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(Draft 24/5 2000: iMAC.)

A new kind of Hankel-Toeplitz type operator connected with the complementary series

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0. Introduction. Let us begin by recalling briefly some salient facts from classical Hankel theory (cf. [Ni]). Consider the Hardy class $H^2(\mathbb{T})$, where $\mathbb{T}$ is the unit circle in $\mathbb{C}$, and let $H^2(\mathbb{T})^\perp$ be its orthogonal complement in the space $L^2(\mathbb{T})$. If ϕ is a holomorphic function, one defines the Hankel operator H_ϕ with symbol ϕ by the formula $H_\phi f = P^\perp \bar{\phi} f$ ($f \in H^2(\mathbb{T})$) where $P^\perp$ stands for orthogonal projection onto $H^2(\mathbb{T})^\perp$. (If we instead project onto $H^2(\mathbb{T})$ we get the (classical) Toeplitz operator.) In 1979 Peller (cf. [Pel]) proved that the Hankel operator H_ϕ is in the Schatten-von Neumann class $\mathfrak{S}_p$ if and only if its symbol ϕ belongs to the Besov space $B_pA(\mathbb{T})$, provided that $1 \le p < \infty$; later, this result was extended to cover the case $0 < p < 1$ also.

A basic property of the Hankel operator is its conformal invariance: the map $\phi \mapsto H_\phi$ intertwines with certain natural actions of the group $\mathrm{SU}(1,1)$. This has lead mathematicians to study more general operators with a similar property. This generalization can be done in several steps: replacing the Hardy class $H^2(\mathbb{T})$ by the family of (generalized) weighted Bergman spaces $\mathcal{A}^\nu = \mathcal{A}^\nu(\mathbb{D})$ ($\nu > 0$) over the unit disc $\mathbb{D}$; considering higher order analogues of Hankel operators; using, instead of operators, bilinear or even multilinear forms; passing to the limit $\nu \to \infty$, which leads to the famous Fock space and a degenerate limit of $\mathrm{SU}(1,1)$, the Heisenberg group; replacing $\mathbb{T}$, or rather its "interior" $\mathbb{D}$, by a symmetric domain – this leads to other semi-simple groups than $\mathrm{SU}(1,1)$ – or other domains, or even complex manifolds, with or without a group action; passing to dual ("compact") situations; etc. (cf. e.g. [AFP], [JP], [JPR], [AFJP], [A], [PZ], [Ro]).

In particular, much of the classical Hankel theory can be extended to the case of the group $\mathrm{SU}(n,1)$; instead of the disc there appears now the ball B in $\mathbb{C}^n$. In this case we have an obvious extension of the previous weighted Bergman spaces; we are going to use the same letter for them, writing thus $\mathcal{A}^\nu = \mathcal{A}^\nu(B)$. (Clearly, if $n = 1$ we have $B = \mathbb{D}$ so the notation $\mathcal{A}^\nu$ is unambiguous.)

Generally speaking, Hankel theory forms a link between operator theory and the theory of representations of Lie groups. Whereas in traditional representation theory the focus is mostly on the identification, and the study *in abstracto* of unitary representations, in Hankel theory other problems appear also, which forces one to leave the Hilbertian scenario, such as – in the case one has a representation realized on a space of linear operators – membership in Schatten-von Neumann classes; boundedness; compactness; finite rank.

In a recent paper [DH], dealing with the group $\mathrm{SU}(n,1)$, it was established that in certain cases the complementary series enters in the decomposition of the tensor product $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$. In this paper we wish to realize the complementary series as a space of operators. This leads to a new type of operator apparently not studied before. Whether it is a Hankel or a Toeplitz operator is an academic issue. Therefore we use henceforth the two names together speaking of Hankel-Toeplitz type operator.

The bulk of the paper (Sections 1-5) will be about the case $n = 1$. In Sections 1-3 we review first some known facts about unitary representations of $\mathrm{SU}(1,1)$. We define then

(Section 4) our Hankel-Toeplitz type operators denoted $\mathfrak{H}_\phi$, acting on the spaces $\mathcal{A}^\nu$ with $0 < \nu < \frac{1}{2}$; the symbol ϕ is now not any more analytic. We verify that these operators have the desired intertwining properties and that they are Hilbert-Schmidt operators, as they should. (Note that, formally, the $\mathfrak{H}_\phi$ reduce to ordinary Toeplitz operators when $\nu = 1$, and these are never Hilbert-Schmidt.) Finally, in Section 5 we begin at least the study of their Schatten-von Neumann properties; in particular, we uncover the presence of a “cut-off”. That is, we show that our Hankel-Toeplitz type operator can be in $\mathfrak{S}_p$ only for sufficiently large values of p; so it can never be of finite rank (no Kronecker theorem).

Remark 0.1. (On the work of Wu.) One should bear in mind that for $0 < \nu < 1$ the spaces $\mathcal{A}^\nu$ behave differently as compared to the case $\nu > 1$ ($\nu = 1$ is the limiting case of the Hardy class). These spaces, and the operators acting on them, have been investigated at length by Wu and others (see e.g. [Wu]; we refrain from making a detailed comparison here). $\square$

Only in Section 6 we turn to the case of dimension > 1. In this situation we can construct Hankel-Toeplitz type operators $\mathfrak{H}_\phi^{(k)}$, where k is integer ≥ 0 and ϕ a non-analytic symbol, with the desired intertwining property. If $\nu < \frac{d}{2} - k$, where dimension is now indicated by the letter d, this produces Hilbert-Schmidt operators so we obtain effectively a unitary equivalence with the complementary series.

Finally, in the last section (Section 7) we show that the operators introduced in Section 6 also can be obtained with the help of certain sesquilinear differential operators which we, by analogy with how this word was used in [JP], refer to as transvectants.

Some notation (for the Reader’s benefit)

dimension: n or d

sets: $(d = 1)$ $\mathbb{D}$, $\mathbb{T}$; $(d > 1)$ B, S

function spaces: $(d = 1)$ $H^2(\mathbb{T})$, $\mathcal{A}^\nu = \mathcal{A}^\nu(\mathbb{D})$, $\mathcal{H}^\lambda$; $(d > 1)$ $\mathcal{A}^\nu = \mathcal{A}^\nu(B)$

Hankel-Toeplitz type operators: H_ϕ (Introduction only), $\mathfrak{H}_\phi$, $\mathfrak{H}_\psi^{(k)}$

other operators or transforms: U^α, S^ν, M_ϕ, T_g^ν, $\mathcal{K}$, $\mathcal{J}$, $\mathcal{R}$, R (means two different things!)

integral kernels: $K^\nu(z, w)$ (reproducing kernel in $\mathcal{A}^\nu$), $H_{\mathcal{H}}$ (ditto in an imbedded Hilbert space $\mathcal{H}$; Section 3 only), $\mathfrak{H}_\phi(z, w)$ (kernel of $\mathfrak{H}_\phi$)

group: $(d = 1)$ $\mathfrak{G} = \mathrm{SU}(1, 1)$; $(d > 1)$ $\mathfrak{G} = \mathrm{SU}(d, 1)$

Schatten-von Neumann class: $\mathfrak{S}_p$

N. B. – In Sections 4-5 $(d = 1)$, ν, α, λ are quantities connected by $\alpha = 2\nu$, $\lambda = 2 - \alpha$; in Sections 6-7 $(d > 1)$, by $\alpha = 2\nu + 2k$, $\lambda = d + 1 - \alpha$.

N. B. – The sign $\square$ is used not only for “end of proof”, but also for “end of remark”.

1. Review of irreducible unitary representations of $\mathrm{SU}(1, 1)$. It will be convenient to set $\mathfrak{G} \stackrel{\text{def}}{=} \mathrm{SU}(1, 1)$; the elements of this group thus are matrices $g = \begin{pmatrix} \bar\delta & \bar\gamma \\ \gamma & \delta \end{pmatrix}$ where γ and δ are complex numbers with $|\delta|^2 - |\gamma|^2 = 1$. The universal covering group of $\mathfrak{G}$ will be denoted $\widetilde{\mathfrak{G}}$.

As is well-known, the irreducible unitary representations of $\mathfrak{G}$ (and its universal cover) come in several series (cf. e.g. [L] or [Su]). Below we recall briefly their definition choosing as the underlying manifold either the unit disc $\mathbb{D}$ or, perhaps even better, its boundary $\mathbb{T} = \partial\mathbb{D}$ with $\mathfrak{G}$ acting on these sets via fractional linear transformations.

1. Principal series. We consider the Hilbert space of square integrable functions over $\mathbb{T}$ with norm

$$\|f\|^2 = \frac{1}{2\pi}\int_{\mathbb{T}} |f(x)|^2 |dx| < \infty. \tag{1.1}$$

Then letting $\mathfrak{G}$ act (with $gx = \dfrac{\alpha x + \beta}{\gamma x + \delta}$) via either

$$f(x) \mapsto f(gx)|\gamma x + \delta|^{-(1+is)}$$

or

$$f(x) \mapsto f(gx)|\gamma x + \delta|^{-(1+is)} \operatorname{sign}(\gamma x + \delta),$$

where s is any real number, we obtain an irreducible unitary representation. There are thus two principal series parametrized by the real number s. We are not going to use any of them; it is only for the Reader's benefit that we have mentioned them here.

2. Complementary series. Let us first define the potential operator U^α acting on smooth functions on $\mathbb{T}$,

$$U^\alpha f(x) = \frac{c_\alpha}{2\pi}\int_{\mathbb{T}} |1 - x\bar{y}|^{-\alpha} f(y)\,|dy|,$$

where $0 < \alpha < 1$. Here c_α is a constant which is normalized by the prescription that $U^\alpha 1 = 1$; it will be computed in the following section.

Letting $\langle\cdot,\cdot\rangle$ stand for the scalar product corresponding to the norm in (1.1) the formula

$$\|f\|_\alpha^2 = \langle U^\alpha f, f\rangle = \frac{c_\alpha}{2\pi}\iint_{\mathbb{T}\times\mathbb{T}} |1 - x\bar{y}|^{-\alpha} f(x)\overline{f(y)}\,|dx|\,|dy|$$

gives a norm. This follows from the fact that the kernel $|1 - x\bar{y}|^{-\alpha}$ is positive definite for α precisely in the interval $0 < \alpha < 1$. It is invariant under the group action

$$f(x) \mapsto f(gx)|\gamma x + \delta|^{-\lambda} \equiv f^g(x)$$

provided that λ is related to α via the formula

$$\boxed{\lambda = 2 - \alpha}.$$

Proof of the invariance. Indeed, changing the variables of integration yields

$$\|f\|_\alpha^2 = \frac{c_\alpha}{2\pi}\iint_{\mathbb{T}\times\mathbb{T}} |1 - gx\overline{gy}|^{-\alpha} f(gx)\overline{f(gy)}\,|dgx|\,|dgy|.$$

Now one has the identities

$$1 - gx\overline{gy} = \frac{1 - x\bar{y}}{(\gamma x + \delta)\overline{(\gamma y + \delta)}} \tag{1.2}$$

and

$$|dgx| = \frac{|dx|}{|\gamma x+\delta|^2}; \quad |dgy| = \frac{|dy|}{|\gamma y+\delta|^2}. \tag{1.3}$$

Thus we obtain, using (1.2) and (1.3),

$$\begin{aligned}\|f\|_\alpha^2 &= \frac{c_\alpha}{2\pi}\iint_{\mathbb{T}\times\mathbb{T}} |1-x\bar{y}|^{-\alpha} f(gx)|\gamma x+\delta|^{-(2-\alpha)}\overline{f(gy)|\gamma y+\delta|^{-(2-\alpha)}}\,|dx|\,|dy| = \\ &= \frac{c_\alpha}{2\pi}\iint_{\mathbb{T}\times\mathbb{T}} |1-x\bar{y}|^{-\alpha} f^g(x)\overline{f^g(y)}\,|dx|\,|dy| = \\ &= \|f^g\|_\alpha^2. \quad \square\end{aligned}$$

Taking the completion of smooth functions in the norm $\|\cdot\|_\alpha$ gives a Hilbert space denoted $\mathcal{H}^\lambda$ on which we have a unitary representation of $\mathfrak{G}$ which can be proved to be irreducible. If we let λ vary in the interval $1<\lambda<2$, these representations make up the complementary series.

3. Holomorphic discrete series. For $\nu>1$ we denote by $\mathcal{A}^\nu$ the Hilbert space of holomorphic functions in $\mathbb{D}$, square integrable with respect to the weighted probability measure $d\mu_\nu$,

$$d\mu_\nu(z) = \frac{\nu-1}{\pi}(1-|z|^2)^{\nu-2}dx\,dy.$$

The norm in $\mathcal{A}^\nu$ is thus given by

$$\|f\|_\nu^2 = \frac{\nu-1}{\pi}\iint_{\mathbb{D}} |f(z)|^2(1-|z|^2)^{\nu-2}dx\,dy. \tag{1.4}$$

The group $\mathfrak{G}$ acts on $\mathcal{A}^\nu$ via the transformations

$$f(x)\mapsto f(gx)(\gamma x+\delta)^{-\nu}.$$

(Note that, strictly speaking, we have, unless ν is integer, a *projective* representation, that is, an action of the covering group $\tilde{\mathfrak{G}}$.)

If $\nu\to 1$ the measure $d\mu_\nu$ tends to the normalized arc length measure on the circumference $\mathbb{T}$,

$$\frac{\nu-1}{\pi}(1-|z|^2)^{\nu-2}dx\,dy \to \frac{1}{2\pi}|dz| \quad (z=x+iy);$$

thus it is natural to define $\mathcal{A}^1 = H^2$.

If $0<\nu<1$ we can still define $\mathcal{A}^\nu$ taking the integral in (1.4) in a generalized sense (analytic continuation of the parameter; this is the method of M. Riesz).

The invariance of the norm can be established by first proving it for $\nu>1$, in a similar way as was done in the case of the complementary series, and then performing an analytic continuation.

4. Anti-holomorphic discrete series. This series arises by taking the complex conjugates of the functions belonging to the previous spaces $\mathcal{A}^\nu$. We denote the corresponding spaces by $\bar{\mathcal{A}}^\nu$; thus, each space $\bar{\mathcal{A}}^\nu$ consists of anti-holomorphic functions $f(\bar{z})$ and the group $\mathfrak{G}$ acts $\bar{\mathcal{A}}^\nu$ on the via the transformations

$$f(x)\mapsto f(gx)\overline{(\gamma x+\delta)}^{-\nu}.$$

In view of the F. Riesz lemma, we can also identify the space $\bar{\mathcal{A}}^\nu$ with the dual of $\mathcal{A}^\nu$, $\bar{\mathcal{A}}^\nu \approx (\mathcal{A}^\nu)'$.

2. More about the spaces $\mathcal{H}^\lambda$ and $\mathcal{A}^\nu$. We begin with the space $\mathcal{H}^\lambda$ and say then, more in passing, a few words about $\mathcal{A}^\nu$ also.

1. The space $\mathcal{H}^\lambda$. First we wish to calculate the norm, in the $\mathcal{H}^\lambda$-metric, of the elements of the standard orthogonal basis $\{x^m\}$ $(m = 0, \pm 1, \pm 2, \dots)$. To this end we need to calculate $U^\alpha x^m$. As, quite generally,

$$\overline{U^\alpha f} = U^\alpha \bar{f},$$

it suffices to take $m \geq 0$.

We have the double series expansion

$$|1 - x\bar{y}|^{-\alpha} = (1 - x\bar{y})^{-\frac{\alpha}{2}} \overline{(1 - x\bar{y})^{-\frac{\alpha}{2}}} = \sum_{k,\ell=0}^{\infty} \frac{(\frac{\alpha}{2})_k}{k!} \frac{(\frac{\alpha}{2})_\ell}{\ell!} x^k \bar{y}^k \bar{x}^\ell y^\ell.$$

Here we used the Pochhammer symbol or factorial:

$$(u)_p = u(u+1)\dots(u+p-1) \qquad (p = 0, 1, 2, \dots),$$

where conventionally $(u)_0 = 1$ for $p = 0$. We have to multiply this by y^m and integrate. Then, in view of the orthogonality, all terms drop out except those for which $k = m + \ell$. Thus we find

$$U^\alpha x^m = c_\alpha \sum_{\ell=0}^{\infty} \frac{(\frac{\alpha}{2})_{\ell+m} (\frac{\alpha}{2})_\ell}{(\ell+m)!\ell!} x^m.$$

Let us write

$$(\ell+m)! = 1 \cdot 2 \cdot \ldots \cdot m(m+1)\dots(m+\ell) = m!(m+1)_\ell$$

and, similarly,

$$(\tfrac{\alpha}{2})_{\ell+m} = \tfrac{\alpha}{2}(\tfrac{\alpha}{2}+1)\dots(\tfrac{\alpha}{2}+m-1)(\tfrac{\alpha}{2}+m)\dots(\tfrac{\alpha}{2}+m+\ell-1) = (\tfrac{\alpha}{2})_m(\tfrac{\alpha}{2}+m)_\ell.$$

Then the last sum can be rewritten as

$$\frac{(\frac{\alpha}{2})_m}{m!} \sum_{\ell=0}^{\infty} \frac{(\frac{\alpha}{2}+m)_\ell (\frac{\alpha}{2})_\ell}{(m+1)_\ell} \frac{1}{\ell!} = \frac{(\frac{\alpha}{2})_m}{m!} {}_2F_1(\tfrac{\alpha}{2}, \tfrac{\alpha}{2}+m; m+1; 1),$$

where ${}_2F_1$ stands for the hypergeometric function. Now recall Gauss's famous formula:

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{2.1}$$

In our case it gives

$$ {}_2F_1(\tfrac{\alpha}{2}, \tfrac{\alpha}{2}+m; m+1; 1) = \frac{m!\,\Gamma(1-\alpha)}{\Gamma(1-\frac{\alpha}{2}+m)\Gamma(1-\frac{\alpha}{2})}.$$

As

$$(\tfrac{\alpha}{2})_m = \frac{\Gamma(\frac{\alpha}{2}+m)}{\Gamma(\frac{\alpha}{2})},$$

this again implies that

$$U^\alpha x^m = c_\alpha \frac{\Gamma(\frac{\alpha}{2}+m)\Gamma(1-\alpha)}{\Gamma(1-\frac{\alpha}{2}+m)\Gamma(1-\frac{\alpha}{2})\Gamma(\frac{\alpha}{2})} x^m.$$

In particular, taking $m=0$ we find that (recall that $U^\alpha 1 = 1$)

$$\boxed{c_\alpha = \frac{(\Gamma(1-\frac{\alpha}{2}))^2}{\Gamma(1-\alpha)}}. \tag{2.2}$$

Thus, eliminating c_α, we can write

$$U^\alpha x^m = \frac{\Gamma(\frac{\alpha}{2}+m)}{\Gamma(1-\frac{\alpha}{2}+m)} : \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} x^m = \frac{(\frac{\alpha}{2})_m}{(1-\frac{\alpha}{2})_m} x^m.$$

Here we have assumed that $m>0$. Upon formally replacing m by $|m|$ we can also write an analogous formula valid regardless of the sign of m.

Hence the desired formula for the α-norm of x^m:

$$\|x^m\|_\alpha^2 = \langle U^\alpha x^m, x^m\rangle = \frac{(\frac{\alpha}{2})_{|m|}}{(1-\frac{\alpha}{2})_{|m|}}.$$

This again gives the expression of the norm of an arbitrary function $f \in \mathcal{H}^\lambda$ in terms of its Fourier coefficients $\hat{f}(m)$:

$$\|f\|_\alpha^2 = \sum_{m=-\infty}^{\infty} \frac{(\frac{\alpha}{2})_{|m|}}{(1-\frac{\alpha}{2})_{|m|}} |\hat{f}(m)|^2.$$

2. The space $\mathcal{A}^\nu$. Finally, we turn to the spaces $\mathcal{A}^\nu$ $(\nu>0)$. In any case we have the following expression for the norm of the powers of z:

$$\|z^n\|_\nu^2 = \frac{n!}{(\nu)_n} \quad (n=0,1,2,\ldots),$$

so the norm of a general element $f \in \mathcal{A}^\nu$, in term of its Fourier (or Maclaurin) coefficients, is given by

$$\|f\|_\nu^2 = \sum_{n=0}^{\infty} \frac{n!}{(\nu)_n} |\hat{f}(n)|^2 \quad (n=0,1,2,\ldots).$$

For $\nu>1$ this is proved by integration and in general by analytic continuation.

It follows that the space $\mathcal{A}_\nu$ has a reproducing kernel given by the formula

$$K(z,\bar{w}) = K^\nu(z,\bar{w}) = (1-z\bar{w})^{-\nu} = \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!}(z\bar{w})^n.$$

3. Decomposition of $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$. The representation of the universal covering $\widetilde{\mathfrak{G}}$ of $\mathfrak{G}$ in the Hilbert space tensor product $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ is trivial on the center of $\widetilde{\mathfrak{G}}$. It therefore factors to a representation of $\mathfrak{G}$. The explicit decomposition of this representation of $\mathfrak{G}$ into a direct integral of irreducible unitary representations was first performed in [ØZ]. Recently it has been shown that $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ is equivalent to a *canonical representation* as defined in [VGG], see also [DH] and [H1]. In that context, they also computed the decomposition.

A very interesting aspect of the result – from our point of view – is the occurrence of the complementary series representation $\mathcal{H}^{2-2\nu}$ as an invariant subspace in $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ when $0 < \nu < \frac{1}{2}$ (this fact is also stated in [Ne1]; see also the interesting paper [Ne2] as well as Remark 4.1 *infra*). This will result in the new kind of Hankel-Toeplitz type operators mentioned in the title.

We would like to spend some words on the computation of the decomposition. The method that will be explained is not the one from [ØZ], but the approach as in [DH], [H1]. It has the advantage that it can be generalized more easily to other groups like $\mathrm{SO}(1,n)$, $\mathrm{SU}(p,q)$, $\mathrm{Sp}(1,n)$ and others, or to tensor product of the form $\mathcal{A}^{\nu_1} \otimes \bar{\mathcal{A}}^{\nu_2}$. See also [DP] and [H2].

The space $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ can be considered as a Hilbert space of functions on $\mathbb{D} \times \mathbb{D}$ that are holomorphic in the first and anti-holomorphic in the second variable, and in which point evaluation is a continuous linear functional. As such it has a reproducing kernel, which is

$$((z_1, z_2), (w_1, w_2)) \mapsto K^\nu(z_1, w_1)\overline{K^\nu(z_2, w_2)}. \tag{3.1}$$

A function $f \in \mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ is completely determined by its restriction to the diagonal $\Delta(\mathbb{D}) = \{(z,z) : z \in \mathbb{D}\}$. Let $\mathcal{R} : \mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu \to C^\infty(\mathbb{D})$ be this restriction map. Now $\mathfrak{G}$ acts in $C^\infty(\mathbb{D})$ by means of

$$f(x) \mapsto f(gx)|\gamma x + \delta|^{-2\nu},$$

and $\mathcal{R}$ is equivariant.

Lemma 3.1. *The image of $\mathcal{R}$ is precisely the Hilbert space $\mathcal{H}(|K^\nu|^2)$ determined by the reproducing kernel*

$$|K^\nu|^2(z,w) = |K^\nu(z,w)|^2 = |1 - z\bar{w}|^{-2\nu}.$$

Proof. By taking $w_1 = w_2$ in (3.1) it is clear that the Hilbert space $\mathcal{H}(|K^\nu|^2)$ is contained in the image of $\mathcal{R}$. Let F be in the orthogonal complement of $\mathcal{R}^{-1}(\mathcal{H}(|K^\nu|^2))$. Because $K^\nu(\cdot, z) \otimes \overline{K^\nu(\cdot, z)}$ is an element of this inverse image,

$$F(z,z) = \langle F, K^\nu(\cdot, z) \otimes \overline{K^\nu(\cdot, z)}\rangle = 0$$

for all $z \in \mathbb{D}$. Therefore $F = 0$. □

Let 1 denote the function constant one in $\mathcal{A}^\nu$.

Corollary. *The vector $1 \otimes 1$ is cyclic in $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ for $\mathfrak{G}$.*

Proof. The function $1 = |K^\nu|^2(\cdot, 0) = \mathcal{R}(1 \otimes 1)$ is cyclic in $\mathcal{H}(|K^\nu|^2)$. □

If $(\pi, \mathcal{H})$ is any unitary representation with cyclic vector v_0, then

$$v \mapsto f_v(g) \stackrel{\text{def}}{=} \langle v, \pi(g)v_0\rangle_{\mathcal{H}} \tag{3.2}$$

gives a continuous $\mathfrak{G}$-equivariant embedding of $\mathcal{H}$ into $C^\infty(\mathfrak{G})$, when $\mathfrak{G}$ acts in the latter space by left translation. By means of the Haar measure on $\mathfrak{G}$, $C^\infty(\mathfrak{G})$ can be identified with a subspace of the space $\mathcal{D}'(\mathfrak{G})$ of distributions on $\mathfrak{G}$. Thus $\mathcal{H}$ embeds continuously and equivariantly into $\mathcal{D}'(\mathfrak{G})$. In the terminology of [S] it is an invariant Hilbert subspace of $\mathcal{D}'(\mathfrak{G})$. As such its reproducing kernel $H_{\mathcal{H}} : \mathcal{D}(\mathfrak{G}) \to \mathcal{D}'(\mathfrak{G})$ is given by $H_{\mathcal{H}}(\varphi) = \varphi * f_v$, where $*$ denotes convolution product.

In our situation, $1 \otimes 1$ is not only cyclic, but it is also left fixed by

$$K = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1)) = \left\{ \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} : 0 \leq \varphi < 2\pi \right\} \simeq \mathrm{U}(1).$$

Because $\mathbb{D} \simeq \mathfrak{G}/K$, $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ can be realized as a Hilbert subspace of $\mathcal{D}'(\mathbb{D})$. Identifying functions on $\mathbb{D}$ with functions on $\mathfrak{G}$ that are right-K-invariant, the reproducing kernel $H_\nu : \mathcal{D}(\mathbb{D}) \to \mathcal{D}'(\mathbb{D})$ is given by $H_\nu(f) = f * \psi_\nu$, where

$$\psi_\nu(g) = \langle 1 \otimes 1, g \cdot 1 \otimes 1 \rangle_{\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu} = |\delta|^{-2\nu}.$$

It satisfies $\psi_\nu(k_1 g k_2) = \psi_\nu(g)$ for all $k_1, k_2 \in K$ and $g \in \mathfrak{G}$.

If A denotes the one-parameter subgroup of the elements

$$a_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \qquad t \in \mathbb{R},$$

then $\mathfrak{G} = KAK$. Thus ψ_ν is completely determined by its restriction:

$$\psi_\nu(a_t) = (\cosh t)^{-2\nu},$$

and this is precisely the function used in [VGG] to define canonical representations. The canonical representation corresponding to ψ_ν is therefore equivalent to the tensor product $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$.

The Hilbert spaces of both the principal series (without "sign") and the complementary series contain the constant function 1. It is cyclic because the representations are irreducible, and it is left fixed by K. Define

$$\varphi_z(g) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} |-\gamma x + \delta|^{-(1+z)} |dx|.$$

Then φ_z is the reproducing kernel f_1 (see (3.2)) of the principal series if $z = is$, $s \in \mathbb{R}$, and of the complementary series $\mathcal{H}^\lambda$ if $z = \alpha = 2 - \lambda$, $1 < \lambda < 2$.

In the weak sense, ψ_ν can be expressed in terms of an integral of φ_z in $\mathcal{D}'(\mathfrak{G})$, $z \in i\mathbb{R} \cup (0,1)$.

Theorem 3.1. *i. If $\nu \geq \frac{1}{2}$, then*

$$\psi_\nu = \frac{1}{\Gamma(\nu)^2} \int_0^\infty \varphi_{is} \cdot a(s) ds,$$

ii. If $0 < \nu < \frac{1}{2}$, then

$$\psi_\nu = 2\pi \cot \pi\nu \frac{\Gamma(2\nu)}{\Gamma(\nu)^2} \cdot \varphi_{1-2\nu} + \frac{1}{\Gamma(\nu)^2} \int_0^\infty \varphi_{is} \cdot a(s) ds. \tag{3.3}$$

Here

$$a(s) = \left| \frac{\Gamma(\frac{is+1}{2})\Gamma(\frac{is-1}{2})}{\Gamma(\frac{is}{2})} \right|^2.$$

The direct integral decomposition of $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ can now be derived from these expressions for ψ_ν, see [S]. In particular one can conclude from (3.3) that the complementary series representation $\mathcal{H}^{2-2\nu}$ is an invariant subspace of $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ for $0 < \nu < \frac{1}{2}$. So there should exist a $\mathfrak{G}$-equivariant map of $\mathcal{H}^{2-2\nu}$ into the Hilbert-Schmidt operators $\mathfrak{S}_2(\mathcal{A}^\nu)$. This map will be $\phi \mapsto \mathfrak{H}_\phi$, where $\mathfrak{H}_\phi$ is the Hankel-Toeplitz type operator to be defined in the next section.

4. The Hankel-Toeplitz type operator. Let $\nu \in (0, \frac{1}{2})$ be fixed and set (recall that $\alpha = 2 - \lambda$)

$$\boxed{\lambda = 2 - 2\nu} \text{ or } \boxed{\alpha = 2\nu}.$$

Thus λ lies in the interval $(1, 2)$. To every element $\phi \in \mathcal{H}^\lambda$ we associate a Hilbert-Schmidt operator $\mathfrak{H}_\phi$ acting from $\mathcal{A}^\nu$ to itself, $\mathfrak{H}_\phi \in \mathfrak{S}_2(\mathcal{A}^\nu)$. Formally, it is given by the integral

$$\mathfrak{H}_\phi f(x) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\phi(y) f(y)}{(1 - x\bar{y})^\nu} |dy| \qquad (f \in A^\nu, x \in \mathbb{D}).$$

This gives $\mathfrak{H}_\phi f$ as an element of A^ν in terms of its boundary values. We call $\mathfrak{H}_\phi$ the *Hankel-Toeplitz type operator* with *symbol* ϕ.

Remark 4.1. Before we proceed to show that the map $\phi \mapsto \mathfrak{H}_\phi$ has the desired properties, let us look at some equivalent versions of this correspondence. This will facilitate our discussion of the unit ball. First, identifying $\mathfrak{S}_2(\mathcal{A}^\nu)$ with $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ gives the map $\mathcal{K}$ sending ϕ to the integral kernel

$$\mathfrak{H}_\phi(z, \bar{w}) = \mathfrak{H}_\phi(k_w)(z) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\phi(y)}{(1 - z\bar{y})^\nu (1 - y\bar{w})^\nu} |dy|,$$

where we have written $k_w(\cdot) = K^\nu(\cdot, w)$. Second, one may consider the adjoint operator $\mathcal{J}$ from $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ to $\mathcal{H}^\lambda$ (note that both representations are self-dual). Formally, it is determined by the identity

$$\langle \mathcal{J}(f \otimes \bar{g}), \phi \rangle = \langle f \otimes \bar{g}, \mathcal{K}\phi \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} \overline{\phi(y)} \langle f, k_y \rangle \overline{\langle g, k_y \rangle} |dy|.$$

Now, for f and g polynomial, the reproducing formula holds also on the boundary, so that

$$\langle \mathcal{J}(f \otimes \bar{g}), \phi \rangle = \langle \mathcal{R}(f \otimes \bar{g}), \phi \rangle_{L^2(\mathbb{T})} = \langle (U^\alpha)^{-1} \mathcal{R}(f \otimes \bar{g}), \phi \rangle_{\mathcal{H}^\lambda},$$

where $\mathcal{R}$ is the operator of restriction to the diagonal of the boundary:

$$\mathcal{R}(f \otimes \bar{g})(x) = f(x)\overline{g(x)} \qquad (x \in \mathbb{T}).$$

Thus we have found that, at least formally,

$$\boxed{\mathcal{J} = (U^\alpha)^{-1} \mathcal{R}}.$$

This is how Neretin constructs these operators in [Ne2]. □

To gain a better feeling for the operator $\mathfrak{H}_\phi$ let us also introduce the following integral operator on $\mathbb{T}$:

$$S^\nu h(x) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{h(y)}{(1 - x\bar{y})^\nu} |dy| \qquad (x \in \mathbb{T}). \tag{4.1}$$

For $\nu = 1$ it reduces to the Szegö projection. Therefore we call S^ν the *Szegö transformation* of *weight* ν. One readily sees what is the effect of S^ν on basis elements:

$$S^\nu x^n = \begin{cases} \dfrac{(\nu)_n}{n!} x^n & n \geq 0 \\ 0 & n < 0 \end{cases}.$$

Denoting further the operation of multiplication by ϕ by M_ϕ we can then write

$$\boxed{\mathfrak{H}_\phi = S^\nu M_\phi}.$$

Lemma 4.1. *If we let h transform with biweight $(1, 1-\nu)$, then S^ν transforms with weight ν.*

Proof. Let us replace, in (1), x by gx making at the same time a change of variables in the integral, $y \mapsto gy$:

$$S^\nu h(gx) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{h(gy)}{(1 - gx\overline{gy})^\nu} |dgy| \qquad (x \in \mathbb{T}).$$

Proceeding as in Section 1 (see (1.1) and (1.2)) one finds

$$S^\nu h(gx)(\gamma x + \delta)^{-\nu} = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{h(gy)(\gamma y + \delta)^{-1}\overline{(\gamma y + \delta)^{-(1-\nu)}}}{(1 - x\bar{y})^\nu} |dy| \qquad (x \in \mathbb{T}).$$

Setting, quite generally,

$$T^g_{(\nu_1, \nu_2)} h(x) = h(gx)(\gamma x + \delta)^{-\nu_1} \overline{(\gamma x + \delta)^{-\nu_2}},$$

with $T^g_{(\nu, 0)} = T^g_\nu$, this can also be stated as

$$\boxed{T^g_\nu S^\nu = S^\nu T^g_{(1, 1-\nu)}},$$

which is all that we need. $\square$

From this lemma we again readily deduce the following.

Corollary 4.1. *Let ϕ and f transform according to the rules $\phi(x) \mapsto \phi(gx)|\gamma x + \delta|^{-\lambda}$ and $f(x) \mapsto f(gx)(\gamma x + \delta)^{-\nu}$ respectively. Then $\mathfrak{H}_\phi f$ transforms with weight ν, that is, we have the formula*

$$\boxed{T^g_\nu \mathfrak{H}_\phi = \mathfrak{H}_{T^g_\lambda \phi} T^g_\nu}. \quad \square$$

Next we verify that we indeed get Hilbert-Schmidt operators.

Lemma 4.2. *The operator $\mathfrak{H}_\phi$ for $\phi \in \mathcal{H}^\lambda$ is Hilbert-Schmidt in $\mathcal{A}^\nu$ and its Hilbert-Schmidt norm is given by $\|\mathfrak{H}_\phi\|^2_{\mathfrak{S}_2} = c_\alpha^{-1} \|\phi\|^2_\alpha$ where c_α is the constant defined in Section 2* (see (2.2)).

Proof. Using the above formula for $S^\nu x^n$, one finds for any integer r

$$\mathfrak{H}_{x^r} z^n = \begin{cases} \dfrac{(\nu)_{n+r}}{(n+r)!} z^{n+r} & n + r \geq 0, \\ 0 & n + r < 0 \end{cases} \qquad (n = 0, 1, 2, \ldots). \tag{4.2}$$

It follows that

$$\langle \mathfrak{H}_\phi z^n, z^m\rangle = \hat{\phi}(m-n).$$

As $z^n/\|z^n\|_\nu$ $(n = 0, 1, 2, \ldots)$ is an orthonormal basis of A^ν, we thus have

$$\|\mathfrak{H}_\phi\|_{\mathfrak{S}_2}^2 = \sum_{m,n=0}^{\infty} \frac{|\langle \mathfrak{H}_\phi z^n, z^m\rangle|^2}{\|z^n\|_\nu^2 \|z^m\|_\nu^2} = \sum_{m,n=0}^{\infty} \frac{|\hat{\phi}(m-n)|^2}{\|z^n\|_\nu^2 \|z^m\|_\nu^2} = \sum_{m,n=0}^{\infty} |\hat{\phi}(m-n)|^2 \frac{(\nu)_n (\nu)_m}{n!\, m!}.$$

However, we saw in Section 2 that

$$c_\alpha \sum_{n=0}^{\infty} \frac{(\nu)_n (\nu)_{n+k}}{n!\,(n+k)!} = \frac{(\nu)_k}{(1-\nu)_k} \qquad (k = 0, 1, 2, \ldots).$$

Consequently,

$$\|\mathfrak{H}_\phi\|_{\mathfrak{S}_2}^2 = c_\alpha^{-1} \sum_{k=-\infty}^{\infty} |\hat{\phi}(k)|^2 \frac{(\nu)_{|k|}}{(1-\nu)_{|k|}} = c_\alpha^{-1} \|\phi\|_\alpha^2. \quad \square$$

Remark 4.2. We indicate an alternative way of deriving the above results, although the underlying idea is essentially the same. We realize the tensor product $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ as a space of functions $F(z, w)$ on $\mathbb{D} \times \mathbb{D}$, holomorphic in z and anti-holomorphic in w. Given such a function $F(z, w)$, corresponding to an element in $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$, the corresponding operator is

$$f(z) \mapsto \langle f(\cdot), \overline{F(z, \cdot)}\rangle_{\mathcal{A}^\nu}, \tag{4.3}$$

where (this can easily be proved) for a fixed $z \in \mathbb{D}$ the function $F(z, \cdot)$ is in $\bar{\mathcal{A}}^\nu$. We obtain then the realization of the complementary series as a space of functions on $\mathbb{D}$ as follows. Generally speaking, let $\phi_\lambda(z)$ be the spherical function on $\mathbb{D}$,

$$\phi_\lambda(z) = \int_{\partial\mathbb{D}} \left(\frac{1-|z|^2}{|1-z\bar{b}|^2}\right)^{\frac{1-\lambda}{2}} |db| = (1-|z|^2)^{\frac{1-\lambda}{2}} {}_2F_1(\tfrac{1-\lambda}{2}, \tfrac{1-\lambda}{2}, 1, |z|^2).$$

For $0 < \lambda < 1$, ϕ_λ is positive definite (see e.g. [He], p. 484 and references given there) and thus we get an irreducible unitary representation of $\mathfrak{G}$, denoted as before by $\mathcal{H}^\lambda$, realized as a space of functions on $\mathbb{D}$ with the regular $\mathfrak{G}$-action. We construct an intertwining operator from $\mathcal{H}^\lambda$ to the tensor product $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ by *polarization.* Notice that $\phi_\lambda(z)$ is the restriction of a (unique) function holomorphic in z and antiholomorphic in w, to wit the function

$$\phi_\lambda(z, w) = \int_{\partial\mathbb{D}} \left(\frac{1-z\bar{w}}{(1-z\bar{b})(1-b\bar{w})}\right)^{\frac{1-\lambda}{2}} |db| = (1-z\bar{w})^{\frac{1-\lambda}{2}} {}_2F_1(\tfrac{1-\lambda}{2}, \tfrac{1-\lambda}{2}; 1; z\bar{w}).$$

There is a formal intertwining operator from $\mathcal{H}^\lambda$ to the space of functions $f(z, w)$ holomorphic in z and antiholomorphic in w, with the same action as in $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$, namely

$$\mathcal{H}^\lambda \to \mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu : \sum_j c_j \phi_\lambda(g_j z) \mapsto (1-z\bar{w})^{-\nu} \sum_j c_j \phi_\lambda(g_j z, g_j w).$$

To prove that $\mathcal{H}^\lambda$ is a discrete part we thus need only to prove that the image of ϕ_λ is in $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$, for $\lambda = 1 - 2\nu$. But then the image is

$$(1 - z\bar{w})^{-\nu}(1 - z\bar{w})^{\frac{1-\lambda}{2}}\,{}_2F_1(\tfrac{1-\lambda}{2}, \tfrac{1-\lambda}{2}; 1; z\bar{w}) = {}_2F_1(\nu, \nu; 1; z\bar{w}) = \sum_{n=0}^{\infty} \frac{(\nu)_n(\nu)_n}{n!}\frac{(z\bar{w})^n}{n!},$$

so its tensor norm square is

$$\sum_{n=0}^{\infty} \left(\frac{(\nu)_n(\nu)_n}{n!}\right)^2 (\nu)_n^{-2} = \sum_{n=0}^{\infty} \left(\frac{(\nu)_n}{n!}\right)^2 = {}_2F_1(\nu, \nu; 1; 1) = c_\alpha^{-1} < \infty$$

by formula (4.3). $\square$

We may summarize our findings as follows.

Theorem 4.1. *For every $\phi \in \mathcal{H}^\lambda$ the Hankel-Toeplitz type operator $\mathfrak{H}_\phi$ is a Hilbert-Schmidt operator on $\mathcal{A}^\nu$ where $\lambda = 1 - 2\nu$ and $0 < \nu < \frac{1}{2}$ with the Hilbert-Schmidt norm given by*

$$\|\mathfrak{H}_\phi\|_{\mathfrak{S}_2}^2 = c_\alpha^{-1}\|\phi\|_{\mathcal{H}^\lambda}^2.$$

The symbol map $\phi \mapsto \mathfrak{H}_\phi$ is an isometry from $\mathcal{H}^\lambda$ into $\mathfrak{S}_2(\mathcal{A}^\nu)$ and it intertwines with the natural actions of the group $\mathfrak{G}$ on these spaces. $\square$

Corollary 4.2. *Making the identification $\mathfrak{S}_2(\mathcal{A}^\nu) = \mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ we have a representation of the complementary series in the Hilbert tensor products $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ with $0 < \nu < \frac{1}{2}$.*

This last fact follows from the results in [DH]. $\square$

5. Cut-off. In this Section we consider membership of our Hankel-Toeplitz type operator $\mathfrak{H}_\phi$ in Schatten-von Neumann classes. (For the theory of Schatten-von Neumann classes see e. g. [Si].) We know already (Section 4) that $\mathfrak{H}_\phi$ is in $\mathfrak{S}_2$ for any symbol $\phi \in \mathcal{H}^\lambda$ (see Theorem 4.1).

We are going to apply a very well-known reasoning. Assume that we have $\mathfrak{H}_\phi \in \mathfrak{S}_p$ where $\phi \not\equiv 0$. As for some Fourier coefficient it holds that $\hat{\phi}(r) \neq 0$, taking averages over rotations about the origin $0 \in \mathbb{D}$, we see that $0 \neq \mathfrak{H}_{x^r} \in \mathfrak{S}_p$ also. The pth power of the $\mathfrak{S}_p$-norm of an operator is the sum of the corresponding powers of its singular values. So we must compute the singular values, say, s_n $(n = 0, 1, 2, \ldots)$ of $\mathfrak{H}_{x^r}$.

For simplicity let $r \geq 0$. From (4.2) one finds

$$\mathfrak{H}_{x^r}^* z^{n+r} = \frac{(\nu)_n}{n!} z^n$$

and hence

$$\mathfrak{H}_{x^r}^* \mathfrak{H}_{x^r} z^n = \frac{(\nu)_{n+r}}{(n+r)!}\frac{(\nu)_n}{n!} z^n.$$

We conclude that, again in view of Stirling's formula,

$$s_n = \sqrt{\frac{(\nu)_{n+r}}{(n+r)!}\frac{(\nu)_n}{n!}} = \frac{1}{\Gamma(\nu)} \cdot \sqrt{\frac{\Gamma(n+r+\nu)}{\Gamma(n+r+1)}\frac{\Gamma(n+\nu)}{\Gamma(n+1)}} \sim \frac{1}{\Gamma(\nu)} n^{-(1-\nu)}.$$

Thus

$$\sum_{n=0}^{\infty} s_n^p < \infty \iff p(1-\nu) > 1 \text{ or } \tfrac{1}{p} < 1 - \nu.$$

Thus, to sum up, we have established the following theorem.

Theorem 5.1. *There exists a Hankel-Toeplitz type operator $\mathfrak{H}_\phi$ with non-vanishing symbol ϕ, $\phi \not\equiv 0$, belonging to the Schatten-von Neumann class $\mathfrak{S}_p(\mathcal{A}^\nu)$ if and only if $\frac{1}{p} < 1 - \nu$. In other words, we have the cut-off at $\frac{1}{p} = 1 - \nu$.* $\square$

Corollary 5.1. *There are no Hankel-Toeplitz type operators with non-vanishing symbol which are of finite rank.* $\square$

Remark 5.1. (on the case $\nu > \frac{1}{2}$) The definition of $\mathfrak{H}_\phi$ makes formally sense for any ν. For instance, we can define it on the vector space of all polynomials. If $\frac{1}{2} < \nu < 1$ we can still conclude that we get $\mathfrak{S}_p$-operators only if $\frac{1}{p} < 1 - \nu$. If $\nu > 1$ the interpretation of our computation is that the operator is never bounded (unless the symbol vanishes). Note also that for $\nu = 1$, $\mathfrak{H}_\phi$ reduces to the classical Toeplitz operators mentioned in the introduction. $\square$

6. The case of the unit ball in $\mathbb{C}^d$. We consider the complex vector space $\mathbb{C}^d$ with its standard Hermitian inner product $\langle \cdot, \cdot \rangle$, denoting the unit ball by $B = B^d$. Let $dm(z)$ be the Lebesgue measure on $\mathbb{C}^d$ and consider, for $\nu > d$, on B the normalized weighted measure $d\mu_\nu$ defined by

$$d\mu_\nu(z) = C_\nu (1 - |z|^2)^{\nu - d - 1} dm(z),$$

where

$$C_\nu = \frac{1}{\pi^d} \frac{\Gamma(\nu)}{\Gamma(\nu - d)}$$

is the normalizing constant. Consider also the Bergman space $\mathcal{A}^\nu = \mathcal{A}^\nu(B)$ of square integrable holomorphic functions with respect to $d\mu_\nu$. It has the reproducing kernel

$$K^\nu(z, w) = (1 - \langle z, w \rangle)^{-\nu}.$$

The group $\mathfrak{G} = \mathrm{SU}(d, 1)$ of biholomorphic mappings of B acts unitarily on the space $\mathcal{A}^\nu$ via (a projective representation)

$$f(z) \mapsto f(gz) J_g(z)^{\frac{\nu}{d+1}},$$

where J_g is the Jacobian determinant of $g \in \mathfrak{G}$. The reproducing kernel $K^\nu(z, w)$ is positive definite for all $\nu > d$. This follows also from the binomial expansion

$$K^\nu(z, w) = \sum_{m=0}^{\infty} \frac{(\nu)_m}{m!} \langle z, w \rangle^m; \tag{6.1}$$

indeed, this shows that K^ν (now taking (6.1) as a definition) is positive definite for all $\nu > 0$. We thus get an analytic continuation of the Hibert spaces $\mathcal{A}^\nu$ for all $\nu > 0$. (When $\nu = 0$ we get the trivial Hilbert space consisting of only the constant functions.) We let $\bar{\mathcal{A}}^\nu$ be the conjugate of the Hilbert space $\mathcal{A}^\nu$, namely the space consisting of the complex conjugates $\bar{f}$ of the functions f in $\mathcal{A}^\nu$. Moreover, the group $\mathfrak{G}$ acts on $\bar{\mathcal{A}}^\nu$ by taking formally complex conjugates.

Our main concern will be the tensor product $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$. Note that the group action of $\mathfrak{G}$ on the tensor product forms a genuine representation of this group. As in the case of the unit disc, there is a family of unitary representations of $\mathfrak{G}$ realizing a complementary series on the spaces $\mathcal{H}^\lambda$ of functions ψ on the unit sphere $S = S^{2d-1} = \partial B$ in $\mathbb{C}^d$, with norm

$$\|\psi\|_\lambda^2 = \iint_{S\times S} \frac{\psi(x)\overline{\psi(y)}}{|1-\langle x,y\rangle|^{d+1-\lambda}} d\sigma(x)d\sigma(y),$$

where $d\sigma$ is the normalized area measure on S. Similarly to the unit disc, this can be written as

$$\|\psi\|_\lambda^2 = \langle U^{d+1-\lambda}\psi, \psi\rangle,$$

where

$$U^\alpha\psi(x) = \int_S |1-\langle x,y\rangle|^{-\alpha}\psi(y)\, d\sigma(y).$$

We now construct the intertwining operators of our complementary series into the tensor product $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$. We view the elements in $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ as functions on $B\times B$, holomorphic in z and anti-holomorphic in w. For any integer $k \geq 0$ and a function ψ on ∂B we set

$$\boxed{\mathfrak{H}_\psi^{(k)}(z,w) = \int_S \frac{(1-\langle z,w\rangle)^k\psi(s)}{(1-\langle z,s\rangle)^{k+\nu}(1-\langle s,w\rangle)^{k+\nu}} d\sigma(s)}\,.$$

Theorem 6.1. *The operator*

$$\mathcal{K}^{(k)} : \psi \mapsto \mathfrak{H}_\psi^{(k)}(z,w)$$

is up to a non-zero constant a unitary intertwining operator from the space $\mathcal{H}^{d+1-2\nu-2k}$ into the tensor product $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$, for $0 \leq k < \frac{d-2\nu}{2}$.

Proof. As is indicated in Section 4, we need only to prove that $\mathfrak{H}_\psi^{(k)}$ is in $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$ for some $\psi \neq 0$ in $\mathcal{H}^{d+1-2\nu-2k}$. The result then follows from Schur's lemma. Taking $\psi = 1$ we have

$$\mathfrak{H}_\psi^{(k)}(z,w) = (1-\langle z,w\rangle)^k \sum_{m,n=0}^\infty \frac{(\nu+k)_m}{m!}\cdot\frac{(\nu+k)_n}{n!}\int_S \langle z,s\rangle^m\langle s,w\rangle^n d\sigma(s).$$

The integral can be calculated directly (see e.g. [Ru], p. 18):

$$\int_S \langle z,s\rangle^m\langle s,w\rangle^n d\sigma(s) = \frac{m!}{(d)_m}\langle z,w\rangle^m\delta_{m,n}$$

where $\delta_{m,n}$ is the Kronecker symbol. Thus

$$\mathfrak{H}_\psi^{(k)}(z,w) = (1-\langle z,w\rangle)^k \sum_{m=0}^\infty \frac{(\nu+k)_m(\nu+k)_m}{m!(d)_m}\langle z,w\rangle^m.$$

Now $(1-\langle z,w\rangle)^k = \sum_{j=0}^k (-1)^j\binom{k}{j}\langle z,w\rangle^j$ and therefore

$$\mathfrak{H}_\psi^{(k)}(z,w) = \sum_{j=0}^k \binom{k}{j}(-1)^j\langle z,w\rangle^j \sum_{m=0}^\infty \frac{(\nu+k)_m(\nu+k)_m}{m!(d)_m}\langle z,w\rangle^m =$$

$$= \sum_{m=0}^\infty C_m\langle z,w\rangle^m$$

with

$$C_m = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\nu+k)_{m-j}(\nu+k)_{m-j}}{(m-j)!(d)_{m-j}}.$$

Let us rewrite the coefficient C_m in terms of the hypergeometric function. Notice that

$$\binom{k}{j} = (-1)^j \frac{(-k)_j}{j!}, \quad (u)_{m-j} = (-1)^j \frac{(u)_m}{(-u-m+1)_j}$$

for any u. Thus we have

$$\begin{aligned} C_m &= \frac{(\nu+k)_m(\nu+k)_m}{(d)_m m!} \sum_{j=0}^{k} \frac{(-k)_j}{j!} \frac{(-m)_j(-d-m+1)_j}{(-\nu-k-m+1)_j(-\nu-k-m+1)_j} \\ &= \frac{(\nu+k)_m(\nu+k)_m}{(d)_m m!} \times \\ &\times {}_3F_2(-k,-m,-d-m+1;-\nu-k-m+1,-\nu-k-m+1;1). \end{aligned}$$

To get the right approximation formula of C_m for large m we use Thomae's transformation formula (see [GR], p. 59),

$${}_3F_2(-n,a,b;c,d;1) = \frac{(d-b)_n}{(d)_n} {}_3F_2(-n,c-a,b;c,1+b-d-n;1),$$

to obtain

$$\begin{aligned} C_m &= \frac{(\nu+k)_m(\nu+k)_m(-\nu-k+d)_k}{(d)_m m!(-\nu-k-m+1)_k} \times \\ &\times {}_3F_2(-k,-\nu-k+1,-d-m+1;-\nu-k-m+1,1+\nu-d;1). \end{aligned}$$

The sum ${}_3F_2(-k,-\nu-k+1,-d-m+1;-\nu-k-m+1,1+\nu-d;1)$ is bounded for large m. Moreover, we have

$$\begin{aligned} \left| \frac{(\nu+k)_m(\nu+k)_m(-\nu-k+d)_k}{(d)_m m!(-\nu-k-m+1)_k} \right| &= \left| \frac{(\nu+k)_m(\nu+k)_m(\nu-d+1)_k}{(d)_m m!(\nu+m)_k} \right| \approx \\ &\approx \frac{1}{m^{d+1-2\nu-k}}, \end{aligned}$$

whence

$$|C_m| = \mathrm{O}\left(\frac{1}{m^{d+1-2\nu-k}} \right). \tag{6.2}$$

We calculate now the Hilbert-Schmidt norm of $\mathfrak{H}_\psi^{(k)}$. From the binomial expansion (6.1) we see that the expression $\frac{(\nu)_m}{m!}\langle z,w\rangle^m$ there corresponds to the orthogonal projection from $\mathcal{A}^\nu$ into the subspace of homogeneous polynomials of degree m, whose dimension is $\binom{m+d-1}{d-1}$. Thus

$$\left\| \frac{(\nu)_m}{m!}\langle z,w\rangle^m \right\|^2_{\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu} = \binom{m+d-1}{d-1} = \frac{(d)_m}{m!},$$

that is,

$$\|\langle z,w\rangle^m\|^2_{\mathcal{A}^\nu\otimes\bar{\mathcal{A}}^\nu} = \frac{(d)_m m!}{(\nu)_m^2}.$$

The elements $\langle z,w\rangle^m$ are pairwise orthogonal. Thus $\mathfrak{H}_\psi^{(k)}$ is in the tensor product space if and only if the series

$$\|\mathfrak{H}_\psi^{(k)}\|^2_{\mathcal{A}^\nu\otimes\bar{\mathcal{A}}^\nu} = \sum_{m=0}^\infty C_m^2 \|\langle z,w\rangle^m\|^2_{\mathcal{A}^\nu\otimes\bar{\mathcal{A}}^\nu} = \sum_{m=0}^\infty C_m^2 \frac{(d)_m m!}{(\nu)_m^2} \tag{6.3}$$

is convergent. Now $\frac{(d)_m m!}{(\nu)_m^2} \approx m^{d+1-2\nu}$ so using (6.2) we see that the above series (6.3) is approximately

$$\sum_{m=1}^\infty \frac{1}{m^{d+1-2\nu-2k}},$$

which series again is convergent if and only if $2\nu+2k<d$. This completes the proof. $\square$

Remark 6.1. The condition on the parameters can also be written $\nu < \frac{d}{2}-k$, so if let formally $d=1$ and take $k=0$ then it reduces to the prevous condition $\nu<\frac{1}{2}$ (see Section 4; remember that we have assumed that $k\geq 0$). $\square$

Remark 6.2. The coefficients C_m appearing in the proof of Theorem 6.1 form a system of orthogonal polynomials. More precisely, let us write

$$p_k(m) = \frac{(d)_m m!}{(\nu)_m^2} C_m.$$

It follows from Thomae's transformation formulas that

$$p_k(m) = {}_3F_2(-k,-m,2\nu+k-d;\nu,\nu;1),$$

which shows that $p_k(m)$ is a polynomial in m of degree k.

To deduce an orthogonality relation, choose two integers k, l in the range $0\leq k,l<\frac{d}{2}-\nu$. It then follows from Schur's lemma that (with $\psi=1$)

$$\sum_{m=0}^\infty \frac{(\nu)_m^2}{(d)_m m!} p_k(m)p_l(m) = \|\mathfrak{H}_\psi^{(k)}\|^2_{\mathcal{A}^\nu\otimes\bar{\mathcal{A}}^\nu}\delta_{k,l}$$

(this reduces to (6.3) when $k=l$). Clearly, the p_k do not form a complete system, since their orthogonality measure has only finitely many finite moments. Therefore, they are not in the Askey Scheme of hypergeometric orthogonal polynomials [KS]. They may, however, be viewed as a limit case of the Racah polynomials, and also as dual to (a special case of) the continuous dual Hahn polynomials. More precisely, the equation $q_k(x(2\nu+x-d)) = p_x(k)$ defines a polynomial q_k of degree k which is essentially a continuous dual Hahn polynomial. The finitely many points $0\leq x<\frac{d}{2}-\nu$, x integer, then correspond to the discrete part of the orthogonality measure for q_k. From a group-theoretic point of view, the polynomials p_k and q_k have an interpretation as a rather special kind of Clebsch–Gordan coefficients.

The constant appearing in (6.3) can be obtained formally as a limit from the norm constant for Racah polynomials or by mimicking the corresponding computation in [AW]. It is given by

$$\|\mathfrak{H}_\psi^{(k)}\|^2_{\mathcal{A}^\nu\otimes\bar{\mathcal{A}}^\nu} = (-1)^k\frac{\Gamma(d)\Gamma(d-2\nu)}{\Gamma(d-\nu)^2}\frac{2\nu-d}{2\nu-d+2k}\frac{(1+\nu-d)_k^2 k!}{(\nu)_k^2(2\nu-d)_k}.$$

Denoting this constant by C we have that $(\sqrt{C})^{-1}\mathcal{K}^{(k)}$ is an isometric operator from $\mathcal{H}^\lambda$ into the tensor product.

Remark 6.3. Identifying the function $\mathfrak{H}_\psi^{(k)}(z,w)$ with an operator on $\mathcal{A}^\nu$ as in Section 3 and 4, it is now a combination of Toeplitz and differential operators. Indeed the operator $\mathfrak{H}_\psi^{(k)}$ on $f \in \mathcal{A}^\nu$ is

$$\mathfrak{H}_\psi^{(k)} f(z) = \left\langle f, \int_S \frac{\overline{\psi(s)}(1-\langle \cdot, z\rangle)^k}{(1-\langle s,z\rangle)^{k+\nu}(1-\langle \cdot, s\rangle)^{k+\nu}} d\sigma(s) \right\rangle_{\mathcal{A}^\nu}.$$

Taking f to be a polynomial and replacing the integral on S by εS, with $0<\varepsilon<1$, in the above inner product and, finally, pulling out the integral sign we obtain the expression

$$\int_{\varepsilon S} \frac{\psi(s)}{(1-\langle z,s\rangle)^{k+\nu}} \left\langle f, \frac{(1-\langle \cdot, z\rangle)^k}{(1-\langle \cdot, s\rangle)^{k+\nu}} \right\rangle_{\mathcal{A}^\nu} d\sigma(s).$$

Observe that

$$\begin{aligned} \frac{(1-\langle w,z\rangle)^k}{(1-\langle w,s\rangle)^{k+\nu}} &= \frac{(1-\langle w,s\rangle + \langle w, s-z\rangle)^k}{(1-\langle w,s\rangle)^{k+\nu}} = \\ &= \sum_{j=0}^k \binom{k}{j} \frac{\langle w, s-z\rangle^j}{(1-\langle w,s\rangle)^{j+\nu}} = \\ &= \sum_{j=0}^k \binom{k}{j} \sum_{|\beta|=j} \frac{j!}{\beta!} \frac{w^\beta(\bar{s}-\bar{z})^\beta}{(1-\langle w,s\rangle)^{j+\nu}}, \end{aligned}$$

where $\beta = (\beta_1, \cdots, \beta_d)$ are multi-indices with nonnegative integer components with the usual convention that $s^\beta = s_1^{\beta_1}\dots s_d^{\beta_d}$ and $\beta! = \beta_1!\dots\beta_d!$, and where the sum is over all β with $\beta_1+\cdots+\beta_n = j$. Therefore, taking w as the independent variable in $\mathcal{A}^\nu$, we obtain

$$\left\langle f, \frac{(1-\langle w,z\rangle)^k}{(1-\langle w,s\rangle)^{k+\nu}} \right\rangle_{\mathcal{A}^\nu} = \sum_{j=0}^k \binom{k}{j} \sum_{|\beta|=j} \frac{j!}{\beta!} (s-z)^\beta \left\langle f(w), \frac{w^\beta}{(1-\langle w,s\rangle)^{j+\nu}} \right\rangle_{\mathcal{A}^\nu},$$

To find the inner product we differentiate the reproducing property

$$f(s) = \langle f(w), (1-\langle w,s\rangle)^{-\nu}\rangle_{\mathcal{A}^\nu}$$

to get

$$\partial^\beta f(s) = (\nu)_j \langle f(w), w^\beta (1-\langle w,s\rangle)^{-\nu-j}\rangle_{\mathcal{A}^\nu}.$$

Putting the formulas together we get

$$\begin{aligned} &\int_{\varepsilon S} \frac{\psi(s)}{(1-\langle s,z\rangle)^{k+\nu}} \left\langle f, \frac{(1-\langle w,z\rangle)^k}{(1-\langle w,s\rangle)^{k+\nu}} \right\rangle_{\mathcal{A}^\nu} d\sigma(s) \\ &= \sum_{j=0}^k \binom{k}{j} \frac{1}{(\nu)_j} \sum_{|\beta|=j} \frac{j!}{\beta!} \int_{\varepsilon S} \frac{\psi(s)(s-z)^\beta \partial^\beta f(s)}{(1-\langle z,s\rangle)^{\nu+k}} d\sigma(s). \end{aligned}$$

Now for polynomial f this integral is absolute convergent when $\varepsilon \to 1$, for fixed $z \in B$. We thus get

$$\boxed{\mathfrak{H}_{\psi}^{(k)} f(z) = \sum_{j=0}^{k} \binom{k}{j} \frac{1}{(\nu)_j} \sum_{|\beta|=j} \frac{j!}{\beta!} \int_S \frac{\psi(s)(s-z)^{\beta} \partial^{\beta} f(s)}{(1-\langle z, s\rangle)^{\nu+k}} d\sigma(s)}. \quad \square$$

Remark 6.4. We can also perform the polarization operator on the spherical function for $\mathfrak{G} = SU(d,1)$ to prove Theorem 6.1. Furthermore, we can consider, more generally, the tensor product of $\mathcal{A}^{\nu_1} \otimes \bar{\mathcal{A}}^{\nu_2}$ with two different parameters ν_1 and ν_2. Its decomposition is done in [H2]. $\square$

7. A sesquilinear transvectant. In [JP], in the case of $\mathbb{D}$, higher order bilinear Hankel forms were constructed with the help of certain bilinear differential operators termed transvectants, a notion borrowed from classical invariant theory and presumably first used by Clebsch. (The extension to the multilinear situation was performed in [Ro].) Now we show, in the case of B, that the operators $\mathfrak{H}_{\phi}^{(k)}$ introduced in Section 6 in a similar way are generated by corresponding sesquilinear differential operators. If $d = 1$ ($B = \mathbb{D}$) and $k = 0$ this has already been indicated in Remark 4.1, in which case the transvectant is nothing but the multiplication operator $f \otimes g \mapsto f\bar{g}$.

So let us place ourselves in the d-dimensional situation of Section 6. If $\mathcal{K}^{(k)}$ denotes the map $\phi \mapsto \mathfrak{H}_{\phi}^{(k)}$, then its adjoint $\mathcal{J}^{(k)}$ admits the factorization

$$\boxed{\mathcal{J}^{(k)} = (U^{\alpha})^{-1} \mathcal{R}^{(k)}},$$

where U^{α} is the same potential transform as in Section 6, $\alpha = 2\nu + 2k$, and $\mathcal{R}^{(k)}$ is given by

$$\mathcal{R}^{(k)}(f, \bar{g})(s) = \left\langle f \otimes \bar{g}, \frac{(1-\langle z, w\rangle)^k}{(1-\langle z, s\rangle)^{\nu+k}(1-\langle s, w\rangle)^{\nu+k}} \right\rangle_{\mathcal{A}^{\nu} \otimes \bar{\mathcal{A}}^{\nu}}.$$

In order to formulate our result let us introduce the radial derivative,

$$Rf(s) = \sum_{j=1}^{d} s_j \partial_j f(s),$$

and the sesquilinear Beltrami differential parameter[1],

$$(\partial \otimes \bar{\partial})(f\bar{g}) = \sum_{j=1}^{d} \partial_j f \, \overline{\partial_j g}.$$

Theorem 7.1. *$\mathcal{R}^{(k)}$ is a local operator and we have the following explicit formula:*

$$\boxed{\mathcal{R}^{(k)}(f\bar{g}) = \frac{1}{(\nu)_k^2} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (\partial \otimes \bar{\partial})^j \big((R+\nu)_{k-j} \otimes (\bar{R}+\nu)_{k-j}\big)(f\bar{g})}.$$

[1] *Historical note.* Classicaly, Eugenio Beltrami defined two "differential parameters"; the first one Δ_1 is the corresponding gadget in the real case, while the second one Δ_2 is just the Laplace operator.

It is remarkable that this sesquilinear differential operator gives a formal intertwining operator from the tensor product to the complementary series.

Proof. That $\mathcal{R}^{(k)}$ is a local operator means simply that it is, at each point $s \in B$, a finite linear combination of products of derivatives. If $s = 0$ then this is obvious, as only finitely many Taylor coefficients intervene. The general case follows then by the covariance.

To establish the explicit presentation we note first that expanding $(1-\langle z, w\rangle)^k$ according the binomial theorem (see proof of Theorem 6.1) it suffices to deal with the expressions

$$\left\langle f \otimes \bar{g}, \frac{\langle z, w\rangle^j}{(1-\langle z, s\rangle)^{\nu+k}(1-\langle s, w\rangle)^{\nu+k}} \right\rangle_{\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu} \qquad (j = 0, 1, \ldots, k). \tag{7.1}$$

Furthermore, we need also some preparations involving the operator R. Differentiating the reproducing property

$$f(s) = \left\langle f(\cdot), \frac{1}{(1-\langle \cdot, s\rangle)^\nu} \right\rangle_{\mathcal{A}^\nu}$$

yields

$$Rf(s) = \nu \left\langle f(\cdot), \frac{\langle \cdot, s\rangle}{(1-\langle \cdot, s\rangle)^{\nu+1}} \right\rangle_{\mathcal{A}^\nu}.$$

Next, writing[2]

$$\frac{\langle z, s\rangle}{(1-\langle z, s\rangle)^{\nu+1}} = \frac{1-(1-\langle z, s\rangle)}{(1-\langle z, s\rangle))^{\nu+1}} = \frac{1}{(1-\langle z, s\rangle)^{\nu+1}} - \frac{1}{(1-\langle z, s\rangle)^{\nu}},$$

we find

$$Rf(s) = -\nu f + \nu \left\langle f(\cdot), \frac{1}{(1-\langle \cdot, s\rangle)^{\nu+1}} \right\rangle_{\mathcal{A}^\nu},$$

or

$$\left\langle f(\cdot), \frac{1}{(1-\langle \cdot, s\rangle)^{\nu+1}} \right\rangle_{\mathcal{A}^\nu} = \frac{1}{\nu}(R+\nu)f(s). \tag{7.2}$$

Using formula (7.2) recursively we have

$$\left\langle f(\cdot), \frac{1}{(1-\langle \cdot, s\rangle)^{\nu+(k-j)}} \right\rangle_{\mathcal{A}^\nu} = \frac{1}{(\nu)_{k-j}}(R+\nu)_{k-j} f(s)$$

and consequently

$$\begin{aligned} &\frac{1}{(\nu)^2_{k-j}}\big((R+\nu)_{k-j} \otimes (\bar{R}+\nu)_{k-j}\big)(f\bar{g})(s) \\ &= \left\langle f(z)\bar{g}(w), \frac{1}{(1-\langle z, s\rangle)^{\nu+(k-j)}(1-\langle s, w\rangle)^{\nu+(k-j)}} \right\rangle_{\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu}. \end{aligned} \tag{7.3}$$

[2] Here we are using one of the perhaps most conspicuous identities in mathematics: $\dfrac{a}{1-a} = -1 + \dfrac{1}{1-a}$.

Let the operator $\partial \otimes \bar{\partial}$ act on (7.3),

$$\frac{1}{(\nu)^2_{k-j}}(\partial \otimes \bar{\partial})^j((R+\nu)_{k-j} \otimes (\bar{R}+\nu)_{k-j})(f\bar{g})(s)$$
$$= (\nu+k-j)^2_j \left\langle f(z)\bar{g}(w), \frac{\langle z,w\rangle^j}{(1-\langle z,s\rangle)^{\nu+k}(1-\langle s,w\rangle)^{\nu+k}} \right\rangle_{\mathcal{A}^\nu \otimes \overline{\mathcal{A}^\nu}};$$

that is,

$$\left\langle f(z)\bar{g}(w), \frac{\langle z,w\rangle^j}{(1-\langle z,s\rangle)^{\nu+k}(1-\langle s,w\rangle)^{\nu+k}} \right\rangle_{\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu}$$
$$= \frac{1}{(\nu)^2_k}(\partial \otimes \bar{\partial})^j \left((R+\nu)_{k-j} \otimes (\bar{R}+\nu)_{k-j}\right)(f\bar{g})(s);$$

this clearly implies the sought formula. $\square$

Remark 7.1. (a different approach) Instead of using $\partial \otimes \bar{\partial}$, R, $\bar{R}$ one can simply write $\mathcal{R}^{(k)}$ with the aid of the Wirtinger operators ∂_j and $\bar{\partial}_j$ ($j = 1, \ldots, d$) with respect to the coordinates.

This time we expand the kernel of $\mathcal{R}^{(k)}$ as

$$\frac{(1-\langle z,w\rangle)^k}{(1-\langle z,s\rangle)^{\nu+k}(1-\langle s,w\rangle)^{\nu+k}} = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{\langle z,w\rangle^j}{(1-\langle z,s\rangle)^{\nu+k}(1-\langle s,w\rangle)^{\nu+k}} =$$
$$= \sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{\langle z,w\rangle^j}{(1-\langle z,s\rangle)^{\nu+j}(1-\langle s,w\rangle)^{\nu+j}}\left(1+\frac{\langle z,s\rangle}{1-\langle z,s\rangle}\right)^{k-j}\left(1+\frac{\langle s,w\rangle}{1-\langle s,w\rangle}\right)^{k-j} =$$
$$= \sum_{j=0}^{k}\sum_{p,q=0}^{k-j}(-1)^j\binom{k}{j}\binom{k-j}{p}\binom{k-j}{q}\frac{\langle z,w\rangle^j\langle z,s\rangle^p\langle s,w\rangle^q}{(1-\langle z,s\rangle)^{\nu+j+p}(1-\langle s,w\rangle)^{\nu+j+q}}.$$

Expanding the three numerator factors by the multinomial theorem gives

$$\sum_{j,p,q}(-1)^{|j|}\frac{k!(k-|j|)!}{(k-|j|-|p|)!(k-|j|-|q|)!j!\,p!\,q!}\frac{\bar{s}^p s^q z^{j+p}\bar{w}^{j+q}}{(1-\langle z,s\rangle)^{\nu+|j+p|}(1-\langle s,w\rangle)^{\nu+|j+q|}},$$

where the sum is now over multi-indices j, p, q with $|j|+|p|,\ |j|+|q| < k$. Since, in general,

$$\partial^j f(s) = (\nu)_{|j|}\left\langle f(\cdot), \frac{(\cdot)^j}{(1-\langle\cdot,s\rangle)^{\nu+|j|}}\right\rangle_{\mathcal{A}^\nu},$$

we obtain the explicit formula

$$\boxed{\mathcal{R}^{(k)}(f\otimes\bar{g})(s) = \sum_{j,p,q}(-1)^{|j|}\frac{k!(k-|j|)!}{(k-|j|-|p|)!(k-|j|-|q|)!j!\,p!\,q!}\frac{s^p\bar{s}^q\partial^{j+p}f(s)\overline{\partial^{j+q}g(s)}}{(\nu)_{|j+p|}(\nu)_{|j+q|}}}. \quad \square$$

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Abstract

Let $\mathcal{A}^\nu$ denote the weighted Bergman spaces either over the disc $\mathbb{D}$ or over the complex ball B in dimension $d > 1$. In a recent paper (G. van Dijk - S. C. Hille, Canonical representations related to hyperbolic spaces. J. Functional Anal. 147 (1997), 109-139) it was established that in certain cases the complementary series enters in the decomposition of the tensor product $\mathcal{A}^\nu \otimes \bar{\mathcal{A}}^\nu$.

In this paper we wish to realize the complementary series as a space of operators. More precisely, we associate, in the case of $\mathbb{D}$, to every "symbol" ϕ a Hankel-Toeplitz type operator $\mathfrak{H}_\phi$ acting in $\mathcal{A}^\nu$ $(0 < \nu < \frac{1}{2})$. We prove that $\mathfrak{H}_\phi$ is Hilbert-Schmidt if and only if ϕ belongs to the Hilbert space $\mathcal{H}^\lambda$ of the complementary series of representations of $\mathrm{SU}(1,1)$, where $\lambda = 2 - 2\nu$. As for Schatten-von Neumann properties of $\mathfrak{H}_\phi$, we establish the presence of a cut-off occuring at p where $\dfrac{1}{p} = 1 - \nu$.

In the case of B we define covariantly k different Hankel-Toeplitz type operators denoted $\mathfrak{H}_\psi^{(k)}$ in $\mathcal{A}^\nu$, where ψ is the symbol and $0 \leq k < \dfrac{d-2\nu}{2}$. We show that the operators $\mathfrak{H}_\psi^{(k)}$ also can be obtained with the help of certain sesquilinear differential operators which we, by analogy with how this word is used in the connection with (bilinear) Hankel forms, refer to as transvectants.

Classification: 22E46; 44A85; 47B10.

Keywords: holomorphic representation; complementary series; Hankel-Toeplitz type operator; generalized weighted Bergman space; Hilbert-Schmidt class; Schatten-von Neumann class; cut-off; sesquilinear transvectant.

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