)}{h_k(q)h_{n-k}(q)}(q-b)\frac{(q+ab)^{k-1}}{\beta_{k-1}}\, h_{k-1}(b)\beta_{n-k}h_{n-k+1}(a)\right. \\
&\qquad \left. + (1+b\gamma_n)\frac{h_n(q)}{h_{k-1}(q)h_{n-k+1}(q)}\, h_{k-1}(b)h_{n-k+1}(a)\right) y^k x^{n-k+1}.
\end{aligned}
$$

Thus, to prove Theorem 1.1, it suffices to prove that the coefficient of $y^k x^{n-k+1}$ in this sum equals

$$
\frac{h_{n+1}(q)}{h_k(q)h_{n-k+1}(q)}\, h_k(b)h_{n-k+1}(a).
$$

Multiplying the corresponding identity with

$$\frac{h_k(q)h_{n-k+1}(q)\beta_n\beta_{k-1}}{h_n(q)h_{n-k+1}(a)h_{k-1}(b)}$$

gives

$$(q-b)(q+ab)^{k-1}(\beta_{n-k}+q\alpha_{n-k})+(\beta_n+b\alpha_n)(\beta_{k-1}+q\alpha_{k-1}) = (\beta_n+q\alpha_n)(\beta_{k-1}+b\alpha_{k-1}).$$

Writing $\beta_{n-k} + q\alpha_{n-k} = \alpha_{n-k+1}$ and simplifying gives

$$(q-b)(q+ab)^{k-1}\alpha_{n-k+1} = (q-b)(\alpha_n\beta_{k-1} - \beta_n\alpha_{k-1}).$$

Thus all that remains is to prove the following lemma.

Lemma 2.3. *For $i \geq j \geq 0$,*

$$\alpha_i\beta_j - \alpha_j\beta_i = (q+ab)^j\alpha_{i-j}.$$

Proof. Since

$$\begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} = A^k \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix},$$

where

$$A = \begin{pmatrix} q & 1 \\ -ab & 1 \end{pmatrix}, \tag{12}$$

it follows from standard properties of determinants that indeed

$$\det \begin{pmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{pmatrix} = \det(A)^j \det \begin{pmatrix} \alpha_{i-j} & \alpha_0 \\ \beta_{i-j} & \beta_0 \end{pmatrix} = (q+ab)^j\alpha_{i-j}.$$

□

3. A commutation rule

In this section we will apply our binomial formula to compute the coefficients in the expansion

$$x^m y^n = \sum_{l=0}^{m+n} C_{lmn}\, y^l x^{m+n-l}.$$

We will see that such an expansion exists if $\beta_2, \dots, \beta_{m+n-1}$ are invertible. We need Lemma 2.1 when $A_0 = D_0 = 0$, $B_0 = C_0 = 1$. In this case

$$A_n = a\alpha_n, \quad B_n = \delta_n, \quad C_n = \beta_n, \quad D_n = -b\alpha_n,$$

where

$$\delta_n = \alpha_{n+1} - \alpha_n,$$

so that the lemma gives

$$(\beta_n x - b\alpha_n y)^m y^n = (a\alpha_m x + \delta_m y)^n x^m. \tag{13}$$

Remark 3.1. One may check that

$$A^n = \begin{pmatrix} \delta_n & \alpha_n \\ -ab\alpha_n & \beta_n \end{pmatrix},$$

where A is the matrix (12). If $q+ab$ is invertible this implies

$$A^{-n}=\frac{1}{(q+ab)^n}\begin{pmatrix}\beta_n & -\alpha_n\\ ab\alpha_n & \delta_n\end{pmatrix},$$

so it is natural to define

$$\begin{cases}\alpha_{-n}=-(q+ab)^{-n}\alpha_n,\\ \beta_{-n}=(q+ab)^{-n}\delta_n,\\ \delta_{-n}=(q+ab)^{-n}\beta_n\end{cases}\tag{14}$$

for negative integers n. Moreover, $A^{m+n}=A^mA^n$ implies

$$\begin{cases}\alpha_{m+n}=\delta_m\alpha_n+\alpha_m\beta_n,\\ \beta_{m+n}=\beta_m\beta_n-ab\alpha_m\alpha_n,\\ \delta_{m+n}=\delta_m\delta_n-ab\alpha_m\alpha_n.\end{cases}\tag{15}$$

The next step is to find an expansion of the form

$$x^m=\sum_{k=0}^m D_{kmn}\,y^k(\beta_nx-b\alpha_ny)^{m-k}.$$

Lemma 3.2. *If c and d are central elements such that $c\beta_k+ad\alpha_k$ is invertible for $0\leq k\leq m-1$, then*

$$x^m=\sum_{k=0}^m\frac{h_m(q)}{h_k(q)h_{m-k}(q)}\frac{d^kh_k(q-1-bc/d)}{c^mh_m(ad/c)}(-1)^k\,y^k(cx+dy)^{m-k}.$$

This should be interpreted as in (10) if c or d is not invertible.

Proof. The starting point is the identity

$$-(a+1)(x+y)y=-a(x+y)^2-(q-a)y(x+y)+(q-a-b-1)y^2,$$

which is easily verified using (5). Thus, if $a+1$ is invertible, then $\tilde{x}=x+y$ and $\tilde{y}=-y$ satisfy relation (5) with a, q, b replaced by

$$\tilde{a}=-\frac{a}{a+1},\quad \tilde{q}=\frac{q-a}{a+1},\quad \tilde{b}=\frac{q-a-b-1}{a+1}.$$

The corresponding elements $\tilde{\gamma}_k$ and polynomials $\tilde{h}_k$ may be computed:

$$\tilde{\gamma}_k=\frac{(1+a)\alpha_k}{\beta_k+a\alpha_k},$$

$$\tilde{h}_k(x)=\frac{h_k(a+(1+a)x)}{h_k(a)}.$$

In particular

$$\tilde{h}_k(\tilde{a})=\frac{1}{h_k(a)},\quad \tilde{h}_k(\tilde{q})=\frac{h_k(q)}{h_k(a)},\quad \tilde{h}_k(\tilde{b})=\frac{h_k(q-1-b)}{h_k(a)},$$

so that Theorem 1.1 gives

$$x^m=\sum_{k=0}^m\frac{h_m(q)}{h_k(q)h_{m-k}(q)}\frac{h_k(q-1-b)}{h_m(a)}(-1)^k\,y^k(x+y)^{m-k}$$

if $\beta_k + a\alpha_k$ is invertible for $0 \le k \le m-1$. This proves the lemma for $c = d = 1$. The general case follows by replacing x, y with cx, dy, as in (9). □

In particular, when $c = \beta_n$, $d = -b\alpha_n$, (15) gives

$$c\beta_k + ad\alpha_k = \beta_{n+k},$$

and one may also check that

$$q - 1 - \frac{bc}{d} = q - 1 + \frac{1}{\gamma_n} = \frac{\delta_n}{\alpha_n}.$$

Thus in this case we obtain

$$x^m = \sum_{k=0}^{m} \frac{h_m(q)}{h_k(q)h_{m-k}(q)} \frac{(b\alpha_n)^k h_k(\delta_n/\alpha_n)}{\beta_n^m h_m(-ab\gamma_n)}\, y^k(\beta_n x - b\alpha_n y)^{m-k},$$

if $\beta_n, \dots, \beta_{m+n-1}$ are invertible. Together with (13) this gives

$$x^m y^n = \sum_{k=0}^{m} \frac{h_m(q)}{h_k(q)h_{m-k}(q)} \frac{(b\alpha_n)^k h_k(\delta_n/\alpha_n)}{\beta_n^m h_m(-ab\gamma_n)}\, y^k(a\alpha_{m-k}x + \delta_{m-k}y)^n x^{m-k},$$

and, by (9),

$$\begin{aligned} x^m y^n = \sum_{k=0}^{m}\sum_{j=0}^{n} & \frac{h_m(q)}{h_k(q)h_{m-k}(q)} \frac{(b\alpha_n)^k h_k(\delta_n/\alpha_n)}{\beta_n^m h_m(-ab\gamma_n)} \frac{h_n(q)}{h_j(q)h_{n-j}(q)} \\ & \times \delta_{m-k}^j h_j\left(ab\frac{\alpha_{m-k}}{\delta_{m-k}}\right)(a\alpha_{m-k})^{n-j} h_{n-j}\left(\frac{\delta_{m-k}}{\alpha_{m-k}}\right) y^{k+j}x^{m+n-k-j}, \end{aligned}$$

assuming that also $\beta_2, \dots, \beta_{n-1}$ are invertible.

Thus we have proved the following theorem.

Theorem 3.3. *If $\beta_2, \dots, \beta_{m+n-1}$ are invertible, there is an expansion*

$$x^m y^n = \sum_{l=0}^{m+n} C_{lmn}\, y^l x^{m+n-l}, \tag{16}$$

where the coefficient C_{lmn} equals

$$\sum_{\substack{j+k=l\\ 0\le j\le n\\ 0\le k\le m}} \frac{h_m(q)h_n(q)\delta_{m-k}^j h_j\left(ab\frac{\alpha_{m-k}}{\delta_{m-k}}\right)(b\alpha_n)^k h_k\left(\frac{\delta_n}{\alpha_n}\right)(a\alpha_{m-k})^{n-j}h_{n-j}\left(\frac{\delta_{m-k}}{\alpha_{m-k}}\right)}{\beta_n^m h_m(-ab\gamma_n)h_j(q)h_k(q)h_{n-j}(q)h_{m-k}(q)}. \tag{17}$$

Remark 3.4. The polynomials h_k appearing in Theorem 3.3 admit simple expressions in terms of the elements α_k, β_k, δ_k. From the first identity in (15) it follows that

$$\alpha_n^k h_k\left(\frac{\delta_n}{\alpha_n}\right) = \prod_{j=0}^{k-1} \frac{\alpha_{n+j}}{\beta_j};$$

in particular,

$$h_k(q) = \prod_{j=0}^{k-1} \frac{\alpha_{j+1}}{\beta_j}.$$

Similarly the second identity in (15) gives

$$\beta_n^k h_k(-ab\gamma_n) = \prod_{j=0}^{k-1} \frac{\beta_{n+j}}{\beta_j}.$$

By an extension of (15) to negative indices,

$$\beta_k \delta_n + ab\alpha_k \alpha_n = (q+ab)^n \beta_{k-n} = (q+ab)^k \delta_{n-k},$$

which leads to

$$\delta_n^k h_k \left(ab \frac{\alpha_n}{\delta_n} \right) = (q+ab)^{nk} \prod_{j=0}^{k-1} \frac{\beta_{j-n}}{\beta_j} = (q+ab)^{\binom{k}{2}} \prod_{j=0}^{k-1} \frac{\delta_{n-j}}{\beta_j},$$

where we must use (14) if some indices are negative.

Remark 3.5. The coefficients C_{lmn} are very simple if a or b is zero. Let us put $a = 0$, $b = h$, so that we consider relation (3). Then, in (17), only the term with $j = n$ is non-zero, and such a term exists only if $l \geq n$ in (16). Moreover, one may check that $\alpha_n = [n]_q$, $\beta_n = 1$, $\delta_n = q^n$ and that

$$\alpha_n^k h_k \left(\frac{\delta_n}{\alpha_n} \right) = \frac{(q^n;q)_k}{(1-q)^k};$$

in particular,

$$h_k(q) = \frac{(q;q)_k}{(1-q)^k}.$$

Plugging this into Theorem 3.3 gives

$$x^m y^n = \sum_{l=n}^{m+n} \frac{(q;q)_m (q^n;q)_{l-n}}{(q;q)_{l-n}(q;q)_{m+n-l}} (1-q)^{n-l} q^{n(m+n-l)} h^{l-n}\, y^l x^{m+n-l},$$

where $xy = qyx + hy^2$. The case $m = 1$ of this identity was used in [B2].

Remark 3.6. Working in a polynomial algebra over the complex numbers, one may solve the recursion (6) explicitly; cf. Section 4. For generic values of the parameters, the coefficients C_{lmn} may then be written as terminating balanced ${}_4\phi_3$ series [GR]; for instance, when $l \leq m$,

$$C_{lmn} = b^l a^n \left(\frac{1+Q}{1-qQ} \right)^{l+n} \frac{(Q;Q)_m (Q^n;Q)_l (Q^{m-l};Q)_n}{(Q;Q)_l (Q;Q)_{m-l} \left(Q^{m-l} \frac{Q-q}{1-qQ}; Q \right)_{l+n}}$$
$$\times\, {}_4\phi_3 \left[\begin{matrix} Q^{-l}, Q^{-n}, Q^{m-l}(Q-q)/(1-qQ), Q^{1+m-l}(1-qQ)/(Q-q) \\ Q^{1-n-l}, Q^{m-l}, Q^{1+m-l} \end{matrix} ; Q, Q \right],$$

where Q satisfies (7). This kind of sums have very rich symmetry properties [GR].

4. Explicit form of the binomial formula

In this section we work in the algebra of polynomials in x, y, over the complex field, with the relation (5), where a, q, and b are scalars. Let us first assume that the matrix

$$A = \begin{pmatrix} q & 1 \\ -ab & 1 \end{pmatrix}$$

is diagonalizable with different eigenvalues λ_1 and λ_2. Then α_n and β_n are linear combinations of λ_1^n and λ_2^n. Since $\alpha_0 = 0$, $\alpha_1 = 1$, we have

$$\alpha_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \tag{18}$$

where

$$\lambda_1 + \lambda_2 = 1 + q, \quad \lambda_1\lambda_2 = q + ab.$$

Remark 4.1. At this point we may find explicit expressions for α_n and β_n as polynomials in ab and q. Using the symmetric function identity [MM]

$$\frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2} = \sum_{k=0}^{n} \lambda_1^k \lambda_2^{n-k} = \sum_{k=0}^{[n/2]} \binom{n-k}{k} (-1)^k (\lambda_1\lambda_2)^k (\lambda_1 + \lambda_2)^{n-k} \tag{19}$$

we find that

$$\alpha_n = \sum_{k=0}^{[(n-1)/2]} \binom{n-k-1}{k} (-1)^k (q+ab)^k (1+q)^{n-k-1}.$$

One may then express β_n as a polynomial by

$$\beta_n = \alpha_{n+1} - q\alpha_n.$$

The identity (19) is easily proved using generating functions; write

$$\frac{1}{(1-\lambda_1 t)(1-\lambda_2 t)} = \frac{1}{1 - t(\lambda_1 + \lambda_2) + t^2\lambda_1\lambda_2}$$

and identify the coefficient of t^n.

Now let $Q = \lambda_2/\lambda_1$. Then Q satisfies

$$Q + Q^{-1} = \frac{(\lambda_1+\lambda_2)^2 - 2\lambda_1\lambda_2}{\lambda_1\lambda_2} = \frac{(1+q)^2 - 2(q+ab)}{q+ab} = \frac{1+q^2-2ab}{q+ab},$$

that is, equation (7). We may write

$$\lambda_1 = \lambda_1 \frac{1+q}{\lambda_1+\lambda_2} = \frac{1+q}{1+Q},$$

$$\lambda_2 = Q\lambda_1 = Q\frac{1+q}{1+Q}.$$

Plugging this into (18) gives

$$\alpha_n = \left(\frac{1+q}{1+Q}\right)^{n-1} \frac{1-Q^n}{1-Q},$$

so that

$$\beta_n = \alpha_{n+1} - q\alpha_n = \frac{(1+q)^{n-1}(1-qQ-(Q-q)Q^n)}{(1+Q)^n(1-Q)},$$

$$\gamma_n = \frac{\alpha_n}{\beta_n} = \frac{(1-Q^n)(1+Q)}{1-qQ-(Q-q)Q^n},$$

$$\delta_n = \alpha_{n+1} - \alpha_n = \frac{(1+q)^{n-1}(q-Q+(1-qQ)Q^n)}{(1+Q)^n(1-Q)}.$$

If $1-qQ$, $1-qQ+x(1+Q)\neq 0$, one may then write

$$\begin{aligned} h_n(x) &= \prod_{i=0}^{n-1}(1+x\gamma_i) = \prod_{i=0}^{n-1}\frac{1-qQ-(Q-q)Q^i+x(1-Q^i)(1+Q)}{1-qQ-(Q-q)Q^i} \\ &= \prod_{i=0}^{n-1}\frac{1-qQ+x(1+Q)-(Q-q+x(1+Q))Q^i}{1-qQ-(Q-q)Q^i} \\ &= \left(\frac{1-qQ+x(1+Q)}{1-qQ}\right)^n \frac{\left(\frac{Q-q+x(1+Q)}{1-qQ+x(1+Q)};Q\right)_n}{\left(\frac{Q-q}{1-qQ};Q\right)_n}. \end{aligned}$$

Inserting this expression into Theorem 1.1 gives Corollary 1.2.

As was remarked in the introduction, the case $(q-1)^2 = 4ab$, $Q=1$ is of special interest. This corresponds to the matrix A not being diagonalizable. In this case, both eigenvalues equal $\lambda = (1+q)/2$, and one has

$$\alpha_n = \lambda^{n-1}n,$$

$$\beta_n = \lambda^n(n+1) - q\lambda^{n-1}n,$$

$$\gamma_n = \frac{2n}{(1-q)n+q+1},$$

$$h_n(x) = \prod_{i=0}^{n-1}(1+x\gamma_i) = \prod_{i=0}^{n-1}\frac{1+q+(1-q+2x)i}{1+q+(1-q)i}.$$

This yields the binomial formula (8).

Finally we make a remark on formal power series identities. The q-binomial formula (2) is equivalent to the identity

$$e_q(x+y) = e_q(y)\,e_q(x), \quad xy = qyx,$$

where

$$e_q(x) = \sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}.$$

A generalization to relation (5) may be obtained from Corollary 1.2 by multiplying with

$$\frac{\left(\frac{Q-q}{1-qQ};Q\right)_n}{(Q;Q)_n}$$

and then summing over n; namely,

$$
\begin{aligned}
{}_1\phi_0\left[\frac{Q-q}{1-qQ};Q,x+y\right] &= {}_1\phi_0\left[\frac{Q-q+b(1+Q)}{1-qQ+b(1+Q)};Q,\frac{1-qQ+b(1+Q)}{1-qQ}\,y\right] \\
&\quad\times {}_1\phi_0\left[\frac{Q-q+a(1+Q)}{1-qQ+a(1+Q)};Q,\frac{1-qQ+a(1+Q)}{1-qQ}\,x\right],
\end{aligned}
$$

where

$$
{}_1\phi_0[a;Q,x] = \sum_{n=0}^{\infty} \frac{(a;Q)_n}{(Q;Q)_n}\,x^n .
$$

It would be interesting to find generalizations of other formal power series identities such as

$$
e_q(x+y-yx) = e_q(x)\,e_q(y), \quad xy = qyx;
$$

cf. [K] for this and other more general identities.

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