MULTIVARIABLE q-HAHN POLYNOMIALS AS COUPLING COEFFICIENTS FOR QUANTUM ALGEBRA REPRESENTATIONS

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ABSTRACT. We study coupling coefficients for a multiple tensor product of highest weight representations of the quantum algebra $\mathcal{U}_{q^{1/2}}(\mathfrak{su}(1,1))$. These are multivariable generalizations of the q-Hahn polynomials.

1. Introduction

It is known that the Clebsch–Gordan coefficients for highest weight representations of $\mathrm{SU}(1,1)$ may be identified with three classical systems of orthogonal polynomials, namely, the Jacobi polynomials, the Hahn polynomials, and the continuous Hahn polynomials [VK]. They correspond to the three conjugacy classes of one-parameter subgroups of $\mathrm{SU}(1,1)$. In our previous papers [R1, R3] we found that many multivariable polynomials occurring in the literature have a similar interpretation, as coupling coefficients for an n-fold tensor product of highest weight representations. This interpretation leads to simple proofs of many properties of such polynomials.

The purpose of this paper is to generalize some of this to the context of quantum groups and q-series. Several authors have found that q-Hahn polynomials appear as Clebsch–Gordan coefficients for the $\mathrm{SU}(2)$ and $\mathrm{SU}(1,1)$ quantum groups [KR, KK, LK, S, V]. More recently, Granovskii and Zhedanov [GZ] and Koelink and Van der Jeugt [KV] found similar interpretations of q-Racah polynomials and Askey–Wilson polynomials, respectively. In this paper, however, we only consider multivariable generalizations of q-Hahn polynomials. This case is more elementary and more similar to the Lie algebra case.

For comparison, we will summarize some of the basic facts of [R1, R3]. Let, for $\nu > 0$, $\mathcal{A}^\nu$ be the Hilbert space of analytic functions on the complex unit disc with norm

$$\|f\|^2 = \sum_{k=0}^{\infty} \frac{k!}{(\nu)_k} |\hat{f}(k)|^2.$$

Here $f(z) = \sum \hat{f}(k) z^k$ and $(\nu)_k = \nu(\nu+1)\cdots(\nu+k-1)$ is the Pochhammer symbol. For $\nu > 1$ one has the integral formula

$$\|f\|^2 = \frac{\nu-1}{\pi} \int_{|z|<1} |f(z)|^2 (1-|z|^2)^{\nu-2}\, dxdy. \tag{1}$$

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The norm of $\mathcal{A}^\nu$ is invariant under the transformations

$$f(z) \mapsto \frac{1}{(cz+d)^\nu} f\left(\frac{az+b}{cz+d}\right), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(1,1), \tag{2}$$

which give rise to a unitary representation of a covering group of $\mathrm{SU}(1,1)$ on each space $\mathcal{A}^\nu$. This representation is irreducible and has the highest weight vector 1 of weight $-\nu$. The derived representation of the complexified Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is given by the densely defined operators

$$X_+ = -\frac{d}{dz}, \qquad X_- = z^2\frac{d}{dz} + \nu z, \qquad H = -\left(\nu + 2z\frac{d}{dz}\right), \tag{3}$$

which satisfy the structure equations

$$[X_+X_-] = H, \qquad [HX_+] = 2X_+, \qquad [HX_-] = -2X_-$$

and the $\mathfrak{su}(1,1)$ reality conditions

$$X_\pm^* = -X_\mp, \qquad H^* = H.$$

Consider a Hilbert tensor product $\mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n}$ of such spaces. It decomposes under the Lie algebra action as

$$\mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n} = \bigoplus_{s=0}^{\infty} \binom{n+s-2}{n-2} \mathcal{A}^{|\nu|+2s} \tag{4}$$

(where $|\nu| = \sum \nu_i$), that is, $\mathcal{A}^{|\nu|+2s}$ occurs with the multiplicity $\binom{n+s-2}{n-2}$. A highest weight vector Q in $\mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n}$ of weight $-(|\nu|+2s)$ is an image of 1 under an intertwining embedding

$$\mathcal{A}^{|\nu|+2s} \to \mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n}.$$

In our realization this means that Q is a homogeneous polynomial of degree s which can be expressed as a function of the differences $z_i - z_j$ of the coordinates. Equivalently, Q satisfies

$$Q(az_1+b, \dots, az_n+b) = a^s Q(z_1, \dots, z_n), \quad a,\, b \in \mathbb{C}.$$

In agreement with (4), the space of such polynomials has dimension $\binom{n+s-2}{n-2}$.

Let, for each highest weight vector Q, $\mathcal{J}_Q$ be the intertwining operator

$$\mathcal{J}_Q : \ \mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n} \to \mathcal{A}^{|\nu|+2s}$$

determined by $\mathcal{J}_Q^* 1 = Q$. In the present realization, $\mathcal{J}_Q$ is a differential operator. In fact, if

$$Q(z) = \sum_{|t|=s} c_t\, z_1^{t_1} \cdots z_n^{t_n}$$

then

$$\mathcal{J}_Q(f_1 \otimes \cdots \otimes f_n)(z) = \sum_{|t|=s} \bar{c}_t\, \frac{f_1^{(t_1)}(z) \cdots f_n^{(t_n)}(z)}{(\nu_1)_{t_1} \cdots (\nu_n)_{t_n}}. \tag{5}$$

In particular, if $n = 2$ and $Q(z_1, z_2) = (z_1 - z_2)^s$ then

$$\mathcal{J}_Q(f_1 \otimes f_2)(z) = \sum_{k=0}^{s} \binom{s}{k} \frac{(-1)^{s-k}}{(\nu_1)_k (\nu_2)_{s-k}}\, f_1^{(k)}(z) f_2^{(s-k)}(z). \tag{6}$$

These operators occur (although usually written in homogeneous coordinates, cf. Appendix 1, and with ν_i replaced by negative integers) in classical invariant theory and are called *transvectants*. They have also been used in the theory of modular forms, where they are called Rankin-Cohen brackets [Z].

It follows from (5) that, writing $e_m(z) = z^m$,

$$\mathcal{J}_Q(e_{m_1} \otimes \cdots \otimes e_{m_n}) = \overline{T_1Q(m)}\, e_{|m|-s},$$

where T_1Q is the polynomial

$$T_1Q(m_1, \dots, m_n) = (-1)^s \sum_{|t|=s} c_t \frac{(-m_1)_{t_1} \cdots (-m_n)_{t_n}}{(\nu_1)_{t_1} \cdots (\nu_n)_{t_n}}.$$

Since the monomials are eigenfunctions of the rotations of the disc, this exhibits T_1Q as a coupling coefficient, with respect to this subgroup, of our representation. If Q' is another highest weight vector then, by an application of Schur's Lemma to $\mathcal{J}_{Q'}\mathcal{J}_Q^*$,

$$\sum_{|m|=k+s} \frac{(\nu_1)_{m_1} \cdots (\nu_n)_{m_n}}{m_1! \cdots m_n!} T_1Q(m)\, \overline{T_1Q'(m)} = \frac{(|\nu| + 2s)_k}{k!} \langle Q, Q' \rangle_{\bigotimes_{i=1}^n \mathcal{A}^{\nu_i}} \tag{7}$$

for $k = 0, 1, 2, \dots$. Thus, given an orthogonal system of highest weight vectors, the transform T_1 gives a corresponding system of discrete orthogonal polynomials. For fixed k one may eliminate one variable and view these as polynomials of $n - 1$ variables. In the case $n = 2$ one obtains in this way the Hahn polynomials.

Replacing the monomials by formal eigenvectors to other one-parameter subgroups gives multivariable Jacobi polynomials

$$T_2Q(x_1, \dots, x_n) = \sum_{|t|=s} c_t \frac{x_1^{t_1} \cdots x_n^{t_n}}{(\nu_1)_{t_1} \cdots (\nu_n)_{t_n}}$$

and multivariable continuous Hahn polynomials

$$T_3Q(x_1, \dots, x_n) = \sum_{|t|=s} c_t \frac{(\frac{\nu_1}{2} - ix_1)_{t_1} \cdots (\frac{\nu_n}{2} - ix_n)_{t_n}}{(\nu_1)_{t_1} \cdots (\nu_n)_{t_n}}.$$

Many orthogonal and biorthogonal polynomial systems occurring in the literature may be obtained by applying the three transforms T_i to specific bases in the space of highest weight vectors; cf. [R3] and references given there.

In this paper we will obtain a q-analogue of the transform T_1, leading to multivariable q-Hahn polynomials. One may also consider more general coupling coefficients connected with so called twisted primitive elements of the quantum algebra. This leads to multivariable generalizations of Askey–Wilson polynomials and q-Racah polynomials; cf. [R5] for some further remarks.

The plan of the paper is as follows. Section 2 contains preliminaries on q-series and quantum algebra. In Section 3 we find a q-analogue of the expression (5). This is used in Section 4 to find analogues of the transform T_1 and of the orthogonality relations (7). In Section 5 we study q-analogues of the *coupling kernels* introduced in [R3]. These are the reproducing kernels for spaces of coupling coefficients. In Section 6 we discuss identities involving multivariable q-Hahn polynomials and coupling kernels which follow from the quantum algebraic interpretation. In Section 7 we consider explicit examples of orthogonal and biorthogonal systems which may be obtained as coupling coefficients. There are two appendices, where we use quasi-commuting homogeneous coordinates to give an algebraic description of the space of highest weight vectors.

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2. Preliminaries

2.1. **Notation.** Throughout the paper, q will be a fixed number in the range $0 < q < 1$. It will be convenient to use the symmetric q-factorials and q-Pochhammer symbols defined by

$$[k]! = [1][2]\cdots[k], \qquad [a]_k = [a][a+1]\cdots[a+k-1],$$

where

$$[a] = \frac{q^{\frac{a}{2}} - q^{-\frac{a}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

This is related to the more standard notation of [GR],

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}), \tag{8}$$

by

$$[a]_k = \frac{(q^a;q)_k}{(1-q)^k}\, q^{-\frac{1}{4}k(2a+k-3)},$$
$$\frac{[a]_k}{[b]_k} = \frac{(q^a;q)_k}{(q^b;q)_k}\, q^{\frac{1}{2}k(b-a)}.$$

Sometimes we will use the multi-index notation

$$[a]_k = [a_1]_{k_1}\cdots[a_n]_{k_n}, \qquad [k]! = [k_1]!\cdots[k_n]!.$$

The basic hypergeometric series ${}_r\phi_s$ is defined by [GR]

$${}_r\phi_s\left[\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s\end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1;q)_k\cdots(a_r;q)_k}{(q;q)_k(b_1;q)_k\cdots(b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k;$$

in particular,

$${}_{r+1}\phi_r\left[\begin{matrix} q^{a_1},\dots,q^{a_{r+1}} \\ q^{b_1},\dots,q^{b_r}\end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{[a_1]_k\cdots[a_{r+1}]_k}{[k]![b_1]_k\cdots[b_r]_k}\, q^{\frac{1}{2}k(|a|-|b|-1)} z^k,$$

where $|x| = \sum x_i$. The q-Hahn polynomials, introduced by Hahn [H], are given by

$$Q_n(q^{-x};\alpha,\beta,N|q) = {}_3\phi_2\left[\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-x} \\ \alpha q, q^{-N}\end{matrix}; q, q\right], \quad n = 0,1,\dots,N; \tag{9}$$

this is a polynomial of degree n in q^{-x}. They form an orthogonal system with respect to a measure supported on $\{1, q, \dots, q^{-N}\}$; cf. [KS]. The dual orthogonality relation may be written as an orthogonality relation for a different system of polynomials, namely, the dual q-Hahn polynomials.

We will write

$$D_q f(z) = \frac{f(q^{\frac{1}{2}}z) - f(q^{-\frac{1}{2}}z)}{q^{\frac{1}{2}}z - q^{-\frac{1}{2}}z}, \tag{10}$$

$$f^{[k]}(z) = D_q^k f(z)$$

for the symmetric q-derivative. It satisfies the q-Leibniz rule

$$D_q(fg)(z) = (D_q f)(z) g(q^{-\frac{1}{2}}z) + f(q^{\frac{1}{2}}z)(D_q g)(z). \tag{11}$$

We will need the following simple lemma in a special case.

Lemma 2.1. *For α_{ij} scalars, f_i formal power series and z_i, w_i formal variables, the following identity holds*:

$$\sum_{k_1,\dots,k_n=0}^{\infty} \frac{w_1^{k_1}\cdots w_n^{k_n}}{[k_1]!\cdots[k_n]!} f_1^{[k_1]}(\alpha_{11}^{k_1}\cdots\alpha_{1n}^{k_n} z_1)\cdots f_n^{[k_n]}(\alpha_{n1}^{k_1}\cdots\alpha_{nn}^{k_n} z_n)$$
$$= \sum_{k_1,\dots,k_n=0}^{\infty} \frac{z_1^{k_1}\cdots z_n^{k_n}}{[k_1]!\cdots[k_n]!} f_1^{[k_1]}(\alpha_{11}^{k_1}\cdots\alpha_{n1}^{k_n} w_1)\cdots f_n^{[k_n]}(\alpha_{1n}^{k_1}\cdots\alpha_{nn}^{k_n} w_n).$$

Proof. It suffices to take $f_i(z) = z^{m_i}$. Since then

$$f_i^{[k_i]}(z) = \frac{[m_i]!}{[m_i-k_i]!} z^{m_i-k_i}$$

the left hand side is given by

$$\sum_{k_1,\dots,k_n=0}^{m_1,\dots,m_n} \frac{[m_1]!\cdots[m_n]!}{[k_1]![m_1-k_1]!\cdots[k_n]![m_n-k_n]!} w_1^{k_1}\cdots w_n^{k_n} z_1^{m_1-k_1}\cdots z_n^{m_n-k_n} \prod_{i,j=1}^{n} \alpha_{ij}^{k_j(m_i-k_i)}.$$

Changing k to $m-k$ gives the same expression with z and w interchanged and α_{ij} replaced by α_{ji}. This proves the lemma. □

For any $x \in \mathbb{R}^n$, we will denote by x^* the element in $\mathbb{R}^n$ with coordinates

$$x_i^* = \sum_{j=1}^{i-1} x_j - \sum_{j=i+1}^{n} x_j, \quad i = 1,\dots,n. \tag{12}$$

Then

$$\{x, y\} = \sum_{i=1}^{n} x_i y_i^* = -\sum_{i=1}^{n} x_i^* y_i = \sum_{i>j} x_i y_j - \sum_{i<j} x_i y_j \tag{13}$$

defines a skew-symmetric form on $\mathbb{R}^n$. It appears naturally in connection with multivariable q-series; for instance, one has the multivariable q-Chu–Vandermonde formula (equivalent to Exercise 1.3 of [GR])

$$\frac{[x_1+\cdots+x_n]_k}{[k]!} = \sum_{i_1+\cdots+i_n=k} \frac{[x_1]_{i_1}\cdots[x_n]_{i_n}}{[i_1]!\cdots[i_n]!} q^{\frac{1}{2}\{i,x\}}. \tag{14}$$

2.2. **Quantum algebra.** The quantum algebra (or quantized universal enveloping algebra) $\mathcal{U} = \mathcal{U}_{q^{1/2}}(\mathfrak{su}(1,1))$ is the associative involutive algebra over the complex numbers defined by generators

$$X_+, \quad X_-, \quad K, \quad K^{-1},$$

relations

$$\begin{cases} KK^{-1} = K^{-1}K = 1, \\ KX_{\pm}K^{-1} = q^{\pm\frac{1}{2}} X_{\pm}, \\ X_+X_- - X_-X_+ = \dfrac{K^2 - K^{-2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \end{cases} \tag{15}$$

and involution

$$(K^{\pm})^* = K^{\pm}, \qquad X_{\pm}^* = -X_{\mp}; \tag{16}$$

here and below we write $K^+ = K$, $K^- = K^{-1}$ when convenient. We refer to [CP] for an introduction to quantum group theory. A unitary representation of $\mathcal{U}$ is a representation by densely defined operators on a Hilbert space, such that the involution $*$ coincides with the Hilbert space adjoint.

The algebra $\mathcal{U}$ has the additional structure of a Hopf algebra. We need to discuss only the *coproduct*, which is the map $\Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$ given on the generators by

$$\Delta(K^{\pm}) = K^{\pm} \otimes K^{\pm}, \qquad \Delta(X_{\pm}) = K \otimes X_{\pm} + X_{\pm} \otimes K^{-1}.$$

This means that, given two unitary representations H_1 and H_2 of $\mathcal{U}$, another one is defined on the Hilbert tensor product $H_1 \otimes H_2$ by

$$K^{\pm} = K^{\pm} \otimes K^{\pm}, \qquad X_{\pm} = X_{\pm} \otimes K + K^{-1} \otimes X_{\pm}.$$

The coproduct is non-cocommutative, that is, the flip $f \otimes g \mapsto g \otimes f$ is not an intertwining map $H_1 \otimes H_2 \to H_2 \otimes H_1$. It is, however, coassociative in the sense that there is only one way to repeat it to obtain a representation on a finite tensor product $H_1 \otimes \dots \otimes H_n$, namely, by

$$\begin{aligned} K^{\pm} &= K^{\pm} \otimes \dots \otimes K^{\pm}, \\ X_{\pm} &= X_{\pm} \otimes K \otimes \dots \otimes K + K^{-1} \otimes X_{\pm} \otimes K \otimes \dots \otimes K \\ &\quad + \dots + K^{-1} \otimes \dots \otimes K^{-1} \otimes X_{\pm}. \end{aligned}$$

It is easily proved by induction on k that

$$\begin{aligned} X_{\pm}^k &= \sum_{j_1+\dots+j_n=k} \frac{[k]!}{[j_1]!\cdots[j_n]!} X_{\pm}^{j_1} K^{j_1^*} \otimes \dots \otimes X_{\pm}^{j_n} K^{j_n^*} \\ &= \sum_{j_1+\dots+j_n=k} \frac{[k]!}{[j_1]!\cdots[j_n]!} K^{j_1^*} X_{\pm}^{j_1} \otimes \dots \otimes K^{j_n^*} X_{\pm}^{j_n}, \end{aligned} \tag{17}$$

where $*$ is given by (12).

2.3. **Holomorphic realization.** We will work with the holomorphic realization of highest weight representations of $\mathcal{U}_{q^{1/2}}(\mathfrak{su}(1,1))$; cf. [J, KL, Ko]. A "geometric" motivation for this realization is provided in Appendix 1.

For $\nu > 0$ and $0 < q < 1$, we denote by $\mathcal{A}_q^\nu$ the Hilbert space of analytic functions with the norm

$$\|f\|^2 = \langle f, f\rangle = \sum_{k=0}^{\infty} \frac{[k]!}{[\nu]_k} \, |\hat{f}(k)|^2,$$

where $f(z) = \sum \hat{f}(k) z^k$. By Cauchy's q-binomial formula [GR], $\mathcal{A}_q^\nu$ has the reproducing kernel

$$k_w(z) = \sum_{k=0}^{\infty} \frac{[\nu]_k}{[k]!} \, (z\bar{w})^k = \prod_{j=0}^{\infty} \frac{1 - q^{\frac{1}{2}+\frac{\nu}{2}+j} z\bar{w}}{1 - q^{\frac{1}{2}-\frac{\nu}{2}+j} z\bar{w}}$$

(here we use that $0 < q < 1$) and thus the natural domain of definition is the disc

$$\left\{ z \in \mathbb{C};\ |z| < q^{\frac{\nu}{4}-\frac{1}{4}} \right\}.$$

We write $e_k(z) = z^k$ for monomials, so that

$$\langle f, g\rangle = \sum_{k=0}^{\infty} \frac{[\nu]_k}{[k]!} \, \langle f, e_k\rangle \langle e_k, g\rangle.$$

There is a unitary representation of $\mathcal{U}_q(\mathfrak{su}(1,1))$ on $\mathcal{A}_q^\nu$, given by

$$\begin{cases} X_+ f(z) = -\dfrac{f(q^{\frac{1}{2}}z) - f(q^{-\frac{1}{2}}z)}{q^{\frac{1}{2}}z - q^{-\frac{1}{2}}z}, \\[2ex] X_- f(z) = z\, \dfrac{q^{\frac{\nu}{2}} f(q^{\frac{1}{2}}z) - q^{-\frac{\nu}{2}} f(q^{-\frac{1}{2}}z)}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \\[2ex] K^{\pm} f(z) = q^{\mp\frac{\nu}{4}} f(q^{\mp\frac{1}{2}}z). \end{cases} \tag{18}$$

In the limit $q \to 1$,

$$\begin{cases} X_+ \to -\dfrac{d}{dz}, \\[2ex] X_- \to z^2 \dfrac{d}{dz} + \nu z, \\[2ex] 4\log_q K^{\pm} \to \mp\left(\nu + 2z\dfrac{d}{dz}\right), \end{cases}$$

so we recover the Lie algebra operators (3).

Though we will not use it, we remark that for $\nu > 1$ there is an integral formula for the norm of $\mathcal{A}_q^\nu$ [KL]. In terms of Jackson's q-integral [GR]

$$\int_0^c f(x)\, d_q x = c(1-q) \sum_{k=0}^{\infty} q^k f(q^k c)$$

it can be written as

$$\|f\|^2 = \frac{[\nu-1](1+q^{-\frac{1}{2}})}{2\pi} \int_{-\pi}^{\pi} \int_0^{q^{\frac{1}{4}(\nu-1)}} r|f(re^{i\theta})|^2 \prod_{j=0}^{\infty} \frac{1 - r^2 q^{\frac{1}{2}(3-\nu)+j}}{1 - r^2 q^{\frac{1}{2}(\nu-1)+j}} \, d_{q^{1/2}} r \, d\theta,$$

which is a q-analogue of (1). In contrast to the case $q=1$, this formula extends to the case $0<\nu<1$, though the mass on the outer circle $|z|=q^{\frac{1}{4}(\nu-1)}$ is then negative.

3. Quantum transvectants

In this section we will obtain an expression for the multilinear transvectants in the present context. As in the case $q=1$, there is a decomposition

$$\mathcal{A}_q^{\nu_1}\otimes\cdots\otimes\mathcal{A}_q^{\nu_n}=\bigoplus_{s=0}^{\infty}\binom{n+s-2}{n-2}\mathcal{A}_q^{|\nu|+2s},$$

where $|\nu|=\sum\nu_i$. We define a transvectant of order s to be an intertwining map

$$\mathcal{J}:\ \mathcal{A}_q^{\nu_1}\otimes\cdots\otimes\mathcal{A}_q^{\nu_n}\to\mathcal{A}_q^{|\nu|+2s};$$

thus the transvectants of order s form a linear space of dimension $\binom{n+s-2}{n-2}$.

A highest weight vector is, by definition, a solution in $\bigotimes_{i=1}^n\mathcal{A}_q^{\nu_i}$ to

$$KQ=q^{-\frac{1}{4}(|\nu|+2s)}Q,\qquad X_+Q=0.$$

These are mapped to constants by the transvectants of order s. The first equation means that they are homogeneous polynomials of degree s. For each highest weight vector Q, we denote by $\mathcal{J}_Q$ the transvectant of order s determined by $\mathcal{J}_Q^*1=Q$. This gives a one-to-one correspondence between highest weight vectors and transvectants.

We will now fix Q and seek a q-analogue of the expression (5) for $\mathcal{J}_Q$. It will be convenient to write $\mu=|\nu|+2s$,

$$\langle f,g\rangle_\nu=\langle f,g\rangle_{\mathcal{A}_q^{\nu_1}\otimes\cdots\otimes\mathcal{A}_q^{\nu_n}},\qquad\langle f,g\rangle_\mu=\langle f,g\rangle_{\mathcal{A}_q^{\mu}}.$$

Since

$$e_k=\frac{1}{[\mu]_k}X_-^k1$$

in $\mathcal{A}_q^\mu$, we must have

$$\mathcal{J}_Q^*e_k=\frac{1}{[\mu]_k}X_-^kQ.$$

For f a polynomial (to avoid questions of convergence) in $\bigotimes_{i=1}^n\mathcal{A}_q^{\nu_i}$, it follows that

$$\begin{aligned}\mathcal{J}_Qf(w)&=\langle\mathcal{J}_Qf,k_w\rangle_\mu=\sum_{k=0}^{\infty}\frac{[\mu]_k}{[k]!}w^k\langle\mathcal{J}_Qf,e_k\rangle_\mu=\sum_{k=0}^{\infty}\frac{[\mu]_k}{[k]!}w^k\langle f,\mathcal{J}_Q^*e_k\rangle_\nu\\&=\sum_{k=0}^{\infty}\frac{w^k}{[k]!}\langle f,X_-^kQ\rangle_\nu=\sum_{k=0}^{\infty}\frac{(-1)^kw^k}{[k]!}\langle X_+^kf,Q\rangle_\nu=\langle\tau_wf,Q\rangle_\nu,\end{aligned}$$

where τ_w is the q-translation operator

$$\tau_w=\sum_{k=0}^{\infty}\frac{(-wX_+)^k}{[k]!}.$$

Lemma 3.1. *As an operator on the subspace of polynomials in* $\bigotimes_{i=1}^n \mathcal{A}_q^{\nu_i}$, τ_w *is given by*

$$\begin{aligned}\tau_w(f_1\otimes\cdots\otimes f_n)(z) &= \sum_{k_1,\dots,k_n=0}^{\infty} \frac{w^{k_1+\cdots+k_n}}{[k_1]!\cdots[k_n]!}\, q^{\frac14\{k,\nu\}} f_1^{[k_1]}(q^{-\frac12 k_1^*} z_1)\cdots f_n^{[k_n]}(q^{-\frac12 k_n^*} z_n)\\ &= \sum_{k_1,\dots,k_n=0}^{\infty} \frac{z_1^{k_1}\cdots z_n^{k_n}}{[k_1]!\cdots[k_n]!}\, f_1^{[k_1]}(q^{\frac12 k_1^*+\frac14\nu_1^*}w)\cdots f_n^{[k_n]}(q^{\frac12 k_n^*+\frac14\nu_n^*}w).\end{aligned}$$

Proof. The first expression follows from (17). The second one then follows from Lemma 2.1, in the case when

$$\alpha_{ij} = \begin{cases} q, & i<j,\\ 1, & i=j,\\ q^{-1}, & i>j.\end{cases}$$

□

Plugging the last expression of the lemma into the equality

$$\mathcal{J}_Q(f_1\otimes\cdots\otimes f_n)(w) = \langle \tau_w(f_1\otimes\cdots\otimes f_n), Q\rangle_\nu$$

gives the following explicit expression for the transvectant.

Theorem 3.2. *If*

$$Q(z_1,\dots,z_n) = \sum_{|t|=s} c_t\, z_1^{t_1}\cdots z_n^{t_n} \tag{19}$$

is a highest weight vector, then

$$\mathcal{J}_Q(f_1\otimes\cdots\otimes f_n)(w) = \sum_{|t|=s} \frac{\bar{c}_t}{[\nu_1]_{t_1}\cdots[\nu_n]_{t_n}}\, f_1^{[t_1]}(q^{\frac14\nu_1^*+\frac12 t_1^*}w)\cdots f_n^{[t_n]}(q^{\frac14\nu_n^*+\frac12 t_n^*}w),$$

where $*$ *is defined by* (12).

As an example, for $n=2$ the space of highest weight vectors of degree s is one-dimensional. It is easy to verify that it is spanned by

$$\begin{aligned}Q(z_1,z_2) &= \prod_{j=0}^{s-1}(q^{\frac{\nu_2}{4}+\frac{j}{2}}z_1 - q^{-\frac{\nu_1}{4}-\frac{j}{2}}z_2)\\ &= \sum_{i+j=s}\frac{[s]!}{[i]![j]!}(-1)^j q^{\frac14((s-1)(i-j)+\nu_2 i-\nu_1 j)} z_1^i z_2^j;\end{aligned} \tag{20}$$

cf. also Section 7 and Appendix 2. The corresponding transvectant is

$$\mathcal{J}_Q(f_1\otimes f_2)(w) = \sum_{i+j=s}\frac{[s]!}{[i]![j]!}(-1)^j q^{\frac14((s-1)(i-j)+\nu_2 i-\nu_1 j)} \frac{f_1^{[i]}(q^{-\frac{\nu_2}{4}-\frac{j}{2}}w)f_2^{[j]}(q^{\frac{\nu_1}{4}+\frac{i}{2}}w)}{[\nu_1]_i[\nu_2]_j},$$

which is a q-analogue of (6).

4. Coupling coefficients for group-like elements

To each highest weight vector Q we associate the function P on $\mathbb{Z}_+^n$ given by

$$\mathcal{J}_Q(e_{m_1}\otimes\cdots\otimes e_{m_n})=\overline{P(m_1,\dots,m_n)}\,e_{|m|-s},$$

where s is the degree of Q. Equivalently,

$$\frac{[|\nu|+2s]_k}{[k]!}\,\mathcal{J}_Q^*e_k=\sum_{|m|=k+s}\frac{[\nu_1]_{m_1}\cdots[\nu_n]_{m_n}}{[m_1]!\cdots[m_n]!}\,P(m)\,e_{m_1}\otimes\cdots\otimes e_{m_n}.$$

This exhibits P as a coupling coefficient for the quantum algebra. If Q is given by (19), it follows from Theorem 3.2 that

$$P(m)=\sum_{|t|=s}c_t\,\frac{[-m_1]_{t_1}\cdots[-m_n]_{t_n}}{[\nu_1]_{t_1}\cdots[\nu_n]_{t_n}}(-1)^sq^{\frac14(\nu_1^*+2t_1^*)(m_1-t_1)+\cdots+\frac14(\nu_n^*+2t_n^*)(m_n-t_n)}.$$

Using that $\sum_i x_ix_i^*=0$, one may rewrite the exponent of q as

$$\tfrac18\{\nu+2m,\nu+2t\},$$

where $\{\,,\,\}$ is given by (13). Thus $P=TQ$, where T is the linear operator defined by

$$T(e_{t_1}\otimes\cdots\otimes e_{t_n})(m)=(-1)^{|t|}\frac{[-m_1]_{t_1}\cdots[-m_n]_{t_n}}{[\nu_1]_{t_1}\cdots[\nu_n]_{t_n}}\,q^{\frac18\{\nu+2m,\nu+2t\}}.$$

Now let Q and Q' be two highest weight vectors. Then, by an application of Schur's Lemma to $\mathcal{J}_{Q'}\mathcal{J}_Q^*$,

$$\begin{aligned}\langle Q,Q'\rangle\|e_k\|^2=\langle\mathcal{J}_Q^*e_k,\mathcal{J}_{Q'}^*e_k\rangle&=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{[\nu_1]_{m_1}\cdots[\nu_n]_{m_n}}{[m_1]!\cdots[m_n]!}\,\langle\mathcal{J}_Q^*e_k,e_m\rangle\langle e_m,\mathcal{J}_{Q'}^*e_k\rangle\\&=\sum_{|m|=k+s}\frac{[\nu_1]_{m_1}\cdots[\nu_n]_{m_n}}{[m_1]!\cdots[m_n]!}\,TQ(m)\,\overline{TQ'(m)}\,\|e_k\|^4.\end{aligned}$$

This gives the following orthogonality property.

Theorem 4.1. *For Q and Q' two highest weight vectors in $\bigotimes_{i=1}^n\mathcal{A}_q^{\nu_i}$ and k a non-negative integer,*

$$\sum_{|m|=k+s}\frac{[\nu_1]_{m_1}\cdots[\nu_n]_{m_n}}{[m_1]!\cdots[m_n]!}\,TQ(m)\,\overline{TQ'(m)}=\frac{[|\nu|+2s]_k}{[k]!}\,\langle Q,Q'\rangle,$$

where s is the degree of Q (or of Q').

The function TQ is not a polynomial. To obtain orthogonal polynomials from Theorem 4.1, we introduce the new variables $x_i=q^{-(m_1+m_2+\cdots+m_i)}$, $i=1,\dots,n$. Switching to standard notation (8) and using

$$|m||t|=\{m,t\}+\sum_i m_it_i+2\sum_{i<j}m_it_j \tag{21}$$

we obtain

$$q^{-\frac{1}{2}s|m|-\frac{1}{4}\{m,\nu\}}\,TQ(m_1,\dots,m_n)$$
$$=\sum_{|t|=s} c_t\,(-1)^s q^{\frac{1}{4}\{\nu,t\}+\frac{1}{2}\sum_{i=1}^n \nu_i t_i} x_1^{t_2}\cdots x_{n-1}^{t_n}\frac{(x_1;q)_{t_1}(x_2/x_1;q)_{t_2}\cdots(x_n/x_{n-1};q)_{t_n}}{(q^{\nu_1};q)_{t_1}(q^{\nu_2};q)_{t_2}\cdots(q^{\nu_n};q)_{t_n}},$$

which is a polynomial in the x_i of total degree $|t|$. Given a complete orthogonal (biorthogonal) family of highest weight vectors, Theorem 4.1 gives an orthogonality (biorthogonality) relation for the corresponding polynomials. For fixed k, one may view this as a system of polynomials in the $n-1$ variables $x_1,\dots,x_{n-1}$. By a dimension count, this system will be complete.

In particular, for $n=2$ and Q as in (20) one has

$$TQ(m_1,m_2)$$
$$=q^{\frac{1}{4}(\nu_1 m_2-\nu_2 m_1)}\sum_{i+j=s}\frac{[s]![-m_1]_i[-m_2]_j}{[i]![j]![\nu_1]_i[\nu_2]_j}(-1)^i q^{\frac{1}{4}(s-1)(i-j)+\frac{1}{2}i(\nu_2+m_2)-\frac{1}{2}j(\nu_1+m_1)}.$$

Using transformation formulas from [GR] one may check that

$$q^{-\frac{1}{2}s(m_1+m_2)-\frac{1}{4}(\nu_1 m_2-\nu_2 m_1)}\,TQ(m_1,m_2)$$
$$=q^{\frac{1}{2}s(\nu_2-\nu_1)-\frac{1}{4}s(s-1)}\frac{(q^{-m_1-m_2};q)_s}{(q^{\nu_2};q)_s}\,Q_s(q^{-m_1};q^{\nu_1-1},q^{\nu_2-1},m_1+m_2|q),$$

where the right hand side is a q-Hahn polynomial as defined in (9). Theorem 4.1 then gives the orthogonality relation for q-Hahn polynomials; cf. [LK, S].

5. Coupling kernels

In [R3] we introduced certain functions called *coupling kernels*. We expressed them explicitly as multivariable hypergeometric sums. The discrete coupling kernels, connected with the transform T_1, were used in [R2] to study Wigner $9j$-symbols. The coupling kernels connected with T_2 were independently introduced by Xu [X], who used them to study Cesàro summability of multivariable Jacobi polynomial expansions. In this section we will generalize the explicit expression for discrete coupling kernels to the quantum algebra case.

Just as coupling coefficients are matrix elements for the intertwining maps $\mathcal{J}_Q:\bigotimes_{i=1}^n\mathcal{A}_q^{\nu_i}\to\mathcal{A}_q^{|\nu|+2s}$, coupling kernels are matrix elements for the intertwining projection Π_s from $\bigotimes_{i=1}^n\mathcal{A}_q^{\nu_i}$ onto the isotypic subspace equivalent to $\binom{n+s-2}{n-2}\mathcal{A}_q^{|\nu|+2s}$. More precisely, for any elements X, Y of the quantum algebra, consider the matrix element

$$\langle\Pi_s X e_k, Y e_l\rangle=\langle\Pi_s X(e_{k_1}\otimes\cdots\otimes e_{k_n}),Y(e_{l_1}\otimes\cdots\otimes e_{l_n})\rangle_{\bigotimes_{i=1}^n\mathcal{A}_q^{\nu_i}}$$

of Π_s. Let $(Q_j)_j$ be an orthonormal basis of the space $\mathcal{V}_s$ of highest weight vectors of degree s. Then, by Hilbert space arguments,

$$\Pi_s=\sum_j\mathcal{J}_{Q_j}^*\mathcal{J}_{Q_j}$$

and thus

$$\langle \Pi_s X e_k, Y e_l\rangle = P_s(k,l)\langle X e_{|k|-s}, Y e_{|l|-s}\rangle_{\mathcal{A}_q^{|\nu|+2s}},$$

where

$$P_s(k,l) = \sum_j TQ_j(k)\,\overline{TQ_j(l)}.$$

We call the functions P_s coupling kernels. Since

$$Q_s(z,\bar w) = \sum_j Q_j(z)\overline{Q_j(w)}$$

is the reproducing kernel for the space $\mathcal{V}_s$, it follows from Theorem 4.1 that P_s is the reproducing kernel for the corresponding space of coupling coefficients, with the reproducing property

$$\frac{[|\nu|+2s]_k}{[k]!}\,TQ(l) = \sum_{|m|=k+s} \frac{[\nu_1]_{m_1}\cdots[\nu_n]_{m_n}}{[m_1]!\cdots[m_n]!}\,TQ(m)P_s(l,m)$$

for $k=0,1,2,\dots$ and $Q\in\mathcal{V}_s$. Choosing $Q=Q_{l'}$ so that $TQ(l)=P_s(l,l')$ gives the addition formula

$$\frac{[|\nu|+2s]_k}{[k]!}\,P_s(l,l') = \sum_{|m|=k+s} \frac{[\nu_1]_{m_1}\cdots[\nu_n]_{m_n}}{[m_1]!\cdots[m_n]!}\,P_s(l,m)P_s(m,l')$$

for coupling kernels. Since, again by Hilbert space arguments,

$$Q_s(z,\bar w) = \sum_{|t|=|u|=s} \frac{[\nu]_t[\nu]_u}{[t]![u]!}\,\langle \Pi_s e_t, e_u\rangle\, z^t \bar w^u,$$

we may also write

$$P_s(l,m) = \sum_{|t|=|u|=s} \frac{[-l]_t[-m]_u}{[t]![u]!}\,\langle \Pi_s e_t, e_u\rangle\, q^{\frac18\{\nu+2l,\nu+2t\}+\frac18\{\nu+2m,\nu+2u\}}. \tag{22}$$

We will use this identity to prove the following theorem.

Theorem 5.1. *The coupling kernel P_s is given by*

$$\begin{aligned} P_s(l,m) &= \frac{(-1)^s[-|l|]_s[-|m|]_s}{[|\nu|+s-1]_s[s]!} \\ &\quad\times \sum_{j_1,\dots,j_n=0}^{\infty} \frac{[|\nu|+s-1]_{|j|}[-s]_{|j|}}{[-|l|]_{|j|}[-|m|]_{|j|}}\,\frac{[-l]_j[-m]_j}{[\nu]_j[j]!}\,q^{\frac14\{\nu+l+m,\nu+2j\}}, \end{aligned}$$

with the convention $0/0=0$, so that the sum is actually finite.

We remark that for $n=2$ the space of coupling coefficients is one-dimensional, and therefore its reproducing kernel P_s factors as a product of two q-Hahn polynomials. This gives a quantum algebraic proof of Rahman's Watson-type product formula for the q-Hahn polynomials [Ra2].

The proof of Theorem 5.1 is similar to the case $q=1$ treated in [R3]. It is based on the following projection formula.

Lemma 5.2. *Let $\mathcal{H}_s$ denote the subspace of $\bigotimes_{i=1}^n \mathcal{A}_q^{\nu_i}$ consisting of homogeneous polynomials of degree s. Then the restriction of Π_s to $\mathcal{H}_s$ is given by*

$$\sum_{k=0}^{s} \frac{(-1)^k}{[k]![2-|\nu|-2s]_k} X_-^k X_+^k. \tag{23}$$

Since we failed to do so in [R3], let us give some history here. In the case $q=1$ and the realization of $\mathfrak{su}(1,1)$ coming from spherical harmonics ($X_+ = \Delta$, $X_- = \sum_i x_i^2$ generate a realization of $(\mathcal{A}^{1/2} \oplus \mathcal{A}^{3/2})^{\otimes n}$ on $\mathrm{L}^2(\mathbb{R}^n)$), Lemma 5.2 goes back to Clebsch [C]; cf. also [E]. For $\mathfrak{su}(2)$, it was rediscovered and used by Löwdin [L] in the context of quantum physics. This was the start of so called projection operator methods in quantum physics and representation theory, developped by Asherova, Smirnov and Tolstoy; cf. [AST, STK, T].

Let us indicate the simple proof of Lemma 5.2. Using the fact that $\mathcal{H}_s$ is an eigenspace of $K^{\pm}$ and the elementary identity

$$[a] + [b][a+b+1] = [b+1][a+b]$$

one proofs by induction on k that

$$X_+ X_-^k\big|_{\mathcal{H}_s} = X_-^k X_+ - [k][|\nu| + 2s + k - 1] X_-^{k-1}. \tag{24}$$

We note that the restriction of Π_s to $\mathcal{H}_s$ is the orthogonal projection onto the subspace $\mathcal{V}_s$ of highest weight vectors. Denoting the operator in (23) by A, it follows from (24) that $X_+ A = 0$, so that $A\mathcal{H}_s \subseteq \mathcal{V}_s$. On the other hand, the image of $\mathrm{Id} - A$ is in $X_-\mathcal{H}_{s-1}$, which is the orthogonal complement of $\mathcal{V}_s$ in $\mathcal{H}_s$. Thus A is indeed the orthogonal projection onto $\mathcal{V}_s$.

Now let t and u be two multi-indices of length s. We want to compute the scalar product $\langle \Pi_s e_t, e_u\rangle$ occurring in (22). Since $X_-^* = -X_+$, Lemma 5.2 gives

$$\langle \Pi_s e_t, e_u\rangle = \sum_{k=0}^{s} \frac{1}{[k]![2-|\nu|-2s]_k} \langle X_+^k e_t, X_+^k e_u\rangle.$$

Now, by (17),

$$X_+^k e_t = \sum_{|j|=k} \frac{[k]!}{[j]!} [-t]_j \, q^{\frac14\{j,\nu+2t\}} e_{t-j} = (-1)^{|t|} \sum_{|j|=|t|-k} \frac{[k]!}{[j]!} [-t]_j \, q^{\frac14\{t-j,\nu+2t\}} e_j,$$

which gives

$$\langle \Pi_s e_t, e_u\rangle = \sum_j \frac{[s-|j|]!}{[2-|\nu|-2s]_{s-|j|}} \frac{[-t]_j[-u]_j}{[j]![\nu]_j} q^{\frac14\{t-j,\nu+2t\}+\frac14\{u-j,\nu+2u\}}.$$

Inserting this expression in (22) gives

$$P_s(l,m) = \sum_{|t|=|u|=s} \sum_j \frac{[s-|j|]!}{[2-|\nu|-2s]_{s-|j|}} \frac{[-l]_t[-m]_u}{[t]![u]!} \frac{[-t]_j[-u]_j}{[j]![\nu]_j}$$
$$\times\, q^{\frac14\{t-j,\nu+2t\}+\frac14\{u-j,\nu+2u\}+\frac18\{\nu+2l,\nu+2t\}+\frac18\{\nu+2m,\nu+2u\}}.$$

Let us replace t, u by $t+j$, $u+j$. The exponent of q may then be expressed as

$$\tfrac14\{\nu+l+m,\nu+2j\} + \tfrac12\{t, j-l\} + \tfrac12\{u, j-m\}.$$

Thus, changing the order of summation, we obtain

$$P_s(l,m)=\sum_j \frac{[s-|j|]!}{[2-|\nu|-2s]_{s-|j|}}\,\frac{[-l]_j[-m]_j}{[j]![\nu]_j}\,q^{\frac14\{\nu+l+m,\nu+2j\}}$$
$$\times\sum_{|t|=s-|j|}\frac{[j-l]_t}{[t]!}\,q^{\frac12\{t,j-l\}}\sum_{|u|=s-|j|}\frac{[j-m]_u}{[u]!}\,q^{\frac12\{u,j-m\}},$$

where the inner sums are of the form (14). This leads to the expression

$$P_s(l,m)=\sum_j\frac{[|j|-|l|]_{s-|j|}[|j|-|m|]_{s-|j|}}{[2-|\nu|-2s]_{s-|j|}[s-|j|]!}\,\frac{[-l]_j[-m]_j}{[j]![\nu]_j}\,q^{\frac14\{\nu+l+m,\nu+2j\}},$$

which may be rewritten as in Theorem 5.1.

6. Convolution and linearization formulas

In the case $q=1$, matrix elements for the group action in the basis of monomials are given by Meixner polynomials [KS]

$$M_n(k;\nu,c)={}_2F_1\left(\begin{matrix}-n,-k\\ \nu\end{matrix}\,\middle|\,1-\frac1c\right),$$

where ${}_2F_1$ is Gauss' hypergeometric function. The interpretation of the polynomials T_1Q as coupling coefficients leads to the convolution formula

$$\frac{(|\nu|+2s)_k}{k!}\,T_1Q(l)\,c^s\,{}_2F_1\left(\begin{matrix}-k,s-|l|\\ |\nu|+2s\end{matrix}\,\middle|\,c\right)$$
$$=\sum_{|m|=k+s}\frac{(\nu_1)_{m_1}\cdots(\nu_n)_{m_n}}{m_1!\cdots m_n!}\,T_1Q(m)\prod_{j=1}^n{}_2F_1\left(\begin{matrix}-m_j,-l_j\\ \nu_j\end{matrix}\,\middle|\,c\right)$$

and the linearization formula

$$\prod_{j=1}^n{}_2F_1\left(\begin{matrix}-k_j,-l_j\\ \nu_j\end{matrix}\,\middle|\,c\right)=\sum_{s=0}^{\min(|k|,|l|)}P_s(k,l)\,c^s\,{}_2F_1\left(\begin{matrix}s-|k|,s-|l|\\ |\nu|+2s\end{matrix}\,\middle|\,c\right)$$

for Meixner polynomials, cf. [R3]. In this section we will generalize these formulas to the present setting.

Let τ_λ be the q-translation operator occurring in Lemma 3.1. We will consider the matrix element

$$\langle\tau_\lambda e_k,\tau_\mu e_l\rangle=\langle\tau_\lambda(e_{k_1}\otimes\cdots\otimes e_{k_n}),\tau_\mu(e_{l_1}\otimes\cdots\otimes e_{l_n})\rangle_{\bigotimes_{i=1}^n\mathcal{A}_q^{\nu_i}},$$

which, by Lemma 3.1, equals

$$\sum_{j_1,\dots,j_n=0}^{\infty}\frac{[-k]_j[-l]_j}{[j]![\nu]_j}\,q^{\frac14\{\nu+k+l,\nu+2j\}}\lambda^{|k|-|j|}\mu^{|l|-|j|}\tag{25}$$
$$=q^{\frac14\{k+l,\nu\}}\lambda^{|k|}\mu^{|l|}\prod_{j=1}^n{}_2\phi_1\left[\begin{matrix}q^{-k_j},q^{-l_j}\\ q^{\nu_j}\end{matrix};q,\frac{q^{\frac12(\nu_j+k_j+l_j+1-\nu_j^*-k_j^*-l_j^*)}}{\lambda\mu}\right].$$

One can give a motivation for this choice of matrix elements in terms of the algebra of functions on the quantum $\mathrm{SU}(1,1)$ group; cf. [K, MM2, VS] for the case of quantum SU(2) and [MM1] for the present case. Here we use the more pedestrian approach of [FV2, KMM, Zh], considering q-exponentials of quantum algebra elements as generalized group elements. See [FV1] for the relation between these approaches (the quantum group and the quantum algebra approach) to q-special functions. We remark that we would have obtained the same results considering instead the matrix elements

$$\langle e_q(-\lambda X_+K)e_k, E_q(-\mu X_+K^{-1})e_l\rangle_{\bigotimes_{i=1}^n \mathcal{A}_q^{\nu_i}},$$

where e_q and E_q are the q-exponential functions [GR]

$$e_q(z)=\sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n}, \qquad E_q(z)=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}z^n}{(q;q)_n}.$$

Consider the identity

$$\langle \tau_\lambda \mathcal{J}_Q^* e_k, \tau_\mu e_l\rangle = \sum_{m_1,\dots,m_n=0}^{\infty}\frac{[\nu_1]_{m_1}\cdots[\nu_n]_{m_n}}{[m_1]!\cdots[m_n]!}\,\langle \mathcal{J}_Q^* e_k, e_m\rangle\langle \tau_\lambda e_m, \tau_\mu e_l\rangle. \tag{26}$$

Since τ_λ is a lowering operator there is no problem with convergence. Since τ_λ commutes with $\mathcal{J}_Q^*$, the left hand side equals

$$TQ(l)\,\langle \tau_\lambda e_k, \tau_\mu e_{|l|-s}\rangle_{\mathcal{A}_q^{|\nu|+2s}};$$

in particular

$$\langle \mathcal{J}_Q^* e_k, e_m\rangle = TQ(m)\langle e_k, e_{|m|-s}\rangle_{\mathcal{A}_q^{|\nu|+2s}} = TQ(m)\frac{[k]!}{[|\nu|+2s]_k}\,\delta_{|m|,k+s}.$$

Inserting the expression (25) for matrix elements and letting $c=1/\lambda\mu$ we obtain the following proposition.

Proposition 6.1. *For Q a highest weight vector in $\bigotimes_{i=1}^n \mathcal{A}_q^{\nu_i}$ of degree s, and for k a non-negative integer, one has the identity*

$$\begin{aligned}&\frac{[|\nu|+2s]_k}{[k]!}\,TQ(l)\,c^s\,{}_2\phi_1\left[\begin{matrix}q^{-k},q^{s-|l|}\\ q^{|\nu|+2s}\end{matrix};q,q^{\frac12(|\nu|+k+|l|+s+1)}c\right]\\ &\quad=\sum_{|m|=k+s}\frac{[\nu]_m}{[m]!}\,q^{\frac14\{m+l,\nu\}}\,TQ(m)\prod_{j=1}^n {}_2\phi_1\left[\begin{matrix}q^{-m_j},q^{-l_j}\\ q^{\nu_j}\end{matrix};q,q^{\frac12(\nu_j+m_j+l_j+1-\nu_j^*-m_j^*-l_j^*)}c\right].\end{aligned}$$

For $n=2$ this is a degenerate case of identities proved in [KV]. We must point out that it is possible to deduce the general case from the case $n=2$ by choosing a basis in the space of highest weight vectors constructed by binary coupling; cf. Section 7. We also remark that for $\lambda=0$, (26) takes the form

$$\frac{[s-|l|]_k}{[k]!}\,TQ(l)=(-1)^s\sum_{|m|=k+s}\frac{[-l]_m}{[m]!}\,q^{\frac18\{\nu+2l,\nu+2m\}}\,TQ(m).$$

A linearization formula may be obtained by considering the identity

$$\langle \tau_\lambda e_k, \tau_\mu e_l\rangle = \sum_{s=0}^{\infty}\langle \Pi_s\tau_\lambda e_k, \tau_\mu e_l\rangle = \sum_{s=0}^{\min(|k|,|l|)} P_s(k,l)\langle \tau_\lambda e_{|k|-s}, \tau_\mu e_{|l|-s}\rangle_{\mathcal{A}_q^{|\nu|+2s}}. \tag{27}$$

This leads to the following proposition.

Proposition 6.2. *One has the identity*

$$q^{\frac{1}{4}\{k+l,\nu\}}\prod_{j=1}^{n}{}_2\phi_1\left[\begin{matrix}q^{-k_j},q^{-l_j}\\ q^{\nu_j}\end{matrix};q,q^{\frac{1}{2}(\nu_j+k_j+l_j+1-\nu_j^*-k_j^*-l_j^*)}c\right] = \sum_{s=0}^{\min(|k|,|l|)}P_s(k,l)\,c^s\,{}_2\phi_1\left[\begin{matrix}q^{s-|k|},q^{s-|l|}\\ q^{|\nu|+2s}\end{matrix};q,q^{\frac{1}{2}(|\nu|+|k|+|l|+1)}c\right]. \tag{28}$$

Inserting the expression for coupling kernels given in Theorem 5.1, this is not hard to prove directly. It can be obtained as a special case of Verma's expansion formula [Ve], cf. also [Sr]. This gives an alternative proof of Theorem 5.1, since the expansion (28) determines the coefficients P_s. For $n=2$, factoring P_s as a product of two Clebsch–Gordan coefficients, this linearization formula occurs in [KMM, S, V] in a similar context.

The special case $\lambda=0$ of (27) is

$$\frac{[-l]_k}{[\nu]_k}\,q^{\frac{1}{8}\{\nu+2l,\nu+2k\}}=(-1)^s\sum_{s=0}^{\min(|k|,|l|)}\frac{[s-|l|]_{|k|-s}}{[|\nu|+2s]_{|k|-s}}\,P_s(k,l),$$

which, for $|l|=|k|$, reduces to

$$\frac{[k]!}{[\nu]_k}\,\delta_{k,l}=\sum_{s=0}^{|k|}\frac{[|k|-s]!}{[|\nu|+2s]_{|k|-s}}\,P_s(k,l).$$

For $n=2$, factoring P_s as a product of two terminating ${}_3\phi_2$ series, this is the orthogonality relation for dual q-Hahn polynomials.

7. Examples

In this section we will consider some examples of orthogonal and biorthogonal multivariable Hahn polynomials which may be obtained by our method.

One way to construct orthogonal bases in the space of coupling coefficients is by binary coupling. This leads to multivariable polynomials which may be factored as products of q-Hahn polynomials. For q a power of a prime, such polynomials arise in connection with finite fields [D]. In the limit $q\to 1$, the corresponding polynomials are of importance in the quantum theory of angular momentum, and they have also been used in certain stochastic models in genetics [KM]. To obtain them as coupling coefficients, we first note that if $Q_1\in\bigotimes_{i=1}^k\mathcal{A}_q^{\nu_i}$ and $Q_2\in\bigotimes_{i=k+1}^n\mathcal{A}_q^{\nu_i}$ are highest weight vectors of degrees t_1, t_2 and weights $-\mu_1$, $-\mu_2$, respectively, and Q_s is a highest weight vector of degree s in $\mathcal{A}_q^{\mu_1}\otimes\mathcal{A}_q^{\mu_2}$ (unique up to a multiplicative constant) then the equation

$$\mathcal{J}_{(Q_1,Q_2)_s}(f_1\otimes\cdots\otimes f_n)=\mathcal{J}_{Q_s}(\mathcal{J}_{Q_1}(f_1\otimes\cdots\otimes f_k)\otimes\mathcal{J}_{Q_2}(f_{k+1}\otimes\cdots\otimes f_n))$$

determines a highest weight vector $(Q_1,Q_2)_s\in\bigotimes_{i=1}^n\mathcal{A}_q^{\nu_i}$. Choosing $f_i(z)=z^{m_i}$ gives

$$T(Q_1,Q_2)_s(m)=TQ_1(m')\,TQ_2(m'')\,TQ_s(|m'|-t_1,|m''|-t_2), \tag{29}$$

where $m'=(m_1,\dots,m_k)$, $m''=(m_{k+1},\dots,m_n)$. Moreover, one has the identity

$$\langle (Q_1,Q_2)_s,(Q_3,Q_4)_t\rangle=\delta_{st}\,\|Q_s\|^2\langle Q_1,Q_3\rangle\langle Q_2,Q_4\rangle.$$

As an example, the polynomials

$$Q_{qr} = ((1,1)_q, 1)_r, \quad q+r=s$$

form an orthogonal basis in the space of highest weight vectors of degree s in a threefold tensor product $\mathcal{A}_q^{\nu_1} \otimes \mathcal{A}_q^{\nu_2} \otimes \mathcal{A}_q^{\nu_3}$. Applying the transform T gives a system of orthogonal polynomials which, by (29), factor as products of two q-Hahn polynomials. The orthogonality relation, as well as the convolution formula in Proposition 6.1, then follow from the corresponding identities for q-Hahn polynomials. The coefficients

$$\langle ((1,1)_q,1)_r, (1,(1,1)_t)_u \rangle$$

are Racah coefficients, or Wigner $6j$-symbols, for $\mathcal{U}_{q^{1/2}}(\mathfrak{su}(1,1))$, which may be identified with q-Racah polynomials [KR]. For an $(n+1)$-fold tensor product, there is a large number of ways to construct orthogonal bases by binary coupling, and the coefficients for a change between two such bases are given by Wigner $3nj$-symbols; cf. [BL].

As a different example of coupling coefficients, we will construct q-analogues of certain biorthogonal multivariable Hahn polynomials introduced by Rahman [Ra1] for $n=3$ and by Tratnik [Tr] in general. For Jacobi-type weights, this kind of polynomials were first studied by Appell; cf. [AK]. In the case $q=1$, they correspond to highest weight vectors of the form

$$(z_1 - z_n)^{t_1} \cdots (z_{n-1} - z_n)^{t_{n-1}}$$

and to the dual basis.

It will be convenient to write $\mathcal{D}_q$ (for Dirichlet, cf. Appendix 1) for the space of polynomials in one variable, viewed as an $\mathcal{U}$-module with the action

$$\begin{cases} X_+ z^k = -[k]\, z^{k-1}, \\ X_- z^k = [k]\, z^{k+1}, \\ K^{\pm} z^k = q^{\mp\frac{1}{2}k} z^k, \end{cases}$$

which may be obtained by formally letting $\nu = 0$ in (18). Let us write δ for the dilation operator $\delta f(z) = f(qz)$, so that $K = q^{-\nu/4}\delta^{-1/2}$ as an operator on $\mathcal{A}_q^{\nu}$. Thus

$$-X_+\big|_{\bigotimes_{i=1}^n \mathcal{A}_q^{\nu_i}} = q^{\frac{1}{4}\nu_1^*} D_q \otimes \delta^{-\frac{1}{2}} \otimes \cdots \otimes \delta^{-\frac{1}{2}} + \cdots + q^{\frac{1}{4}\nu_n^*} \delta^{\frac{1}{2}} \otimes \cdots \otimes \delta^{\frac{1}{2}} \otimes D_q,$$

where D_q is the q-derivative (10), while

$$-X_+\big|_{\mathcal{D}_q^{\otimes n}} = D_q \otimes \delta^{-\frac{1}{2}} \otimes \cdots \otimes \delta^{-\frac{1}{2}} + \cdots + \delta^{\frac{1}{2}} \otimes \cdots \otimes \delta^{\frac{1}{2}} \otimes D_q.$$

Applying the commutation rule

$$D_q\, \delta^{\lambda} = q^{\lambda} \delta^{\lambda} D_q$$

gives the following lemma.

Lemma 7.1. *The operators* $X_+\big|_{\bigotimes_{i=1}^n \mathcal{A}_q^{\nu_i}}$ *and* $X_+\big|_{\mathcal{D}_q^{\otimes n}}$ *are connected by*

$$X_+\big|_{\bigotimes_{i=1}^n \mathcal{A}_q^{\nu_i}} = S^{-1} X_+\big|_{\mathcal{D}_q^{\otimes n}} S,$$

where

$$S = \delta^{\frac{1}{4}\nu_1^*} \otimes \cdots \otimes \delta^{\frac{1}{4}\nu_n^*}.$$

In particular, this shows that if $\mathcal{V}_s$ is the space of highest weight vectors of degree s in $\bigotimes_{i=1}^n \mathcal{A}_q^{\nu_i}$, then $S\mathcal{V}_s$ is independent of the parameters ν_i.

Lemma 7.2. *Let j be an integer with $1 \leq j \leq n$. If f and g are two elements in $\mathcal{D}_q^{\otimes n}$ such that f depends only on the variables $z_1, \dots, z_j$ and g only on $z_j, \dots, z_n$, then*

$$X_+(fg) = X_+ f K g + K^{-1} f X_+ g.$$

Proof. For $n = 1$ this is (11). To prove it in general, decompose $\mathcal{D}_q^{\otimes n}$ as $\mathcal{D}_q^{\otimes (j-1)} \otimes \mathcal{D}_q \otimes \mathcal{D}_q^{\otimes (n-j)}$. By linearity, one may assume $f = f_1 \otimes f_2 \otimes 1$, $g = 1 \otimes g_2 \otimes g_3$. The lemma then follows from a computation, using the case $n = 1$ and the coproduct rule

$$X_+ = X_+ \otimes K \otimes K + K^{-1} \otimes X_+ \otimes K + K^{-1} \otimes K^{-1} \otimes X_+.$$

□

The lemma also follows from the results of Appendix 2, where we describe a product $\bullet$ on $\mathcal{D}_q^{\otimes n}$ which satisfies

$$X_+(f \bullet g) = X_+ f \bullet K g + K^{-1} f \bullet X_+ g$$

and which is easily seen to agree with the usual product of polynomials under the assumptions of the lemma.

Proposition 7.3. *For any multi-index $t = (t_1, \dots, t_{n-1})$, the polynomial*

$$\tilde{Q}_t(z) = \prod_{j=0}^{t_1-1} (z_1 - q^{-(j+t_2+\dots+t_{n-1})} z_n) \prod_{j=0}^{t_2-1} (z_2 - q^{-(j+t_3+\dots+t_{n-1})} z_n) \cdots \prod_{j=0}^{t_{n-1}-1} (z_{n-1} - q^{-j} z_n)$$

is annihilated by $X_+\big|_{\mathcal{D}_q^{\otimes n}}$, and thus

$$Q_t(z) = q^{\frac{1}{4}(\nu_1^* t_1 + \dots + \nu_{n-1}^* t_{n-1})} S^{-1} \tilde{Q}_t(z) = q^{\frac{1}{4}\{(t,0),\nu\}} \tilde{Q}_t(q^{-\frac{1}{4}\nu_1^*} z_1, \dots, q^{-\frac{1}{4}\nu_n^*} z_n)$$

is a highest weight vector in $\bigotimes_{i=1}^n \mathcal{A}_q^{\nu_i}$.

Note that

$$Q_t\Big|_{z_n=0} = z_1^{t_1} \cdots z_{n-1}^{t_{n-1}}. \tag{30}$$

Since the dimension of the space of monomials of degree s in $n-1$ variables equals the dimension $\binom{n+s-2}{n-2}$ of the space of highest weight vectors of degree s, the polynomials $(Q_t)_{|t|=s}$ form a basis of the latter space. In particular, it follows that Q_t is the unique highest weight vector satisfying (30).

Proof of Proposition 7.3. Let j be the smallest number such that $t_j \neq 0$. Then

$$\tilde{Q}_t(z_1, \dots, z_n) = (z_j - q^{1-|t|} z_n)\, \tilde{Q}_{t'}(z_1, \dots, z_n),$$

where

$$t' = (0, \dots, 0, t_j - 1, t_{j+1}, \dots, t_n).$$

Proceeding by induction on $|t|$, we may assume that $X_+ \tilde{Q}_{t'} = 0$. Since $\tilde{Q}_{t'}$ depends only on the variables $z_j, \dots, z_n$, Lemma 7.2 gives

$$\begin{aligned} X_+ \tilde{Q}_t &= X_+(z_j\, \tilde{Q}_{t'}) - q^{1-|t|} X_+(\tilde{Q}_{t'}\, z_n) \\ &= X_+ z_j\, K \tilde{Q}_{t'} + K^{-1} z_j\, X_+ \tilde{Q}_{t'} - q^{1-|t|} X_+ \tilde{Q}_{t'}\, K z_n - q^{1-|t|} K^{-1} \tilde{Q}_{t'}\, X_+ z_n. \end{aligned}$$

Since $X_+\tilde{Q}_{t'} = 0$, $X_+ z_j = X_+ z_n = -1$ and $K^{\pm}\tilde{Q}_{t'} = q^{\mp\frac{1}{2}(|t|-1)}\tilde{Q}_{t'}$, it follows that $X_+\tilde{Q}_t = 0$. $\square$

To obtain an expression for the coupling coefficients $P_t = TQ_t$, we expand Q into monomials. Using the terminating q-binomial formula

$$\prod_{j=0}^{s-1}(x - q^{-j}y) = (-1)^s \sum_{l=0}^{s} \frac{[-s]_l}{[l]!}\, q^{-\frac{1}{2}(s-l)(s-1)}\, x^l y^{s-l}$$

one obtains

$$\tilde{Q}_t(z) = (-1)^s \sum_{l_1,\dots,l_{n-1}} \frac{[-t_1]_{l_1}\cdots[-t_{n-1}]_{l_{n-1}}}{[l_1]!\cdots[l_{n-1}]!} \times q^{-\frac{1}{2}\sum_i (t_i-l_i)(t_i-1) - \sum_{i<j}(t_i-l_i)t_j}\, z_1^{l_1}\cdots z_{n-1}^{l_{n-1}} z_n^{s-l_1-\cdots-l_{n-1}},$$

where $s = |t|$. Let us put $l_n = s - l_1 - \cdots - l_{n-1}$. Using (21) one may then rewrite the exponent of q as

$$\tfrac{1}{2}\, l_n + \tfrac{1}{2}\{(t,0), l\}.$$

Writing also

$$\tfrac{1}{2}\{(t,0), l\} + \tfrac{1}{4}\{(t,0), \nu\} + \tfrac{1}{4}\{\nu, l\} = \tfrac{1}{8}\{\nu + 2(t,0), \nu + 2l\}$$

gives

$$Q_t(z) = (-1)^s \sum_{l_1+\cdots+l_n=s} \frac{[-t_1]_{l_1}\cdots[-t_{n-1}]_{l_{n-1}}}{[l_1]!\cdots[l_{n-1}]!}\, q^{\frac{1}{2}l_n + \frac{1}{8}\{\nu+2(t,0),\nu+2l\}}\, z_1^{l_1}\cdots z_n^{l_n}$$

and thus

$$P_t(m) = \sum_{l_1+\cdots+l_n=s} \frac{[-t_1]_{l_1}\cdots[-t_{n-1}]_{l_{n-1}}}{[l_1]!\cdots[l_{n-1}]!} \times \frac{[-m_1]_{l_1}\cdots[-m_n]_{l_n}}{[\nu_1]_{l_1}\cdots[\nu_n]_{l_n}}\, q^{\frac{1}{2}l_n + \frac{1}{4}\{\nu+m+(t,0),\nu+2l\}}. \tag{31}$$

Next we consider the dual basis of $(P_t)_{|t|=s}$. It can be expressed in terms of coupling kernels. To see this we first observe that, because of (30),

$$\langle Q_t, z_1^{u_1}\cdots z_{n-1}^{u_{n-1}}\rangle = \frac{[t_1]!\cdots[t_{n-1}]!}{[\nu_1]_{t_1}\cdots[\nu_{n-1}]_{t_{n-1}}}\,\delta_{t,u}$$

for arbitrary u_i. Thus the dual basis (Q'_t) of (Q_t) is given by

$$Q'_t = \frac{[\nu_1]_{t_1}\cdots[\nu_{n-1}]_{t_{n-1}}}{[t_1]!\cdots[t_{n-1}]!}\,\Pi_s(z_1^{t_1}\cdots z_{n-1}^{t_{n-1}}),$$

where, as above, Π_s is the orthogonal projection onto the space of highest weight vectors. It follows from (22) that, in general,

$$T\Pi_s(z_1^{t_1}\cdots z_n^{t_n})(m) = P_s(t,m), \quad |t| = s,$$

which gives

$$P'_t(m) = TQ'_t(m) = \frac{[\nu_1]_{t_1}\cdots[\nu_{n-1}]_{t_{n-1}}}{[t_1]!\cdots[t_{n-1}]!}\, P_s(t_1,\dots,t_{n-1},0,m_1,\dots,m_n).$$

We insert the explicit expression for P_s from Theorem 5.1. Then the summation variable j_n may be put equal to zero. Writing $\hat{x}=(x_1,\dots,x_{n-1})$ for $x\in\mathbb{R}^n$, we obtain

$$\begin{aligned}P_t'(m)&=\frac{[-|m|]_s}{[|\nu|+s-1]_s}\,\frac{[\hat{\nu}]_t}{[t]!}\\ &\quad\times\sum_{j_1,\dots,j_{n-1}=0}^{\infty}\frac{[|\nu|+s-1]_{|j|}}{[-|m|]_{|j|}}\,\frac{[-t]_j[-\hat{m}]_j}{[\hat{\nu}]_j[j]!}\,q^{\frac14\{\nu+m+(t,0),\nu+2(j,0)\}}\end{aligned}\tag{32}$$

(which should be interpreted as a finite sum, as in Theorem 5.1). Thus, as a consequence of Theorem 4.1, we have the following fact.

Theorem 7.4. *The coupling coefficients $P_t(m)$ and $P_u'(m)$ given by* (31) *and* (32), *respectively, satisfy the relations*

$$\sum_{|m|=s+k}\frac{[\nu_1]_{m_1}\cdots[\nu_n]_{m_n}}{[m_1]!\cdots[m_n]!}\,P_t(m)P_u'(m)=\frac{[|\nu|+2s]_k}{[k]!}\,\delta_{t,u},$$

where $s=|t|$ (or $s=|u|$).

As indicated in Section 4, we may view this as a biorthogonality relation for $(n-1)$-variable polynomials after fixing $|m|$. It may be worth writing this out explicitly, in standard notation. To facilitate comparison with the one-variable case, we view the q^{ν_i-1} as parameters and normalize the polynomials so that they take the value 1 at $m_1=\dots=m_{n-1}=0$. Thus, we write

$$\frac{q^{-\frac12|t||m|-\frac14\{m,\nu\}}P_t(m)}{q^{-\frac12|t||m|-\frac14|m||\hat{\nu}|}P_t(0,|m|)}=p_t(q^{-m_1},q^{-(m_1+m_2)},\dots,q^{-(m_1+\dots+m_{n-1})};q^{\nu_1-1},\dots,q^{\nu_n-1},|m|;q),$$

and p_t' for the polynomials similarly obtained from P_t'. Replacing n by $n+1$, and writing $\Phi^{1:2}_{1:1}$ for the q-Kampé de Fériet function

$$\begin{aligned}&\Phi^{1:2}_{1:1}\left(\begin{matrix}a\\ d\end{matrix}:\begin{matrix}b_1,c_1\\ e_1\end{matrix};\dots;\begin{matrix}b_n,c_n\\ e_n\end{matrix};q\,;\,x_1,\dots,x_n\right)\\ &\qquad=\sum_{k_1,\dots,k_n=0}^{\infty}\frac{(a;q)_{|k|}}{(d;q)_{|k|}}\,\frac{(b_1;q)_{k_1}(c_1;q)_{k_1}}{(q;q)_{k_1}(e_1;q)_{k_1}}\cdots\frac{(b_n;q)_{k_n}(c_n;q)_{k_n}}{(q;q)_{k_n}(e_n;q)_{k_n}}\,x_1^{k_1}\cdots x_n^{k_n},\end{aligned}$$

we have

$$\begin{aligned}&p_t(x_1,\dots,x_n;a_1,\dots,a_n,b,N;q)=x_n^{|t|}\frac{(x_n^{-1}q^{-N};q)_{|t|}}{(q^{-N};q)_{|t|}}\\ &\quad\times\Phi^{1:2}_{1:1}\left(\begin{matrix}b^{-1}q^{-|t|}\\ x_nq^{1+N-|t|}\end{matrix}:\begin{matrix}q^{-t_1},x_1\\ a_1q\end{matrix};\begin{matrix}q^{-t_2},x_2/x_1\\ a_1q\end{matrix};\dots;\begin{matrix}q^{-t_n},x_n/x_{n-1}\\ a_nq\end{matrix};q\,;\right.\\ &\qquad\left.a_1\cdots a_nbq^{N+|t|+n+1},a_2\cdots a_nbx_1q^{N+|t|-t_1+n},\dots,a_nbx_{n-1}q^{N+t_n+2}\right)\end{aligned}$$

and

$$p'_t(x_1,\dots,x_n;a_1,\dots,a_n,b,N;q)=\Phi^{1:2}_{1:1}\left(\begin{matrix}a_1\cdots a_nbq^{|t|+n}\\ q^{-N}\end{matrix}:\right.$$

$$\left.\begin{matrix}q^{-t_1},x_1\\ a_1q\end{matrix};\begin{matrix}q^{-t_2},x_2/x_1\\ a_1q\end{matrix};\dots;\begin{matrix}q^{-t_n},x_n/x_{n-1}\\ a_nq\end{matrix};q\,;q,\frac{x_1}{a_1}\,q^{-t_1},\dots,\frac{x_{n-1}}{a_1\cdots a_{n-1}}\,q^{2-n-|t|+t_n}\right).$$

In terms of these polynomials, Theorem 7.4 may be rewritten as

$$\begin{aligned}&\sum_{0\le m_1\le\cdots\le m_n\le N}\frac{(a_1q;q)_{m_1}(a_2q;q)_{m_2-m_1}\cdots(a_nq;q)_{m_n-m_{n-1}}(bq;q)_{N-m_n}}{(q;q)_{m_1}(q;q)_{m_2-m_1}\cdots(q;q)_{m_n-m_{n-1}}(q;q)_{N-m_n}}\prod_{j=1}^n(a_jq)^{m_j}\\ &\quad\times p_t(q^{-m_1},\dots,q^{-m_n};a_1,\dots,a_n,b,N;q)\,p'_u(q^{-m_1},\dots,q^{-m_n};a_1,\dots,a_n,b,N;q)\\ &=\delta_{tu}\,\frac{(q;q)_{t_1}\cdots(q;q)_{t_n}}{(a_1q;q)_{t_1}\cdots(a_nq;q)_{t_n}}\frac{(bq;q)_{|t|}}{(q^{-N};q)_{|t|}}\frac{(a_1\cdots a_nbq^{n+|t|};q)_{N+1}}{(q;q)_N}\frac{1}{1-a_1\cdots a_nbq^{n+2|t|}}\\ &\quad\times(-1)^{|t|}q^{\binom{|t|}{2}-N|t|}\,(a_1\cdots a_nq^n)^{-N}\prod_{1\le i\le j\le n}(a_jq)^{t_i}\,.\end{aligned}$$

Appendix 1. Quasi-commuting homogeneous coordinates

In this appendix we will use quasi-commuting homogeneous coordinates to give a "geometric" motivation for the holomorphic realization, where geometric may be understood in the sense of non-commutative geometry; cf. [Ma]. Let us first recall the classical case. If f is a function defined in the unit disc, one may (at least for integer ν) introduce a function F defined in the cone $|x|<|y|$ in $\mathbb{C}^2$ by

$$F(x,y)=\frac{1}{y^\nu}\,f\left(\frac{x}{y}\right).$$

The natural right action on F by $\mathrm{SU}(1,1)$,

$$F(x,y)\mapsto F(ax+by,cx+dy),\qquad\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SU}(1,1)$$

then gives rise to the action (2) on f.

In the quantum case we must replace the coordinate functions x, y by *quasi-commuting* elements, that is, quantities satisfying $xy=q^{1/2}yx$. Since we need fractional powers, we let $\mathcal{F}_q$ be the associative algebra over $\mathbb{C}$ generated by x^α, y^α, $\alpha\in\mathbb{R}$, with relations

$$x^0=y^0=1,\qquad x^\alpha x^\beta=x^{\alpha+\beta},\qquad y^\alpha y^\beta=y^{\alpha+\beta},\qquad x^\alpha y^\beta=q^{\frac12\alpha\beta}y^\beta x^\alpha.$$

To proceed, we need a q-analogue of the action of $\mathrm{SL}(2,\mathbb{C})$ on functions on $\mathbb{C}^2$. This will be the action of $\mathcal{U}$ on $\mathcal{F}_q$ given by

$$\begin{cases}X_+(x^\alpha y^\beta)=-q^{\frac14(1+\beta-\alpha)}[\alpha]\,x^{\alpha-1}y^{\beta+1},\\ X_-(x^\alpha y^\beta)=-q^{\frac14(1+\alpha-\beta)}[\beta]\,x^{\alpha+1}y^{\beta-1},\\ K^\pm(x^\alpha y^\beta)=q^{\pm\frac14(\beta-\alpha)}x^\alpha y^\beta.\end{cases}\tag{33}$$

In fact, this makes $\mathcal{F}_q$ into a $\mathcal{U}$-module algebra, that is, the equations

$$\begin{cases} K^{\pm}(1) = 1, \qquad X_{\pm}(1) = 0, \\ K^{\pm}(fg) = K^{\pm}(f)K^{\pm}(g), \\ X_{\pm}(fg) = X_{\pm}(f)K(g) + K^{-1}(f)X_{\pm}(g) \end{cases}$$

are satisfied. The existence of this module algebra is well-known.

We now fix $\nu > 0$, consider the $\mathcal{U}$-submodule of $\mathcal{F}_q$ generated by $(x^k y^{-\nu-k})_{k=0}^{\infty}$, pass to the inhomogeneous notation

$$z^k = q^{\frac{1}{4}k(\nu+k)} x^k y^{-\nu-k} = q^{-\frac{1}{4}k(\nu+k)} y^{-\nu-k} x^k$$

and identify the z^k with powers of a complex variable. The action (33) is then given by

$$\begin{cases} X_+ z^k = -[k]\, z^{k-1}, \\ X_- z^k = [\nu+k]\, z^{k+1}, \\ K^{\pm} z^k = q^{\mp\frac{1}{4}(\nu+2k)} z^k \end{cases}$$

or equivalently by (18). If this defines a unitary action on a Hilbert space in the sense of (16), it is easily seen that one must have (assuming the normalizing condition $\|1\| = 1$)

$$\|z^k\|^2 = \frac{[k]!}{[\nu]_k}.$$

Thus we recover the unitary representation of $\mathcal{U}$ on the space $\mathcal{A}_q^{\nu}$.

Similarly, for $\nu = 0$ we obtain the space $\mathcal{D}_q$ considered in Section 7. Since

$$q^{\frac{1}{4}k^2} x^k y^{-k} = (q^{\frac{1}{4}} x y^{-1})^k,$$

the map $z^k \mapsto q^{k^2/4} x^k y^{-k}$ embeds $\mathcal{D}_q$ as a sub-module algebra of $\mathcal{F}_q$. We will exploit this fact in the next appendix. We also mention that $\mathcal{D}_q/\mathbb{C}$ is a unitarizable $\mathcal{U}_{q^{1/2}}(\mathfrak{su}(1,1))$-module. Its completion is the quantum Dirichlet space, consisting of analytic functions on the disc $|z| < q^{-\frac{1}{2}}$, modulo constants, with the norm

$$\|f\|^2 = \sum_{k=1}^{\infty} [k] |\hat{f}(k)|^2 = \|D_q f\|^2_{\mathcal{A}_q^2}.$$

It is equivalent to the quantum Bergman space $\mathcal{A}_q^2$, an intertwining map being the q-derivative D_q.

Remark 7.5. Since $\mathcal{F}_q$ is a $\mathcal{U}$-module algebra, the multiplication $\mathcal{F}_q^{\otimes n} \to \mathcal{F}_q$ is intertwining. Since degree of homogeneity is additive under multiplication, it induces an intertwining operator $\bigotimes_{i=1}^n \mathcal{A}_q^{\nu_i} \to \mathcal{A}_q^{|\nu|}$. By the results of Section 3, this operator is a constant multiple of the transvectant $\mathcal{J}_1$,

$$\mathcal{J}_1(f_1 \otimes \cdots \otimes f_n)(w) = f_1(q^{\frac{1}{4}\nu_1^*} w) \cdots f_n(q^{\frac{1}{4}\nu_n^*}).$$

This is easy to check directly.

Appendix 2. The algebra of highest weight vectors

In this section we will describe the kernel of X_+ in $\bigotimes_{i=1}^n \mathcal{A}_q^{\nu_i}$. In the case $q=1$, $X_+ = -\sum_{i=1}^n \frac{\partial}{\partial z_i}$ satisfies the Leibniz rule $X_+(fg) = X_+(f)g + fX_+(g)$, so the kernel is formally an associative algebra. (The product is unbounded, but one can obtain a well-defined algebra for instance as the subspace of polynomials.) In this appendix we will find a product $\bullet$ on the space of polynomials in $\bigotimes_{i=1}^n \mathcal{A}_q^{\nu_i}$ which satisfies

$$X_+(f \bullet g) = X_+(f) \bullet K(g) + K^{-1}(f) \bullet X_+(g),$$

so that it gives an algebra structure to the space of polynomials annihilated by X_+, that is, to the span of the highest weight vectors.

First we recall the universal R-matrix

$$R = q^{\frac{1}{4}(H\otimes H)} \sum_{j=0}^{\infty} \frac{(1-q^{-1})^j}{[j]!} q^{\frac{1}{4}j(j-1)} (KX_+)^j \otimes (K^{-1}X_-)^j,$$

where H satisfies $q^{\frac{1}{4}H} = K$. It may be viewed as an element of a suitable extension of $\mathcal{U} \otimes \mathcal{U}$. For "nice" $\mathcal{U}$-modules V_1 and V_2,

$$\Psi = \sigma \circ R : \ V_1 \otimes V_2 \to V_2 \otimes V_1$$

is intertwining, where σ is the flip $\sigma(f \otimes g) = g \otimes f$. We will only need this when $V_1 = V_2 = \mathcal{D}_q$.

There is a canonical way to give a module algebra structure to a tensor product of module algebras, known as the braided tensor product [M]. Namely, if A_1 and A_2 are $\mathcal{U}$-module algebras with products

$$m_i : \ A_i \otimes A_i \to A_i, \quad i = 1, 2,$$

then $A_1 \otimes A_2$ is another one with the product

$$(m_1 \otimes m_2) \circ (1 \otimes \Psi \otimes 1). \tag{34}$$

This construction is associative in the sense that regardless of how it is iterated to define a module algebra structure on a finite tensor product $A_1 \otimes \cdots \otimes A_n$, the result is the same.

Let us consider the space $\mathcal{D}_q \otimes \mathcal{D}_q$, consisting of polynomials in two variables z_1 and z_2, and view it as a sub-module algebra of the braided tensor product $\mathcal{F}_q \otimes \mathcal{F}_q$. Thus we write

$$z_1 = q^{\frac{1}{4}} xy^{-1} \otimes 1, \qquad z_2 = 1 \otimes q^{\frac{1}{4}} xy^{-1}.$$

Denoting by $\bullet$ the product defined by (34), we have

$$z_1 \bullet z_2 = q^{\frac{1}{4}} xy^{-1} \otimes q^{\frac{1}{4}} xy^{-1} = z_1 z_2,$$

where the product on the right hand side is the usual product of polynomials, while

$$\begin{aligned} z_2 \bullet z_1 &= \Psi\left(q^{\frac{1}{4}} xy^{-1} \otimes q^{\frac{1}{4}} xy^{-1}\right) = q^{\frac{3}{2}} xy^{-1} \otimes xy^{-1} + (1-q)x^2y^{-2} \otimes 1 \\ &= q z_1 z_2 + (1-q) z_1^2. \end{aligned}$$

By associativity, this determines the product $\bullet$ uniquely. One may prove by induction that, more generally, the product on $\mathcal{D}_q^{\otimes n}$ considered as a sub-module algebra of $\mathcal{F}_q^{\otimes n}$ is determined by

$$z_i \bullet z_j = \begin{cases} z_i z_j, & i \leq j, \\ q z_j z_i + (1-q) z_j^2, & i > j. \end{cases}$$

With this product, the kernel of X_+ is a a graded algebra

$$\mathrm{Ker} X_+ \big|_{\mathcal{D}_q^{\otimes n}} = \sum_{s=0}^{\infty} \tilde{\mathcal{V}}_s,$$

where $\tilde{\mathcal{V}}_s$ consists of homogeneous polynomials of degree s. It is generated by $\tilde{\mathcal{V}}_0$, which is the space of constants, together with

$$\tilde{\mathcal{V}}_1 = \{a_1 z_1 + \cdots + a_n z_n;\ a_1 + \cdots + a_n = 0\}.$$

By Lemma 7.1, these observations carry over to the space $\bigotimes_{i=1}^n \mathcal{A}^{\nu_i}$. The span of all highest weight vectors is a graded algebra generated by 1 and $\mathcal{V}_1 = S^{-1}\tilde{\mathcal{V}}_1$, with the product

$$Q_1 \bullet Q_2 = S^{-1}(SQ_1 \bullet SQ_2)$$

given on z_i by

$$z_i \bullet z_j = \begin{cases} z_i z_j, & i \leq j, \\ q z_j z_i + (1-q) q^{\frac{1}{4}(\nu_i^* - \nu_j^*)} z_j^2, & i > j, \end{cases}$$

where

$$\nu_i^* - \nu_j^* = \nu_j + 2(\nu_{j+1} + \cdots + \nu_{i-1}) + \nu_i.$$

For $n = 2$, the space $\tilde{\mathcal{V}}_s$ is one-dimensional and generated by $(z_1 - z_2)^{\bullet s}$. Now, by a generalized binomial formula due to Benaoum [B], cf. also [R4], one has

$$(z_1 - z_2)^s = \sum_{k=0}^{s} \frac{[s]!}{[k]![s-k]!} q^{\frac{1}{2}k(s-1)} (-1)^{s-k} z_1^k z_2^{s-k}$$

for variables satisfying $z_2 z_1 = q z_1 z_2 + (1-q) z_1^2$. However, for *commuting* variables, the right hand side of this equation equals

$$\prod_{j=0}^{s-1} (q^j z_1 - z_2),$$

which thus generates $\tilde{\mathcal{V}}_s$. Applying S^{-1} gives, up to a multiplicative constant, the polynomial (20).

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