# MULTIVARIABLE ORTHOGONAL POLYNOMIALS AND COUPLING COEFFICIENTS FOR DISCRETE SERIES REPRESENTATIONS

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ABSTRACT. We study polynomials of several variables which occur as coupling coefficients for the analytic continuation of the holomorphic discrete series of $\mathrm{SU}(1,1)$. There are three types of such polynomials, one corresponding to each conjugacy class of one-parameter subgroups. They may be viewed as multivariable generalizations of Hahn, Jacobi, and continuous Hahn polynomials and include many orthogonal and biorthogonal families occurring in the literature. We give a simple and unified approach to these polynomials using the group theoretic interpretation. We prove many formal properties, in particular a number of convolution and linearization formulas. We also develop the corresponding theory for the Heisenberg group, leading to multivariable generalizations of Krawtchouk and Hermite polynomials.

## 1. INTRODUCTION

In [R], we introduced three classes of polynomials in several variables. They appeared there as coupling coefficients for the group $\mathrm{SU}(1,1) \simeq \mathrm{SL}(2,\mathbb{R})$ and its covering groups. Special cases occur in the work of many authors, cf. [A1], [A2], [AK], [E], [Ex], [FL], [KMT], [KM1], [KM2], [K1], [KS], [LT], [M], [MP], [Pr], [Ra], [T1], [T2], [T4], [T5], and [V]. In this paper we give a simple and unified approach to the study of these polynomials. We also consider the case of the oscillator or Heisenberg group, leading to two more classes of polynomials.

In many cases such multivariable polynomials arose naturally from problems in physics or statistics. In view of the group theoretic interpretation this is not surprising, since symmetry groups play an important role in these sciences. There are also close connections with spherical harmonics, as well as with the Wigner symbols (recoupling coefficients) occurring in quantum mechanics.

We will work with the analytic continuation of the holomorphic discrete series of $\mathrm{SU}(1,1) \simeq \mathrm{SL}(2,\mathbb{R})$. We realize these representations on the Hilbert spaces $\mathcal{A}^\nu$ ($\nu > 0$) of analytic functions on the complex unit disc, with the scalar product

$$\langle f, g\rangle = \sum_{k=0}^{\infty} \frac{k!}{(\nu)_k} \hat{f}(k)\overline{\hat{g}(k)},$$

where $f(z) = \sum \hat{f}(k) z^k$ and

$$(\nu)_k = \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \nu(\nu+1)\cdots(\nu+k-1)$$

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is the Pochhammer symbol. The norm is invariant under the transformations

$$f(z) \mapsto f\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)\frac{1}{(\gamma z+\delta)^\nu}, \qquad \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{SU}(1,1).$$

This may be used to define a unitary representation of the universal covering group of $\mathrm{SU}(1,1)$ on each space $\mathcal{A}^\nu$, cf. [Sa] for the details.

We consider a Hilbert tensor product $\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n}$ of such spaces. It decomposes under the group action as

$$\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n} \simeq \bigoplus_{s=0}^{\infty}\binom{n+s-2}{n-2}\mathcal{A}^{|\nu|+2s} \tag{1}$$

(where $|\nu|=\sum\nu_i$). A highest weight vector $Q$ is an image of the constant function 1 under an intertwining embedding

$$\mathcal{A}^{|\nu|+2s} \to \mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n}. \tag{2}$$

In our realization, this means that $Q$ is a homogeneous polynomial of degree $s$, which may be expressed as a function of differences $z_i-z_j$ of the coordinates. Equivalently, $Q$ satisfies the identity

$$Q(az_1+b,\dots,az_n+b) = a^s Q(z_1,\dots,z_n), \qquad a,b\in\mathbb{C}. \tag{3}$$

In agreement with (1), the space of such polynomials has dimension $\binom{n+s-2}{n-2}$.

In [R], we introduced three transforms which we will denote here by $T_1$, $T_2$, and $T_3$. Their role is to send highest weight vectors to coupling coefficients. They are defined on $z^k\in\mathcal{A}^\nu$ by

$$T_1z^k(m) = (-1)^k\frac{(-m)_k}{(\nu)_k}, \quad T_2z^k(\xi)=\frac{\xi^k}{(\nu)_k}, \quad T_3z^k(X)=(-1)^k\frac{(\frac{\nu}{2}-iX)_k}{(\nu)_k} \tag{4}$$

and extended to power series by linearity. We will also denote by $T_i$ the extension $T_i\otimes\cdots\otimes T_i$ to a tensor product $\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n}$. Thus if we let

$$Q(z)=\sum_{|t|=s} c_t\, z_1^{t_1}\cdots z_n^{t_n}$$

be a highest weight vector in $\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n}$ (throughout the paper we use multi-index notation freely), the three transforms of $Q$ are given by

$$T_1Q(m_1,\dots,m_n) = (-1)^s\sum_{|t|=s}c_t\,\frac{(-m_1)_{t_1}\cdots(-m_n)_{t_n}}{(\nu_1)_{t_1}\cdots(\nu_n)_{t_n}}, \tag{5}$$

$$T_2Q(\xi_1,\dots,\xi_n) = \sum_{|t|=s}c_t\,\frac{\xi_1^{t_1}\cdots\xi_n^{t_n}}{(\nu_1)_{t_1}\cdots(\nu_n)_{t_n}}, \tag{6}$$

$$T_3Q(X_1,\dots,X_n) = (-1)^s\sum_{|t|=s}c_t\,\frac{(\frac{\nu_1}{2}-iX_1)_{t_1}\cdots(\frac{\nu_n}{2}-iX_n)_{t_n}}{(\nu_1)_{t_1}\cdots(\nu_n)_{t_n}}. \tag{7}$$

Note that the $T_iQ$ are again polynomials of degree $s$, though only $T_2Q$ is homogeneous. Moreover, $T_1$ and $T_3$ are connected by the equation

$$T_3Q(X) = T_1Q\left(iX-\frac{\nu}{2}\right). \tag{8}$$

One may also prove that

$$T_3Q(X) = (-1)^s\, T_3Q(-X), \tag{9}$$

which is not obvious from the definition.

In [R] we proved that if $Q$ and $Q'$ are two highest weight vectors, then the quantities

$$\sum_{m_1+\cdots+m_n=M} \frac{(\nu_1)_{m_1}\cdots(\nu_n)_{m_n}}{m_1!\cdots m_n!}\, T_1Q(m_1,\ldots,m_n)\,\overline{T_1Q'(m_1,\ldots,m_n)}$$

for $M = 0, 1, 2, \ldots,$

$$\int_{\xi\in\mathbb{R}_+^n\,:\,|\xi|=C} T_2Q(\xi)\,\overline{T_2Q'(\xi)}\,\xi_1^{\nu_1-1}\cdots\xi_n^{\nu_n-1}\,d\xi,$$

where $|\xi| = \xi_1+\cdots+\xi_n$ and $C>0$ is arbitrary, and

$$\int_{X\in\mathbb{R}^n\,:\,|X|=C} T_3Q(X)\,\overline{T_3Q'(X)}\,\left|\Gamma\left(\frac{\nu_1}{2}+iX_1\right)\right|^2\cdots\left|\Gamma\left(\frac{\nu_n}{2}+iX_n\right)\right|^2 dX,$$

where $C\in\mathbb{R}$ is arbitrary, are all proportional to the scalar product

$$\langle Q, Q'\rangle_{\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n}}.$$

In particular, given an orthogonal basis for the highest weight vectors, our transforms give us three systems of orthogonal polynomials. The form of the orthogonality relations suggests that one eliminate one variable and consider, for instance, for fixed $M$,

$$T_1Q(m_1,\ldots,m_{n-1},M-m_1-\cdots-m_{n-1})$$

as a polynomial in $n-1$ variables. This is how these polynomials usually occur in the literature.

In the case when all the $\nu_i$ are half-integers, there is a connection with spherical harmonics. If we introduce the polynomial

$$R(x_1,\ldots,x_{2|\nu|}) = (T_2Q)(x_1^2+\cdots+x_{2\nu_1}^2,\ldots,x_{2(\nu_1+\cdots+\nu_{n-1})+1}^2+\cdots+x_{2|\nu|}^2),$$

then the orthogonality relations for $T_2Q$ give corresponding orthogonality relations for $R$ on the unit sphere of $\mathbb{R}^{2\nu}$. Moreover, the conditions on $Q$ ensure that $R$ is harmonic. In fact one obtains precisely the space of harmonic polynomials of degree $s$ invariant with respect to the subgroup

$$\mathrm{O}(2\nu_1)\times\cdots\times\mathrm{O}(2\nu_n)\subseteq\mathrm{O}(2|\nu|),$$

called *polyspherical harmonics* [V]. In [R] we gave an explanation of this, which involved the Howe dual pair $\mathrm{Mp}(1)\times\mathrm{O}(k)\subseteq\mathrm{Mp}(k)$.

The plan of the paper is as follows. In Section 2 we describe some special cases of our polynomials, in particular those that we have found in the literature. In Section 3 we review some fundamental facts on matrix elements of our representation due to Basu and Wolf [BW1]. In Section 4 we obtain the polynomials $T_iQ$ as coupling coefficients for our representation. The proof is based on certain factorizations of the Fourier transforms on $\mathcal{A}^\nu$ with respect to one-parameter subgroups. This is different from the approach in [R]. In Section 5 we prove convolution formulas for our polynomials. These generalize formulas recently found by Koelink and Van der Jeugt [KV1]. In Section 6 we introduce some polynomials in $2n$ variables which we call coupling kernels. Whereas the coupling coefficients $T_iQ$ are connected with matrix elements of the intertwining embeddings (2) (or of the corresponding projections $\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n}\to\mathcal{A}^{|\nu|+2s}$), the coupling kernels are connected with matrix elements for the projection of $\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n}$ onto the isotypic subspace equivalent to $\binom{n+s-2}{n-2}\mathcal{A}^{|\nu|+2s}$ in (1). In Section 7 we study formal properties of the coupling kernels. These include some linearization formulas, which generalize the classical Burchnall–Chaundy formula. In Section 8 we prove that the coupling kernels may be expressed in terms of certain generalized hypergeometric functions. Finally, in Section 9 we give the analogues of our results when $\mathrm{SU}(1,1)$ is replaced by the Heisenberg group and the spaces $\mathcal{A}^{\nu}$ are replaced by Fock spaces of entire functions.

## 2. Special cases

We will now describe some special cases of our polynomials. We have found many occurrences of them in the literature after writing [R], so we will give more complete references here. We first recall the definition of the generalized hypergeometric function

$$ {}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\,\middle|\, x\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k\,x^k}{(b_1)_k\cdots(b_q)_k\,k!}. $$

When $a$ and $k$ are vectors, we write

$$ (a)_k=(a_1)_{k_1}\cdots(a_n)_{k_n},\qquad k!=k_1!\cdots k_n!. $$

We will also encounter Karlsson's generalized Kampé de Fériet functions [Ka], [SK], which we may then write, with a slight variation of Karlsson's notation, as

$$ \begin{aligned} &F^{p:q}_{r:s}\left(\begin{matrix}a_1,\dots,a_p & : & b_1,\dots,b_q\\ c_1,\dots,c_r & : & d_1,\dots,d_s\end{matrix}\,\middle|\, x\right)\\ &\quad=\sum_{k_1,\dots,k_n=0}^{\infty}\frac{(a_1)_{|k|}\cdots(a_p)_{|k|}(b_1)_k\cdots(b_q)_k\,x^k}{(c_1)_{|k|}\cdots(c_r)_{|k|}(d_1)_k\cdots(d_s)_k\,k!}, \end{aligned} \tag{10} $$

where the $a_i$ and $c_i$ are scalar parameters while the $b_i$, $d_i$, and $x$ are vectors of the same dimension $n$. For $n=1$, $F^{p:q}_{r:s}$ reduces to the function ${}_{p+q}F_{r+s}$.

### 2.1. **The bilinear case.**

The simplest case is when $n=2$. Then the space of highest weight vectors of each degree $s$ is one-dimensional and spanned by the single element

$$ Q(z_1,z_2)=(z_1-z_2)^s. $$

In this case, the polynomials $T_1Q$ are Clebsch–Gordan coefficients, which may be expressed in terms of Hahn polynomials. In fact,

$$ \begin{aligned} T_1Q(m_1,m_2)&=\sum_{k=0}^{s}\binom{s}{k}(-1)^k\frac{(-m_1)_k(-m_2)_{s-k}}{(\nu_1)_k(\nu_2)_{s-k}}\\ &=\frac{(-m_1-m_2)_s}{(\nu_2)_s}\,Q_s(m_1;\nu_1-1,\nu_2-1;m_1+m_2), \end{aligned} $$

where $Q_s$ is the Hahn polynomial of degree $s$ in the notation of [VK]. The polynomials $T_2Q$ and $T_3Q$ are Clebsch–Gordan coefficients with respect to continuous bases; they may be expressed in terms of Jacobi polynomials and "Hahn polynomials of an imaginary argument," respectively; cf. [MR], [BW2], compare also [P2] and [VK], Section 8.7. Still in the notation of [VK], we have

$$\begin{aligned}
T_2Q(\xi_1,\xi_2) &= \sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k}\frac{\xi_1^k\,\xi_2^{s-k}}{(\nu_1)_k(\nu_2)_{s-k}}\\
&= \frac{(-1)^s s!}{(\nu_1)_s(\nu_2)_s}(\xi_1+\xi_2)^s P_s^{(\nu_1-1,\nu_2-1)}\left(\frac{\xi_2-\xi_1}{\xi_1+\xi_2}\right),\\
T_3Q(X_1,X_2) &= \sum_{k=0}^{s}\binom{s}{k}(-1)^{k}\frac{(\frac{\nu_1}{2}-iX_1)_k(\frac{\nu_2}{2}-iX_2)_{s-k}}{(\nu_1)_k(\nu_2)_{s-k}}\\
&= \frac{(\frac{\nu_1+\nu_2}{2}-i(X_1+X_2))_s}{(\nu_2)_s}\,q_s(X_1;\tfrac{\nu_1}{2},\tfrac{\nu_2}{2}-i(X_1+X_2)).
\end{aligned}$$

Even in this case, the present paper gives new and simple deductions of properties of these polynomials.

2.2. **Polynomials of Appell-type.** In the case of general $n$, the polynomials

$$Q_t(z)=(z_1-z_n)^{t_1}\cdots(z_{n-1}-z_n)^{t_{n-1}},\qquad t_1+\cdots+t_{n-1}=s \tag{11}$$

form a basis for the space of highest weight vectors of degree $s$. Expanding $Q_t$ by the binomial theorem, one finds that the polynomials $T_iQ_t$ may be expressed in terms of Karlsson's generalized Kampé de Fériet functions (10) with the vector parameters of dimension $n-1$. In fact, one has

$$\begin{aligned}
&T_1Q_t(m_1,\dots,m_n)\\
&=\frac{(-m_n)_s}{(\nu_n)_s}\sum_{k_1,\dots,k_{n-1}=0}^{\infty}\frac{(1-\nu_n-s)_{|k|}(-t_1)_{k_1}\cdots(-t_{n-1})_{k_{n-1}}(-m_1)_{k_1}\cdots(-m_{n-1})_{k_{n-1}}}{(1+m_n-s)_{|k|}(\nu_1)_{k_1}\cdots(\nu_{n-1})_{k_{n-1}}k_1!\cdots k_{n-1}!}\\
&=\frac{(-m_n)_s}{(\nu_n)_s}F^{1:2}_{1:1}\left(\begin{matrix}1-\nu_n-s & : & -t,-(m_1,\dots,m_{n-1})\\ 1+m_n-s & : & (\nu_1,\dots,\nu_{n-1})\end{matrix}\;\middle|\;1\right)
\end{aligned}$$

(where, as is customary, we write the variable of the $F^{1:2}_{1:1}$-function as $1=(1,\dots,1)$). By (8), a similar expression is valid for $T_3Q_t$, while $T_2Q_t$ is given by

$$\begin{aligned}
&T_2Q_t(\xi_1,\dots,\xi_n)\\
&\quad=\frac{(-\xi_n)^s}{(\nu_n)_s}F^{1:1}_{0:1}\left(\begin{matrix}1-\nu_n-s & : & -t\\ - & : & (\nu_1,\dots,\nu_{n-1})\end{matrix}\;\middle|\;-\frac{\xi_1}{\xi_n},\dots,-\frac{\xi_{n-1}}{\xi_n}\right)
\end{aligned}\tag{12}$$

(Karlsson's $F^{1:1}_{0:1}$-function is also known as the Lauricella $F_A$-function).

2.3. **Polynomials biorthogonal to the polynomials of Appell-type.** The family (11) is obviously nonorthogonal for $n\geq 3$. Thus it is a problem to find the dual basis. Applying the $T_i$ to these two bases then gives biorthogonal polynomial systems. Though we have found no very simple expression for the basis dual to (11), the polynomials obtained after applying the $T_i$ are also expressible in terms of Karlsson's functions; cf. Corollary 6.2 below.

The polynomials (12) were introduced by Appell [A1] for $n=3$. The dual basis was found by him [A2] when $\nu_3=1$ and by Engelis [E] and, independently, by Fackerell and Littler [FL] in general. (For the work of Appell, see also the book [AK].) Lam and Tratnik [LT] generalized these polynomials to general $n$. The corresponding spherical harmonics occur in [KMT]. In the special case when all the $\nu_i$ equal 1, these harmonics go back to Hermite, Didon, and Kampé de Fériet; cf. again [AK]. The biorthogonal systems obtained using $T_1$ were introduced by Rahman [Ra] for $n = 3$ and by Tratnik [T2] in general. For the transform $T_3$, the corresponding polynomials are also due to Tratnik [T1].

2.4. **Polynomials of Proriol-type.** There seems to be no canonical choice of an orthogonal basis in the space of highest weight vectors. There is, however, a general procedure, known as binary coupling (the coupling of angular momenta in quantum physics), that allows one to construct many such bases. The corresponding polynomials $T_iQ$ are given by products of $n-1$ factors, each factor being a Clebsch–Gordan coefficient, that is, a Hahn, Jacobi, or continuous Hahn polynomial.

As an example, let us consider the case $n = 3$. Let $K_s^{\nu_1,\nu_2}$ denote the intertwining embedding

$$\mathcal{A}^{\nu_1+\nu_2+2s} \to \mathcal{A}^{\nu_1} \otimes \mathcal{A}^{\nu_2}$$

such that $K_s^{\nu_1,\nu_2}1 = (z_1 - z_2)^s$. Then the polynomials

$$Q_{st} = (K_s^{\nu_1,\nu_2} \otimes \mathrm{id}) \circ K_t^{\nu_1+\nu_2+2s,\nu_3}1, \qquad s + t = N$$

form an orthogonal basis for the space of highest weight vectors in $\mathcal{A}^{\nu_1} \otimes \mathcal{A}^{\nu_2} \otimes \mathcal{A}^{\nu_3}$ of degree $N$. In [R] we found that

$$\begin{aligned}
Q_{st}(z) &= (z_1 - z_2)^s \sum_{i+j=t} \frac{t!}{i!\,j!} \frac{(\nu_1+s)_i(\nu_2+s)_j}{(\nu_1+\nu_2+2s)_t}(z_1-z_3)^i(z_2-z_3)^j, \\
T_1Q_{st}(m) &= \frac{(-m_1-m_2)_s(s-m_1-m_2-m_3)_t}{(\nu_2)_s(\nu_3)_t}\, Q_s(m_1;\nu_1-1,\nu_2-1;m_1+m_2) \\
&\quad \times Q_t(m_1+m_2-s;\nu_1+\nu_2+2s-1,\nu_3-1;m_1+m_2+m_3-s), \\
T_2Q_{st}(\xi) &= \frac{(-1)^{s+t}s!\,t!(\xi_1+\xi_2)^s(\xi_1+\xi_2+\xi_3)^t}{(\nu_1)_s(\nu_2)_s(\nu_1+\nu_2+2s)_t(\nu_3)_t} \\
&\quad \times P_s^{(\nu_1-1,\nu_2-1)}\left(\frac{\xi_2-\xi_1}{\xi_1+\xi_2}\right) P_t^{(\nu_1+\nu_2+2s-1,\nu_3-1)}\left(\frac{\xi_3-\xi_1-\xi_2}{\xi_1+\xi_2+\xi_3}\right), \\
T_3Q_{st}(X) &= \frac{(\frac{\nu_1+\nu_2}{2} - i(X_1+X_2))_s(\frac{\nu_1+\nu_2+\nu_3+2s}{2} - i(X_1+X_2+X_3))_t}{(\nu_2)_s(\nu_3)_t} \\
&\quad \times q_s\left(X_1; \frac{\nu_1}{2}, \frac{\nu_2}{2} - i(X_1+X_2)\right) \\
&\quad \times q_t\left(X_1+X_2; \frac{\nu_1+\nu_2+2s}{2}, \frac{\nu_3}{2} - i(X_1+X_2+X_3)\right).
\end{aligned}$$

The polynomials $T_2Q_{st}$, with $Q_{st}$ as above, were introduced by Proriol in [Pr]. They were independently obtained and applied to genetics by Karlin and McGregor in [KM1] and have also found applications in physics [M], [MP]. See the survey [K1] for these and other two-variable analogues of Jacobi polynomials. The spherical harmonics constructible by this method were, for general $n$, introduced in [V], cf. also [VK], Section 10.5, and generalized further in [KMT]. For the special coupling procedure

$$Q_{s_1,\dots,s_{n-1}} = (K_{s_1}^{\nu_1,\nu_2} \otimes \mathrm{id}) \circ (K_{s_2}^{\nu_1+\nu_2+2s_1,\nu_3} \otimes \mathrm{id}) \circ \cdots \circ K_{s_{n-1}}^{\nu_1+\cdots+\nu_{n-1}+2(s_1+\cdots+s_{n-2}),\nu_n} 1,$$

the polynomials $T_1 Q_{s_1,\dots,s_{n-1}}$ were defined by Karlin and McGregor in [KM2], cf. also [T5]; again they appear in the context of genetics. The corresponding polynomials $T_3 Q_{s_1,\dots,s_{n-1}}$ were introduced by Tratnik in [T4]. An interesting product formula for the polynomials $T_2 Q_{s_1,\dots,s_{n-1}}$ is given in [KS].

## 3. Matrix elements

In this section we write down some basic facts, due to Basu and Wolf [BW1], cf. also [K2], about matrix elements of our representation. Whereas these authors work with the oscillator realization on $\mathrm{L}^2(\mathbb{R}_+)$, we will reformulate their results in terms of the analytic function spaces $\mathcal{A}^\nu$. This has computational advantages, allowing us to work with power series expansions.

Let $\mathbf{D}$ be the unit disc of $\mathbb{C}$ and $G \simeq \mathrm{SU}(1,1)/\{\pm 1\}$ be the group of conformal automorphisms of $\mathbf{D}$. There are three conjugacy classes of one-parameter subgroups in $G$. They may be described in terms of the fixed point sets in $\mathbb{C} \cup \{\infty\}$ of the corresponding Möbius maps. There are the *elliptic* elements, which fix one point in $\mathbf{D}$ and one in the outside, the *hyperbolic* elements, which fix two points on the boundary of $\mathbf{D}$, and finally the *parabolic* elements, which have a single fixed point on the boundary.

We choose a representative subgroup in each conjugacy class, namely, the elliptic subgroup

$$\phi_t^1 = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix},$$

corresponding to rotations $z \mapsto e^{it} z$ of the disc, the parabolic subgroup

$$\phi_t^2 = \begin{pmatrix} 1 + it/2 & it/2 \\ -it/2 & 1 - it/2 \end{pmatrix},$$

and the hyperbolic subgroup

$$\phi_t^3 = \begin{pmatrix} \cosh(t/2) & -\sinh(t/2) \\ -\sinh(t/2) & \cosh(t/2) \end{pmatrix}.$$

One may check that

$$\phi_s^i \phi_t^i = \phi_{s+t}^i, \qquad i = 1, 2, 3.$$

Note that if we choose $\psi$ as the Cayley transform

$$\psi(z) = \frac{i - z}{i + z}$$

of the upper half-plane onto $\mathbf{D}$, then

$$(\psi^{-1} \phi_t^2 \psi)(z) = z + t,$$

so we may realize the parabolic subgroup as translations of the half-plane. Similarly

$$(\gamma^{-1} \phi_t^3 \gamma)(z) = z + t,$$

where

$$\gamma(z) = \frac{i - e^z}{i + e^z},$$

which realizes the hyperbolic subgroup as translations of the strip $\{z \,|\, 0 < \mathrm{Im} z < \pi\}$. (Hyperbolic transformations may also be realized as dilatations of the half-plane.)

To clarify the group theoretic meaning of our formulas, we employ certain notational conventions, which we will now state explicitly. Various objects connected with the subgroup $\phi_t^i$ $(i = 1, 2, 3)$ will be labelled with the index $i$. The letter $z$ will denote variables ranging in the disc (or polydisc). We will be concerned with three versions of the Fourier transform, one for each subgroup. For the elliptic subgroup, the variable dual to $z$ is discrete; it will be denoted by $k, l, m$. For the parabolic and hyperbolic subgroup, the dual variables are continuous; they will be denoted by $\xi, \eta$ and $X, Y$, respectively. The equations (5), (6), and (7) exemplify these conventions.

Recall now the function spaces $\mathcal{A}^\nu$ $(\nu > 0)$ with scalar product

$$\langle f, g\rangle = \sum_{k=0}^{\infty} \frac{k!}{(\nu)_k} \hat{f}(k)\overline{\hat{g}(k)}.$$

We will use this notation also when $f$ or $g$ is not in $\mathcal{A}^\nu$, provided that the sum on the right-hand side converges. This space has the reproducing kernel

$$k_w(z) = \sum_{k=0}^{\infty} \frac{(\nu)_k}{k!} (z\bar{w})^k = \frac{1}{(1 - z\bar{w})^\nu},$$

that is,

$$f(w) = \langle f, k_w\rangle, \qquad f \in \mathcal{A}^\nu, \ w \in \mathbb{D}.$$

We denote the group action by

$$(f * \phi)(z) = f(\phi(z))\phi'(z)^{\frac{\nu}{2}} = f\left(\frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}\right) \frac{1}{(\bar{\beta} z + \bar{\alpha})^\nu},$$

where $\phi \in G$ is given by

$$\phi(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}, \qquad |\alpha|^2 - |\beta|^2 = 1.$$

We also write

$$\mathcal{U}_\phi(f) = f * \phi^{-1}$$

for the corresponding left action. Because of the indeterminacy of the power function, one must pass to the universal covering group to get a unitary representation on each space $\mathcal{A}^\nu$. (If $\nu \in 2\mathbb{Z}/k$ for an integer $k$, the $k$-fold cover is sufficient.)

It is easy to find joint eigenfunctions of our subgroups. Namely, since $e^{i\xi z}$ are eigenfunctions of the translations, it follows that

$$e^{i\xi\psi^{-1}(z)}((\psi^{-1})'(z))^{\frac{\nu}{2}},$$

with $\psi$ as above, are eigenfunctions of $\phi_t^2$. It will be convenient to normalize them so that they take the value 1 at the origin. This gives the functions

$$e_\xi^2(z) = \frac{1}{(1 + z)^\nu} \exp\left(\frac{\xi z}{1 + z}\right),$$

which indeed satisfy

$$e_\xi^2 * \phi_t^2 = e^{it\xi} e_\xi^2.$$

Replacing $\psi$ by $\gamma$ in this argument leads to the functions

$$e_X^3(z) = \frac{2^{\frac{\nu}{2}}}{(1-z)^{\frac{\nu}{2}-iX}(1+z)^{\frac{\nu}{2}+iX}}$$

(the factor $2^{\frac{\nu}{2}}$ being chosen for convenience), which satisfy

$$e_X^3 * \phi_t^3 = e^{itX} e_X^3.$$

Finally we write

$$e_k^1(z) = z^k$$

for the eigenfunctions of the elliptic subgroup. We call $(e_k^1)_{k=0}^\infty$ the elliptic basis of $\mathcal{A}^\nu$, while $(e_\xi^2)_{\xi>0}$ and $(e_X^3)_{X\in\mathbb{R}}$ are called the parabolic and the hyperbolic (generalized) basis, respectively.

Note that, whereas clearly $e_k^1$ is an element of $\mathcal{A}^\nu$, this is not true for $e_\xi^2$ or $e_X^3$. However, one may still define $\langle f, e_x^i\rangle$ $(i = 2,3)$ for $f$ in a dense subspace of $\mathcal{A}^\nu$. Natural choices are the space of polynomials and the linear hull of all reproducing kernels $k_z$, $z \in \mathbf{D}$. With either of these choices the following Plancherel theorem is true.

**Proposition 3.1.** *The operator $(\mathcal{F}_2 f)(\xi) = \langle f, e_\xi^2\rangle$ extends to a Hilbert space isomorphism*

$$\mathcal{F}_2 : \mathcal{A}^\nu \to \mathrm{L}^2\left(\mathbb{R}_+, \frac{1}{\Gamma(\nu)}\, e^{-\xi}\xi^{\nu-1}\, d\xi\right).$$

*Similarly, the operator $(\mathcal{F}_3 f)(X) = \langle f, e_X^3\rangle$ extends to an isomorphism*

$$\mathcal{F}_3 : \mathcal{A}^\nu \to \mathrm{L}^2\left(\mathbb{R}, \frac{1}{2\pi\Gamma(\nu)}\left|\Gamma\left(\frac{\nu}{2}+iX\right)\right|^2 dX\right).$$

The transforms $\mathcal{F}_2$ and $\mathcal{F}_3$ are connected to the classical Fourier transform. For instance, $\mathcal{F}_2 f$ is essentially the Fourier transform of the boundary value distribution of the function $f(\psi(z))\psi'(z)^{\nu/2}$ defined in the upper half-plane (they differ by a multiplicative factor). The proposition may be proved using this observation, cf. [P2], or alternatively using the orthogonality relations for Laguerre and Meixner–Pollaczek polynomials (whose role is made clear below).

By polarization, we have

$$\begin{aligned} \langle f, g\rangle &= \sum_{k=0}^\infty \frac{(\nu)_k}{k!}\langle f, e_k^1\rangle\langle e_k^1, g\rangle \\ &= \frac{1}{\Gamma(\nu)}\int_0^\infty \langle f, e_\xi^2\rangle\langle e_\xi^2, g\rangle\, e^{-\xi}\xi^{\nu-1}\, d\xi \qquad (13) \\ &= \frac{1}{2\pi\Gamma(\nu)}\int_{-\infty}^\infty \langle f, e_X^3\rangle\langle e_X^3, g\rangle \left|\Gamma\left(\frac{\nu}{2}+iX\right)\right|^2 dX, \end{aligned}$$

with a suitable interpretation of the last two expressions. Choosing $g = k_z$, it follows that

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \frac{(\nu)_k}{k!} (\mathcal{F}_1 f)(k) e_k^1(z) \\ &= \frac{1}{\Gamma(\nu)} \int_0^{\infty} (\mathcal{F}_2 f)(\xi) e_\xi^2(z) e^{-\xi} \xi^{\nu-1}\, d\xi \qquad (14) \\ &= \frac{1}{2\pi\Gamma(\nu)} \int_{-\infty}^{\infty} (\mathcal{F}_3 f)(X) e_X^3(z) \left| \Gamma\left(\frac{\nu}{2} + iX\right)\right|^2 d\xi, \end{aligned}$$

where

$$\mathcal{F}_1 f(k) = \langle f, e_k^1 \rangle = \frac{k!}{(\nu)_k} \hat{f}(k).$$

We now introduce the *matrix elements*

$$K^{ij}(x,y,\phi) = \langle e_x^i, e_y^j * \phi \rangle, \qquad i,j = 1,2,3,$$

where $\phi \in G$ is such that this makes sense. By unitarity,

$$K^{ij}(x,y,\phi) = \overline{K^{ji}(y,x,\phi^{-1})},$$

so there are six cases to consider. As was remarked above, the $K^{ij}$ were computed by Basu and Wolf [BW1]. We summarize their results in the following proposition.

**Proposition 3.2.** *The matrix elements of $\mathcal{A}^\nu$ are given by*

$$K^{11}(k,l,\phi) = \begin{cases} \dfrac{\beta^k \bar{\beta}^l}{\alpha^{\nu+k+l}} (-1)^k \, {}_2F_1\left( \begin{matrix} -k, -l \\ \nu \end{matrix} \,\middle|\, -\dfrac{1}{|\beta|^2} \right) & \beta \neq 0, \\ \dfrac{k!}{(\nu)_k \alpha^{\nu+2k}} \delta_{kl} & \beta = 0, \end{cases}$$

$$K^{12}(k,\xi,\phi) = \frac{(-1)^k (\bar{\alpha}+\beta)^k}{(\alpha+\bar{\beta})^{\nu+k}} \exp\left(\frac{\bar{\beta}\xi}{\bar{\beta}+\alpha}\right) {}_1F_1\left( \begin{matrix} -k \\ \nu \end{matrix} \,\middle|\, \frac{\xi}{|\alpha+\bar{\beta}|^2}\right),$$

$$K^{13}(k,X,\phi) = \frac{2^{\frac{\nu}{2}} (\bar{\alpha}-\beta)^k}{(\alpha+\bar{\beta})^{\frac{\nu}{2}-iX} (\alpha-\bar{\beta})^{\frac{\nu}{2}+iX+k}} \, {}_2F_1\left( \begin{matrix} -k, \frac{\nu}{2}-iX \\ \nu \end{matrix} \,\middle|\, \frac{2}{(\alpha+\bar{\beta})(\bar{\alpha}-\beta)}\right),$$

$$\begin{aligned} K^{22}(\xi,\eta,\phi) &= \frac{1}{(2i\mathrm{Im}(\alpha-\beta))^\nu} \exp\left(-\frac{(\bar{\alpha}+\beta)\xi + (\bar{\alpha}-\bar{\beta})\eta}{2i\mathrm{Im}(\alpha-\beta)}\right) \\ &\quad \times {}_0F_1\left( \begin{matrix} - \\ \nu \end{matrix} \,\middle|\, -\frac{\xi\eta}{4(\mathrm{Im}(\alpha-\beta))^2}\right), \end{aligned}$$

$$\begin{aligned} K^{23}(\xi,X,\phi) &= \frac{2^{\frac{\nu}{2}}}{(2\mathrm{Re}(\alpha-\beta))^{\frac{\nu}{2}+iX} (2i\mathrm{Im}(\alpha-\beta))^{\frac{\nu}{2}-iX}} \exp\left(\frac{(\bar{\alpha}-\beta)\xi}{2\mathrm{Re}(\alpha-\beta)}\right) \\ &\quad \times {}_1F_1\left( \begin{matrix} \frac{\nu}{2}-iX \\ \nu \end{matrix} \,\middle|\, \frac{i\xi}{2\mathrm{Re}(\alpha-\beta)\mathrm{Im}(\alpha-\beta)}\right), \end{aligned}$$

$$\begin{aligned} K^{33}(X,Y,\phi) &= \frac{(i\mathrm{Im}(\alpha+\beta))^{i(X-Y)}}{(\mathrm{Re}(\alpha-\beta))^{\frac{\nu}{2}+iX} (\mathrm{Re}(\alpha+\beta))^{\frac{\nu}{2}-iY}} \\ &\quad \times {}_2F_1\left( \begin{matrix} \frac{\nu}{2}+iX, \frac{\nu}{2}-iY \\ \nu \end{matrix} \,\middle|\, \frac{1}{\mathrm{Re}(\alpha+\beta)\mathrm{Re}(\alpha-\beta)}\right), \end{aligned}$$

*where* $\phi(z) = (\alpha z + \beta)/(\bar{\beta} z + \bar{\alpha})$, $|\alpha|^2 - |\beta|^2 = 1$.

One may check that these expressions are well defined up to a choice of $(\phi')^{\nu/2}$. The matrix elements may be identified with classical orthogonal functions as follows, cf. [K2]:

| | | |
|---|---|---|
| $K^{11}$ | : | Meixner polynomials, |
| $K^{12}$ | : | Laguerre polynomials, |
| $K^{13}$ | : | Meixner–Pollaczek polynomials, |
| $K^{22}$ | : | Bessel functions, |
| $K^{23}$ | : | Laguerre functions, |
| $K^{33}$ | : | Meixner–Pollaczek functions. |

By inserting basis elements $e^i_x * \phi$ into (13), one may deduce a host of addition theorems and orthogonality relations for these functions; cf. [VK], Section 7.7.

The proposition does not give the matrix elements $K^{22}$, $K^{23}$, or $K^{33}$ for $\phi = \mathrm{id}$. However, by (14) it is natural to define

$$K^{22}(\xi, \eta, \mathrm{id}) = \Gamma(\nu) e^{\xi} \xi^{1-\nu} \delta(\xi - \eta), \tag{15}$$

$$K^{33}(X, Y, \mathrm{id}) = 2\pi\Gamma(\nu) \left| \Gamma\left(\frac{\nu}{2} + iX\right) \right|^{-2} \delta(X - Y), \tag{16}$$

where $\delta$ is the Dirac measure. Similarly one may define

$$K^{23}(\xi, X, \mathrm{id}) = \frac{\Gamma(\nu) e^{\frac{\xi}{2}}}{\Gamma(\frac{\nu}{2} + iX) 2^{iX}} \, \xi^{-\frac{\nu}{2} + iX}$$

since this kernel has the property

$$e^3_X(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \overline{K^{23}(\xi, X, \mathrm{id})} \, e^2_\xi(z) e^{-\xi} \xi^{\nu-1} \, d\xi.$$

(This reduces to the usual definition of the gamma function after a change of variables.) These identities can be made precise by extending the transforms $\mathcal{F}_i$ to suitable classes of distributions.

As for the proof, we remark that in the present realization, *all* of Proposition 3.2 may be deduced from Meixner's expansion formula [Me]

$$\sum_{k=0}^\infty \frac{(\nu)_k}{k!} \, {}_2F_1\left( \begin{matrix} -k, \alpha \\ \nu \end{matrix} \middle| \, x \right) {}_2F_1\left( \begin{matrix} -k, \beta \\ \nu \end{matrix} \middle| \, y \right) z^k \tag{17}$$

$$= \frac{1}{(1-z)^{\nu-\alpha-\beta}(1-z+xz)^\alpha (1-z+yz)^\beta} \, {}_2F_1\left( \begin{matrix} \alpha, \beta \\ \nu \end{matrix} \middle| \, \frac{xyz}{(1-z+xz)(1-z+yz)} \right).$$

The case $x = 0$ gives the expressions for $K^{11}$ and $K^{13}$. To proceed, one may use the expansion

$$K^{33}(X, Y, \phi) = \sum_{k=0}^\infty \frac{(\nu)_k}{k!} \overline{K^{13}(k, X, \mathrm{id})} \, K^{13}(k, Y, \phi),$$

which follows from the first identity in (13). The expression for $K^{33}$ given in Proposition 3.2 now follows from the general case of Meixner's formula.

The matrix elements involving the parabolic subgroup are computed similarly, using two degenerate cases of Meixner's formula. If we replace $y$ by $y/\beta$ in (17) and let $\beta$ tend to infinity, we obtain the expansion

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(\nu)_k}{k!}\,{}_2F_1\left(\left.\begin{matrix}-k,\alpha\\ \nu\end{matrix}\right|x\right){}_1F_1\left(\left.\begin{matrix}-k\\ \nu\end{matrix}\right|y\right)z^k\\ &\quad=\frac{1}{(1-z)^{\nu-\alpha}(1-z+xz)^{\alpha}}\exp\left(-\frac{yz}{1-z}\right){}_1F_1\left(\left.\begin{matrix}\alpha\\ \nu\end{matrix}\right|\frac{xyz}{(1-z+xz)(1-z)}\right),\end{aligned}\tag{18}$$

due to Weisner. Repeating this once more gives the Hardy–Hille formula

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(\nu)_k}{k!}\,{}_1F_1\left(\left.\begin{matrix}-k\\ \nu\end{matrix}\right|x\right){}_1F_1\left(\left.\begin{matrix}-k\\ \nu\end{matrix}\right|y\right)z^k\\ &\quad=\frac{1}{(1-z)^{\nu}}\exp\left(-\frac{z(x+y)}{1-z}\right){}_0F_1\left(\left.\begin{matrix}-\\ \nu\end{matrix}\right|\frac{xyz}{(1-z)^2}\right).\end{aligned}$$

We refer to the book [SM] for a large number of proofs of these formulas.

## 4. Coupling coefficients

In this section we will obtain the polynomials $T_iQ$ as coupling coefficients of our representation. We consider transforms of the type

$$\mathcal{F}_i^{\phi}f(x)=\mathcal{F}_i\mathcal{U}_{\phi}f(x)=\langle f,e_x^i*\phi\rangle,\qquad i=1,2,3,\ \phi\in G,$$

that is, the Fourier transform with respect to an arbitrary one-parameter subgroup of $G$. We will factorize these into operators which interact nicely with the Lie algebra action. More precisely, we will find factorizations of the type

$$\mathcal{F}_i^{\phi}=M_i\,T_i\,\delta_a\tau_b,$$

where $\delta_a$ and $\tau_b$ are the dilatation and translation operators defined by

$$\delta_af(z)=f(az),\qquad \tau_bf(z)=f(z+b),$$

$M_i$ are the multiplication operators

$$M_if(x)=\overline{e_x^i*\phi(0)}f(x)=(\mathcal{F}_i^{\phi}1)(x)f(x),$$

and $T_i$ are the transforms defined in (4).

We will use the explicit expressions for the matrix elements given in Proposition 3.2 to obtain our factorizations. Note, however, that we only need them in the easiest special case, corresponding to $x=0$ in (17) and (18).

**Proposition 4.1.** *In the notation above, we have for $\phi=\left(\begin{smallmatrix}\alpha&\beta\\ \bar\beta&\bar\alpha\end{smallmatrix}\right)$ the following factorizations of densely defined operators*:

$$\begin{aligned}\mathcal{F}_1^{\phi}&=M_1\,T_1\,\delta_{1/\alpha\bar\beta}\,\tau_{-\beta/\alpha}\qquad(\beta\neq0),\\ \mathcal{F}_2^{\phi}&=M_2\,T_2\,\delta_{1/(\alpha+\bar\beta)^2}\,\tau_{-(\bar\alpha+\beta)/(\alpha+\bar\beta)},\\ \mathcal{F}_3^{\phi}&=M_3\,T_3\,\delta_{2/(\alpha+\bar\beta)(\alpha-\bar\beta)}\,\tau_{(\bar\alpha-\beta)/(\alpha-\bar\beta)}.\end{aligned}\tag{19}$$

*Proof.* Test the action of both sides on the function $e_k^1(z) = z^k$. In general,

$$(\delta_a \tau_b \, e_k^1)(z) = (az+b)^k = b^k \sum_{j=0}^{k} \frac{(-k)_j}{j!} \left(-\frac{az}{b}\right)^j .$$

Thus one has for instance

$$\begin{aligned}
M_1 \, T_1 \, \delta_{1/\alpha\bar\beta} \, \tau_{-\beta/\alpha} \, e_k^1(m) &= \frac{\bar\beta^m}{\alpha^{\nu+m}} T_1 \left( \left(-\frac{\beta}{\alpha}\right)^k \sum_{j=0}^{k} \frac{(-k)_j}{j!} \left(\frac{z}{|\beta|^2}\right)^j \right)(m) \\
&= \frac{\bar\beta^m(-\beta)^k}{\alpha^{\nu+m+k}} \sum_{j=0}^{k} \frac{(-k)_j(-m)_j}{j!(\nu)_j} \left(-\frac{1}{|\beta|^2}\right)^j \\
&= \frac{\bar\beta^m(-\beta)^k}{\alpha^{\nu+k+m}} \, {}_2F_1\left( \begin{matrix} -k, -m \\ \nu \end{matrix} \,\middle|\, -\frac{1}{|\beta|^2} \right) \\
&= K^{11}(k,m,\phi) = \mathcal{F}_1^{\phi} e_k^1(m),
\end{aligned}$$

which proves (19). The other two identities follow similarly. $\square$

Now let us apply these factorizations to a general tensor product $\mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n}$. As in the case $n = 1$, we write

$$(f_1 \otimes \cdots \otimes f_n) * \phi(z) = f_1(\phi(z_1)) \cdots f_n(\phi(z_n)) \phi'(z_1)^{\frac{\nu_1}{2}} \cdots \phi'(z_n)^{\frac{\nu_n}{2}}$$

for the group action and $\mathcal{F}_i^{\phi}$ for the transforms

$$(\mathcal{F}_i^{\phi} f)(x_1, \ldots, x_n) = \langle f, (e_{x_1}^i \otimes \cdots \otimes e_{x_n}^i) * \phi \rangle, \qquad i = 1,2,3, \ \phi \in G.$$

Applying Proposition 4.1 in each variable gives factorizations of these transforms.

We fix a highest weight polynomial $Q$ of degree $s$ and write

$$\mu = |\nu| + 2s.$$

Then there is a unique intertwining embedding $\mathcal{K}_Q$ of $\mathcal{A}^{\mu}$ into $\mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n}$ such that $\mathcal{K}_Q 1 = Q$. We will henceforth denote by $\langle\, , \rangle_\mu$ and $\langle\, , \rangle_\nu$ the scalar product of $\mathcal{A}^{\mu}$ and $\mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n}$, respectively.

**Theorem 4.2.** *For $g \in \mathcal{A}^{\mu}$ and $\phi \in G$, we have the equalities*

$$\begin{aligned}
(\mathcal{F}_1^{\phi} \mathcal{K}_Q g)(m) &= T_1 Q(m) (\mathcal{F}_1^{\phi} g)(|m| - s), \\
(\mathcal{F}_2^{\phi} \mathcal{K}_Q g)(\xi) &= T_2 Q(\xi) (\mathcal{F}_2^{\phi} g)(|\xi|), \\
(\mathcal{F}_3^{\phi} \mathcal{K}_Q g)(X) &= T_3 Q(X) (\mathcal{F}_3^{\phi} g)(|X|).
\end{aligned}$$

Thus we may write

$$\langle \mathcal{K}_Q g, (z_1^{m_1} \cdots z_n^{m_n}) * \phi \rangle_\nu = T_1 Q(m_1, \ldots, m_n) \langle g, z^{|m|-s} * \phi \rangle_\mu,$$

while for $T_2$ and $T_3$ similar equalities are valid with a suitable interpretation. In particular the matrix elements

$$\langle \mathcal{K}_Q e_x^i, (e_{y_1}^j \otimes \cdots \otimes e_{y_n}^j) * \phi \rangle_\nu$$

of the embedding $\mathcal{K}_Q$ factor as $T_j Q(y)$ times a matrix element (in the sense of Section 3) for $\mathcal{A}^{|\nu|+2s}$.

*Proof.* Since $\mathcal{K}_Q$ commutes with the group action, we may assume that $\phi = \mathrm{id}$. Moreover, it suffices to choose $g = k_w$ as a reproducing kernel. The first equality then reduces to

$$\langle \mathcal{K}_Q k_w, e^1_{m_1} \otimes \cdots \otimes e^1_{m_n} \rangle_\nu = T_1 Q(m) \bar{w}^{|m|-s}. \tag{20}$$

First note that

$$k_w = (1 - |w|^2)^{-\frac{\mu}{2}} \, 1 * \phi_w^{-1},$$

where $\phi_w$ is the disc automorphism $\phi_w(z) = (z + w)/(1 + \bar{w}z)$, corresponding to the matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{smallmatrix}\right)$, with

$$\alpha = \frac{1}{\sqrt{1 - |w|^2}}, \qquad \beta = \frac{w}{\sqrt{1 - |w|^2}}.$$

This gives

$$\mathcal{K}_Q k_w = (1 - |w|^2)^{-\frac{\mu}{2}} \, Q * \phi_w^{-1}.$$

Thus we have

$$\begin{aligned} \langle \mathcal{K}_Q k_w, e^1_{m_1} \otimes \cdots \otimes e^1_{m_n} \rangle_\nu &= (1 - |w|^2)^{-\frac{\mu}{2}} \langle Q * \phi_w^{-1}, e^1_{m_1} \otimes \cdots \otimes e^1_{m_n} \rangle_\nu \\ &= (1 - |w|^2)^{-\frac{\mu}{2}} (\mathcal{F}_1^{\phi_w} Q)(m_1, \dots, m_n). \end{aligned}$$

We now apply the factorization (19) to each variable. Note that (3) may be written as

$$(\delta_a \tau_b)^{\otimes n} Q = a^s Q.$$

This gives

$$\begin{aligned} \langle \mathcal{K}_Q k_w, e^1_{m_1} \otimes \cdots \otimes e^1_{m_n} \rangle_\nu &= (1 - |w|^2)^{-\frac{\mu}{2}} \frac{\bar\beta^{|m|}}{\alpha^{|\nu| + |m|}} (T_1 (\delta_{1/\alpha\bar\beta} \, \tau_{-\beta/\alpha})^{\otimes s} Q)(m) \\ &= (1 - |w|^2)^{-\frac{\mu}{2}} \frac{\bar\beta^{|m|-s}}{\alpha^{|\nu| + |m| + s}} T_1 Q(m) = \bar{w}^{|m|-s} T_1 Q(m), \end{aligned}$$

which proves (20). The rest of the theorem follows from the equalities

$$\mathcal{F}_2^{\phi_w} Q(\xi) = (1 - |w|^2)^{\frac{\mu}{2}} \, T_2 Q(\xi) \, \overline{e^2_{|\xi|}(w)},$$

$$\mathcal{F}_3^{\phi_w} Q(X) = (1 - |w|^2)^{\frac{\mu}{2}} \, T_3 Q(X) \, \overline{e^3_{|X|}(w)},$$

which are likewise easily obtained from Proposition 4.1. □

Choosing as in the proof $g = k_w$, $\phi = \mathrm{id}$, and expressing the left-hand sides by means of the adjoint $\mathcal{K}_Q^*$, one sees that the theorem is equivalent to the equalities

$$\begin{aligned} \mathcal{K}_Q^* \, e^1_{m_1} \otimes \cdots \otimes e^1_{m_n} &= \overline{T_1 Q(m)} \, e^1_{|m|-s}, \\ \mathcal{K}_Q^* \, e^2_{\xi_1} \otimes \cdots \otimes e^2_{\xi_n} &= \overline{T_2 Q(\xi)} \, e^2_{|\xi|}, \\ \mathcal{K}_Q^* \, e^3_{X_1} \otimes \cdots \otimes e^3_{X_n} &= \overline{T_3 Q(X)} \, e^3_{|X|}. \end{aligned}$$

This may also be proved using the explicit expression for $\mathcal{K}_Q^*$ as a differential operator (*transvectant*) given in [R]; cf. [P2] for the case $n = 2$.

We now combine Theorem 4.2 with the equalities (14), choosing $f = \mathcal{K}_Q e^i_x$. This will give the expansions of $\mathcal{K}_Q e^i_x$ in tensor products of the $e^j_y$. By the theorem, the coefficients will involve the polynomial $T_j Q$ times the matrix element $K^{ij}$. For $i = j$ this exhibits the polynomials $T_i Q$ as coupling coefficients of our representation. We summarize this in a corollary.

**Corollary 4.3.** *In the notation above, we have the generalized Clebsch–Gordan formulas*

$$\frac{(\mu)_k}{k!}\mathcal{K}_Q e_k^1(z_1,\dots,z_n) = \sum_{|m|=k+s} \frac{(\nu_1)_{m_1}\cdots(\nu_n)_{m_n}}{m_1!\cdots m_n!} T_1Q(m) z_1^{m_1}\cdots z_n^{m_n},$$

$$\begin{aligned}(21)\qquad &\frac{\xi^{\mu-1}}{\Gamma(\mu)}\mathcal{K}_Q e_\xi^2(z_1,\dots,z_n)\\ &= \frac{1}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)} \int_{\eta\in\mathbb{R}_+^n : |\eta|=\xi} T_2Q(\eta)\, e^2_{\eta_1}(z_1)\cdots e^2_{\eta_n}(z_n)\eta_1^{\nu_1-1}\cdots\eta_n^{\nu_n-1}\,d\eta,\end{aligned}$$

$$\begin{aligned}\frac{\left|\Gamma\left(\frac{\mu}{2}+iX\right)\right|^2}{\Gamma(\mu)}\mathcal{K}_Q e_X^3(z_1,\dots,z_n) &= \frac{1}{(2\pi)^{n-1}\Gamma(\nu_1)\cdots\Gamma(\nu_n)}\\ \times\int_{Y\in\mathbb{R}^n : |Y|=X} & T_3Q(Y)\, e^3_{Y_1}(z_1)\cdots e^3_{Y_n}(z_n)\left|\Gamma\left(\frac{\nu_1}{2}+iY_1\right)\cdots\Gamma\left(\frac{\nu_n}{2}+iY_n\right)\right|^2 dY.\end{aligned}$$

To obtain the latter two equalities, one may use formally (15) and (16). This can be justified either by a duality argument or by first writing down an analogous expansion for $\mathcal{K}_Q e_x^i * \phi$ and then letting $\phi$ tend to the identity.

Choosing $i\neq j$ gives six additional expansion formulas. These follow, however, from formulas which have been obtained already. For instance, $i=1$, $j=2$ gives

$$\mathcal{K}_Q e_k^1(z) = \frac{(-1)^k}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)} \int_{\mathbb{R}_+^n} {}_1F_1\left(\begin{matrix}-k\\ \mu\end{matrix}\,\middle|\, |\eta|\right) T_2Q(\eta)\, e^{-|\eta|}\prod_{r=1}^n e^2_{\eta_r}(z_r)\eta_r^{\nu_r-1}\,d\eta.$$

Integrating over the hypersurfaces $|\eta|=\xi$ and assuming (21), we see that this reduces to

$$\mathcal{K}_Q e_k^1(z) = \frac{(-1)^k}{\Gamma(\mu)}\int_0^\infty {}_1F_1\left(\begin{matrix}-k\\ \mu\end{matrix}\,\middle|\, \xi\right)\mathcal{K}_Q e_\xi^2(z) e^{-\xi}\xi^{\mu-1}\,d\xi.$$

However, this follows from applying $\mathcal{K}_Q$ to each side of

$$e_k^1(z) = \frac{1}{\Gamma(\mu)}\int_0^\infty K^{12}(k,\xi,\mathrm{id})\, e_\xi^2(z) e^{-\xi}\xi^{\mu-1}\,d\xi,$$

which is a special case of (14).

As a consequence of the interpretation as coupling coefficients, we recover the orthogonality relations for the polynomials $T_iQ$.

**Corollary 4.4.** *Let $Q$ and $\tilde{Q}$ be two highest weight polynomials in*

$$\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n},$$

*not necessarily of the same degree. Then, if $s$ is the degree of $Q$ (or of $\tilde{Q}$) and $\mu=|\nu|+2s$, the quantities*

$$\frac{k!}{(\mu)_k}\sum_{|m|=k+s}\frac{(\nu_1)_{m_1}\cdots(\nu_n)_{m_n}}{m_1!\cdots m_n!} T_1Q(m)\,\overline{T_1\tilde{Q}(m)},$$

$$\frac{\Gamma(\mu)B^{1-\mu}}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)}\int_{\xi\in\mathbb{R}_+^n : |\xi|=B} T_2Q(\xi)\,\overline{T_2\tilde{Q}(\xi)}\,\xi_1^{\nu_1-1}\cdots\xi_n^{\nu_n-1}\,d\xi,$$

$$\frac{\Gamma(\mu)}{(2\pi)^{n-1}\left|\Gamma\left(\frac{\mu}{2}+iC\right)\right|^2}\int_{X\in\mathbb{R}^n\,:\,|X|=C} T_3Q(X)\,\overline{T_3\tilde{Q}(X)}\prod_{r=1}^{n}\frac{\left|\Gamma\left(\frac{\nu_r}{2}+iX_r\right)\right|^2}{\Gamma(\nu_r)}\,dX$$

*are, for each $k=0,1,2,\dots$, $B\in\mathbb{R}_+$ and $C\in\mathbb{R}$, all equal to the scalar product $\langle Q,\tilde{Q}\rangle_\nu$.*

*Proof.* First note that, as a consequence of Schur's lemma,

$$\langle\mathcal{K}_Q f,\mathcal{K}_{\tilde{Q}}g\rangle_\nu=\langle Q,\tilde{Q}\rangle_\nu\langle f,g\rangle_\mu,$$

where we have used that $\mathcal{K}_Q 1=Q$. If we put $\xi=B$ in the equality (21) and then take the scalar product with $\mathcal{K}_{\tilde{Q}}e_B^2*\phi$ for a suitable $\phi$, the left-hand side thus equals

$$\frac{B^{\mu-1}}{\Gamma(\mu)}\langle Q,\tilde{Q}\rangle_\nu\langle e_B^2,e_B^2*\phi\rangle_\mu,$$

while the right-hand side equals

$$\begin{aligned}&\frac{1}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)}\int_{\eta\in\mathbb{R}_+^n\,:\,|\eta|=B} T_2Q(\eta)\langle e_{\eta_1}^2\otimes\cdots\otimes e_{\eta_n}^2,\mathcal{K}_{\tilde{Q}}e_B^2*\phi\rangle_\nu\,\eta_1^{\nu_1-1}\cdots\eta_n^{\nu_n-1}\,d\eta\\&=\frac{1}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)}\int_{\eta\in\mathbb{R}_+^n\,:\,|\eta|=B} T_2Q(\eta)\,\overline{T_2\tilde{Q}(\eta)}\langle e_B^2,e_B^2*\phi\rangle_\mu\,\eta_1^{\nu_1-1}\cdots\eta_n^{\nu_n-1}\,d\eta.\end{aligned}$$

Dividing out the matrix element $\langle e_B^2,e_B^2*\phi\rangle_\mu$ we obtain the second part of the corollary. The remaining parts are similar. ☐

## 5. Convolution formulas

In this section we will obtain some convolution formulas for the polynomials $T_iQ$. The most general formulas of this type would follow from considering a scalar product

$$\langle\mathcal{K}_Q e_x^i*\phi,(e_{y_1}^j\otimes\cdots\otimes e_{y_n}^j)*\psi\rangle_\nu$$

and expanding it by applying formula number $k$ in (13) in each variable. For instance, with $(i,j,k)=(1,2,3)$ this gives

$$\begin{aligned}&\langle\mathcal{K}_Q e_k^1*\phi,(e_{\xi_1}^2\otimes\cdots\otimes e_{\xi_n}^2)*\psi\rangle_\nu\\&=\int_{\mathbb{R}^n}\langle\mathcal{K}_Q e_k^1*\phi,e_{X_1}^3\otimes\cdots\otimes e_{X_n}^3\rangle_\nu\langle e_{X_1}^3\otimes\cdots\otimes e_{X_n}^3,(e_{\xi_1}^2\otimes\cdots\otimes e_{\xi_n}^2)*\psi\rangle_\nu\,dm(X),\end{aligned}$$

where

$$dm(X)=\frac{1}{(2\pi)^n\Gamma(\nu_1)\cdots\Gamma(\nu_n)}\left|\Gamma\left(\frac{\nu_1}{2}+iX_1\right)\cdots\Gamma\left(\frac{\nu_n}{2}+iX_n\right)\right|^2 dX_1\cdots dX_n.$$

Now, by Theorem 4.2, this may be written as

$$T_2Q(\xi)\,K^{12}(k,|\xi|,\psi\phi^{-1})=\int_{\mathbb{R}^n}T_3Q(X)\,K^{13}(k,|X|,\phi^{-1})\prod_{r=1}^{n}K^{32}(X_r,\xi_r,\psi)\,dm(X).$$

Inserting the expressions for matrix elements from Proposition 3.2, we obtain an integral formula linking $T_2Q$ and $T_3Q$.

We will not write down all such formulas since they may be deduced from a few special cases. Using addition formulas for the matrix elements we may reduce ourselves to the case $\phi=\mathrm{id}$. As in the previous section, we may also assume that $i=k$.

In this case, the formulas follow from applying the transform $\mathcal{F}_j^\psi$ to both sides of the three generalized Clebsch–Gordan formulas of Corollary 4.3. Moreover, the formulas with $j=3$ follow from those with $j=1$, by means of (8). After these reductions, there remain the six cases

$$(i,j)=(1,1),\ (1,2),\ (2,1),\ (2,2),\ (3,1),\ (3,2),$$

which will give the following convolution formulas.

**Theorem 5.1.** *Let, as above, $Q$ be a highest weight polynomial in $\mathcal{A}^{\nu_1}\cdots\otimes\mathcal{A}^{\nu_n}$ of degree $s$, and let $\mu=|\nu|+2s$. Then the following formulas hold*:

$$\frac{(\mu)_k}{k!}c^s{}_2F_1\left(\begin{matrix}-k,s-|l|\\ \mu\end{matrix}\,\middle|\,c\right)T_1Q(l) \tag{22}$$
$$=\sum_{|m|=k+s}\frac{(\nu_1)_{m_1}\cdots(\nu_n)_{m_n}}{m_1!\cdots m_n!}\,T_1Q(m)\prod_{r=1}^n{}_2F_1\left(\begin{matrix}-m_r,-l_r\\ \nu_r\end{matrix}\,\middle|\,c\right),$$

$$\frac{(\mu)_k}{k!}\,{}_1F_1\left(\begin{matrix}-k\\ \mu\end{matrix}\,\middle|\,|\xi|\right)T_2Q(\xi)$$
$$=(-1)^s\sum_{|m|=k+s}\frac{(\nu_1)_{m_1}\cdots(\nu_n)_{m_n}}{m_1!\cdots m_n!}\,T_1Q(m)\prod_{r=1}^n{}_1F_1\left(\begin{matrix}-m_r\\ \nu_r\end{matrix}\,\middle|\,\xi_r\right),$$

$$\frac{\xi^{\mu-1}}{\Gamma(\mu)}\,{}_1F_1\left(\begin{matrix}s-|m|\\ \mu\end{matrix}\,\middle|\,\xi\right)T_1Q(m) \tag{23}$$
$$=\frac{(-1)^s}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)}\int_{\eta\in\mathbb{R}_+^n\,:\,|\eta|=\xi}T_2Q(\eta)\prod_{r=1}^n{}_1F_1\left(\begin{matrix}-m_r\\ \nu_r\end{matrix}\,\middle|\,\eta_r\right)\eta_1^{\nu_1-1}\cdots\eta_n^{\nu_n-1}\,d\eta,$$

$$\frac{\xi^{\mu-1}}{\Gamma(\mu)}\,{}_0F_1\left(\begin{matrix}-\\ \mu\end{matrix}\,\middle|\,-\xi|\zeta|\right)T_2Q(\zeta)$$
$$=\frac{(-1)^s}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)}\int_{\eta\in\mathbb{R}_+^n\,:\,|\eta|=\xi}T_2Q(\eta)\prod_{r=1}^n{}_0F_1\left(\begin{matrix}-\\ \nu_r\end{matrix}\,\middle|\,-\eta_r\zeta_r\right)\eta_1^{\nu_1-1}\cdots\eta_n^{\nu_n-1}\,d\eta,$$

$$\frac{\left|\Gamma\left(\frac{\mu}{2}+iX\right)\right|^2}{\Gamma(\mu)}c^s{}_2F_1\left(\begin{matrix}s-|m|,\frac{\mu}{2}+iX\\ \mu\end{matrix}\,\middle|\,c\right)T_1Q(m) \tag{24}$$
$$=\frac{1}{(2\pi)^{n-1}}\int_{Y\in\mathbb{R}^n\,:\,|Y|=X}T_3Q(Y)\prod_{r=1}^n\frac{\left|\Gamma\left(\frac{\nu_r}{2}+iY_r\right)\right|^2}{\Gamma(\nu_r)}\,{}_2F_1\left(\begin{matrix}-m_r,\frac{\nu_r}{2}+iY_r\\ \nu_r\end{matrix}\,\middle|\,c\right)dY,$$

$$\frac{\left|\Gamma\left(\frac{\mu}{2}+iX\right)\right|^2}{\Gamma(\mu)}\,{}_1F_1\left(\begin{matrix}\frac{\mu}{2}+iX\\ \mu\end{matrix}\,\middle|\,|\xi|\right)T_2Q(\xi)$$
$$=\frac{i^s}{(2\pi)^{n-1}}\int_{Y\in\mathbb{R}^n\,:\,|Y|=X}T_3Q(Y)\prod_{r=1}^n\frac{\left|\Gamma\left(\frac{\nu_r}{2}+iY_r\right)\right|^2}{\Gamma(\nu_r)}\,{}_1F_1\left(\begin{matrix}\frac{\nu_r}{2}+iY_r\\ \nu_r\end{matrix}\,\middle|\,i\xi_r\right)dY.$$

The parameter $c$ occurring in (22) and (24) depends on the choice of $\psi$. More precisely, in the first of these identities $c=-1/|\beta|^2$, where $\psi(z)=(\alpha z+\beta)/(\bar\beta z+\bar\alpha)$, and in the second one $c = 2/(\alpha+\bar\beta)(\bar\alpha-\beta)$. Since both sides are polynomials in $c$ they are valid for arbitrary $c$. For the remaining four identities, different choices of $\psi$ give essentially the same result.

Quite a lot of information is contained in these formulas. Many known identities follow from the three interesting special cases when (i) $Q = 1$, (ii) $n = 2$, or (iii) a numerator parameter in the hypergeometric function on the left-hand side is 0, so that this function reduces to 1. In particular, the case $n = 2$ of the first two identities occur in [KV1] with a similar group theoretic interpretation; cf. also [Su], [VdJ].

It should be noted that if $Q$ is constructed by binary coupling as in Subsection 2.4, the theorem may be proved using the case $n = 2$ and induction on $n$. Since one may construct a basis for the space of highest weight vectors in this way, this actually gives an alternative proof of the general case.

Some degenerate cases of our convolution formulas seem interesting enough to state explicitly. First, the polynomials $T_1Q$ (and thus $T_3Q$) satisfy some related identities.

**Corollary 5.2.** *In the notation of Theorem* 5.1*, the following identities are valid*:

$$\frac{(\mu+|l|-s)_k}{k!}\,T_1Q(l) = \sum_{|m|=k+s}\frac{(\nu_1+l_1)_{m_1}\cdots(\nu_n+l_n)_{m_n}}{m_1!\cdots m_n!}\,T_1Q(m), \tag{25}$$

$$\frac{(\mu+|l|-s)_{k+s}}{(k+s)!}\,T_1Q(l) = \sum_{|m|=k+s}\frac{(\nu_1+l_1)_{m_1}\cdots(\nu_n+l_n)_{m_n}}{m_1!\cdots m_n!}\,T_1Q(m+l), \tag{26}$$

$$\frac{(s-|l|)_k}{k!}\,T_1Q(l) = (-1)^s\sum_{|m|=k+s}\frac{(-l_1)_{m_1}\cdots(-l_n)_{m_n}}{m_1!\cdots m_n!}\,T_1Q(m), \tag{27}$$

$$\frac{(s-|l|)_{k+s}}{(k+s)!}\,T_1Q(l) = \sum_{|m|=k+s}\frac{(-l_1)_{m_1}\cdots(-l_n)_{m_n}}{m_1!\cdots m_n!}\,T_1Q(l-m). \tag{28}$$

*Proof.* This follows from the convolution formula (22). As was remarked above, it is valid for arbitrary values of $c$, though in the proof $c = -1/|\beta|^2$ is negative. Putting $c = 1$ and using the Chu–Vandermonde formula

$${}_2F_1\left(\begin{matrix}-m,-l\\ \nu\end{matrix}\,\middle|\,1\right) = \frac{(\nu)_{m+l}}{(\nu)_m(\nu)_l}$$

gives the equality (25). If we multiply both sides of (22) by $c^{-k-s}$ and use the limit relation

$$\lim_{c\to\infty}\frac{1}{c^m}\,{}_2F_1\left(\begin{matrix}-m,-l\\ \nu\end{matrix}\,\middle|\,c\right) = (-1)^m\frac{(-l)_m}{(\nu)_m}\qquad (m,\,l = 0,1,2,\dots),$$

we obtain (27). Incidentally, (27) also follows from (25) together with

$$T_1Q(m) = (-1)^s\,T_1Q(-\nu-m), \tag{29}$$

which in turn follows from (8) and (9).

If we instead multiply both sides of (22) by $c^{-|l|}$ and let $c\to\infty$, we get

$$(-1)^s\frac{(\mu)_k(-k)_{|l|-s}}{k!(\mu)_{|l|-s}}\,T_1Q(l) = \sum_{|m|=k+s}\frac{(\nu_1)_{m_1}\cdots(\nu_n)_{m_n}(-m_1)_{l_1}\cdots(-m_n)_{l_n}}{m_1!\cdots m_n!\,(\nu_1)_{l_1}\cdots(\nu_n)_{l_n}}\,T_1Q(m).$$

Replacing $k$ by $k+|l|$ and $m$ by $m+l$ gives (26). The equality (28) follows from (26) together with (29). $\square$

We may also obtain two integral formulas for $T_1Q$ from Theorem 5.1. In the bilinear case, the first of these is a known formula for the Hahn polynomials, cf. [VK], equation 8.2.4(7), while we have not found the second one in the literature.

**Corollary 5.3.** *In the notation of Theorem* 5.1*, the following identities are valid*:

$$\begin{aligned}
&\frac{1}{\Gamma(\mu+|m|-s)}\,T_1Q(m)\\
&\quad= \frac{1}{\Gamma(\nu_1+m_1)\cdots\Gamma(\nu_n+m_n)}\int_{\eta\in\mathbb{R}^n_+\,:\,|\eta|=1} T_2Q(\eta)\,\eta_1^{m_1+\nu_1-1}\cdots\eta_n^{m_n+\nu_n-1}\,d\eta,\\
&\frac{\Gamma\left(\frac{\mu}{2}+iX\right)\Gamma\left(\frac{\mu}{2}-iX+|m|-s\right)}{\Gamma(\mu+|m|-s)}\,T_1Q(m)\\
&\quad= \frac{1}{(2\pi)^{n-1}}\int_{Y\in\mathbb{R}^n\,:\,|Y|=X} T_3Q(Y)\prod_{r=1}^{m}\frac{\Gamma\left(\frac{\nu_r}{2}+iY_r\right)\Gamma\left(\frac{\nu_r}{2}-iY_r+m_r\right)}{\Gamma(\nu_r+m_r)}\,dY.
\end{aligned}$$

*Proof.* This follows from the convolution formulas (23) and (24), respectively. In the integral in (23), replace $\eta$ by $\xi\eta$ so that the integration is over $\{|\eta|=1\}$. Then let $\xi\to\infty$ and use the homogeneity of $T_2Q$ and the limit relation

$$\lim_{\xi\to\infty}\frac{1}{\xi^m}\,{}_1F_1\left(\begin{matrix}-m\\ \nu\end{matrix}\,\middle|\,\xi\eta\right)=\frac{(-1)^m}{(\nu)_m}\eta^m \qquad (m=0,1,2,\dots).$$

The other half of the corollary is the case $c=1$ of (24). $\square$

## 6. Coupling kernels

In this section we introduce some polynomials in $2n$ variables which we call *coupling kernels*. Let $V_s$ be the subspace of $\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n}$ consisting of highest weight vectors of degree $s$, and let $Q_s(z,\bar w)$ be its reproducing kernel. The coupling kernels are obtained by applying one of the transforms $T_i$ to each variable in $Q_s(z,\bar w)$. If

$$Q_s(z,\bar w)=\sum_{|t|=|u|=s} c_{t,u}\,\bar w^t z^u,$$

we define the coupling kernel $P_s^{ij}$ by

$$P_s^{ij}(x,y)=\sum_{|t|=|u|=s} c_{t,u}\,\overline{(T_iw^t)(x)}\,(T_jz^u)(y), \qquad i,j=1,2,3.$$

Recall the decomposition

$$\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n}\simeq\bigoplus_{s=0}^{\infty}\binom{n+s-2}{n-2}\mathcal{A}^{|\nu|+2s}.$$

Let $\Pi_s$ denote the orthogonal projection of $\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n}$ onto the isotypic subspace equivalent to $\binom{n+s-2}{n-2}\mathcal{A}^{|\nu|+2s}$. By general Hilbert space arguments, we may express the reproducing kernel $Q_s$ as

$$Q_s(z,\bar{w}) = \sum_{|t|=|u|=s} \frac{(\nu)_t(\nu)_u}{t!\,u!} \langle \Pi_s z^t, z^u\rangle_\nu \, \bar{w}^t z^u$$

or as

$$Q_s(z,\bar{w}) = \sum_{k\in\Lambda_s} \overline{Q_k(w)}\, Q_k'(z),$$

where $(Q_k)_{k\in\Lambda_s}$ is any basis of $V_s$ and $(Q_k')_{k\in\Lambda_s}$ is the dual basis (here $\Lambda_s$ is a suitable index set of cardinality $\binom{n+s-2}{n-2}$). Thus the coupling kernels are given by

$$\begin{aligned} P_s^{ij}(x,y) &= \sum_{|t|=|u|=s} \frac{(\nu)_t(\nu)_u}{t!\,u!} \langle \Pi_s z^t, z^u\rangle_\nu \overline{(T_i w^t)(x)}\,(T_j z^u)(y) && (30)\\ &= \sum_{k\in\Lambda_s} \overline{T_i Q_k(x)}\, T_j Q_k'(y). && (31) \end{aligned}$$

It will turn out that the coupling kernels may be expressed more explicitly in terms of Karlsson's generalized Kampé de Fériet functions (10). Moreover, we will recover as very special cases the dual Appell polynomials described in Subsection 2.3.

To motivate the introduction of the coupling kernels, consider the matrix elements of the projection $\Pi_s$:

$$\langle \Pi_s(e^i_{x_1}\otimes\cdots\otimes e^i_{x_n}), (e^j_{y_1}\otimes\cdots\otimes e^j_{y_n})*\phi\rangle_\nu, \qquad i,j=1,2,3. \tag{32}$$

One may express $\Pi_s$ as

$$\Pi_s = \sum_{k\in\Lambda_s} \mathcal{K}_{Q_k'}\mathcal{K}^*_{Q_k},$$

where $(Q_k)$ and $(Q_k')$ are as above. Using Theorem 4.2 one finds that the matrix element (32) equals a matrix element (in the sense of Section 3) for $\mathcal{A}^{|\nu|+2s}$ times the coupling kernel $P_s^{ij}(x,y)$. For instance, one has

$$\begin{aligned} &\langle \Pi_s(e^1_{l_1}\otimes\cdots\otimes e^1_{l_n}), (e^2_{\xi_1}\otimes\cdots\otimes e^2_{\xi_n})*\phi\rangle_\nu\\ &\quad= \sum_{k\in\Lambda_s} \langle \mathcal{K}^*_{Q_k} e^1_{l_1}\otimes\cdots\otimes e^1_{l_n}, \mathcal{K}^*_{Q_k'}(e^2_{\xi_1}\otimes\cdots\otimes e^2_{\xi_n})*\phi\rangle_{|\nu|+2s}\\ &\quad= \sum_{k\in\Lambda_s} \overline{T_1 Q_k(l)}\, T_2 Q_k'(\xi) \langle e^1_{|l|-s}, e^2_{|\xi|}*\phi\rangle_{|\nu|+2s}\\ &\quad= K^{12}(|l|-s,|\xi|,\phi)\, P_s^{12}(l,\xi). \end{aligned}$$

We will now give explicit expressions for the coupling kernels. It is clear that

$$P_s^{ij}(x,y) = \overline{P_s^{ji}(y,x)}, \tag{33}$$

so we may assume that $i\leq j$. Moreover, it follows from (8) that

$$P_s^{i3}(x,Y) = P_s^{i1}\left(x, iY - \frac{\nu}{2}\right), \tag{34}$$

which leaves us with the three cases $P_s^{11}$, $P_s^{12}$, and $P_s^{22}$.

**Theorem 6.1.** *The coupling kernels are given in terms of Karlsson's functions* (10) *by*

$$\begin{aligned} P_s^{11}(k,l) &= \frac{(-1)^s(-|k|)_s(-|l|)_s}{(|\nu|+s-1)_s s!}\, F^{2:2}_{2:1}\left(\begin{matrix} |\nu|+s-1,-s & : & -k,-l \\ -|k|,-|l| & : & \nu \end{matrix}\,\middle|\, 1\right), \\ P_s^{12}(k,\xi) &= \frac{(-|k|)_s|\xi|^s}{(|\nu|+s-1)_s s!}\, F^{2:1}_{1:1}\left(\begin{matrix} |\nu|+s-1,-s & : & -k \\ -|k| & : & \nu \end{matrix}\,\middle|\, \frac{\xi}{|\xi|}\right), \\ P_s^{22}(\xi,\eta) &= \frac{(-1)^s|\xi|^s|\eta|^s}{(|\nu|+s-1)_s s!}\, F^{2:0}_{0:1}\left(\begin{matrix} |\nu|+s-1,-s & : & - \\ - & : & \nu \end{matrix}\,\middle|\, \frac{\xi\eta}{|\xi||\eta|}\right). \end{aligned}$$

The proof of this theorem will be deferred to Section 8. The variable of the $F^{2:0}_{0:1}$-function should be interpreted as

$$\frac{\xi\eta}{|\xi||\eta|} = \frac{1}{(\xi_1+\cdots+\xi_n)(\eta_1+\cdots+\eta_n)}\,(\xi_1\eta_1,\ldots,\xi_n\eta_n)\,.$$

We also remark that Karlsson's $F^{2:0}_{0:1}$-function is Lauricella's $F_C$-function.

These expressions should be compared with what one gets from choosing a particular basis $(Q_k)$ in (31). This is particularly interesting when $n=2$, when the sum has only one term. In the case of $P_s^{22}$ one gets an expression for the product of two Jacobi polynomials as an $F^{2:0}_{0:1}$-series of two variables, that is, as an Appell $F_4$-series. This is a classical result of Watson [W]. For $P_s^{11}$ one gets a similar expression for the product of two Hahn polynomials due to Gasper [G1]; cf. also [G2]. For $P_s^{12}$ one gets the identity

$$\begin{aligned} &\frac{(-1)^s s!}{(\nu_2)_s}\, P_s^{(\nu_1-1,\nu_2-1)}\left(\frac{\xi_2-\xi_1}{\xi_1+\xi_2}\right) Q_s(k_1;\nu_1-1,\nu_2-1;k_1+k_2) \\ &\quad = F^{2:1}_{1:1}\left(\begin{matrix} \nu_1+\nu_2+s-1,-s & : & -(k_1,k_2) \\ -k_1-k_2 & : & (\nu_1,\nu_2) \end{matrix}\,\middle|\, \frac{\xi_1}{\xi_1+\xi_2},\frac{\xi_2}{\xi_1+\xi_2}\right) \end{aligned}$$

or equivalently

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} -n,n+a \\ b \end{matrix}\,\middle|\, z\right) {}_3F_2\left(\begin{matrix} -n,n+a,c \\ b,d \end{matrix}\,\middle|\, 1\right) \\ &\quad = \frac{(-1)^n(a-b+1)_n}{(b)_n}\, F^{2:1}_{1:1}\left(\begin{matrix} -n,n+a & : & (c,d-c) \\ d & : & (b,a-b+1) \end{matrix}\,\middle|\, z,1-z\right), \end{aligned}$$

which we have not found in the literature. For general $n$ one may obtain generalizations of these formulas by choosing in (31) the basis (11) of $V_s$ or a basis constructed by binary coupling as in Subsection 2.4.

As was mentioned above, the dual Appell polynomials described in Subsection 2.3 may be viewed as special cases of coupling kernels. This might first seem surprising, but is easily understood.

**Corollary 6.2.** *Let* $(Q_t)$ *be the basis*

$$Q_t(z) = (z_1-z_n)^{t_1}\cdots(z_{n-1}-z_n)^{t_{n-1}}, \qquad |t| = t_1+\cdots+t_{n-1} = s$$

*of the space of highest weight polynomials in* $\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n}$ *of degree* $s$. *Let* $(Q_t')$ *be the dual basis. For* $x=(x_1,\ldots,x_n)$, *let* $\hat{x}=(x_1,\ldots,x_{n-1})$. *Then the polynomials*

*$T_iQ'_t$ may be expressed in terms of Karlsson's functions as*

$$\begin{aligned} T_1Q'_t(m) &= \frac{(\hat\nu)_t(-|m|)_s}{t!\,(|\nu|+s-1)_s}\, F^{1:2}_{1:1}\left(\begin{matrix} |\nu|+s-1 & : & -t,-\hat m \\ -|m| & : & \hat\nu \end{matrix}\;\middle|\; 1\right), \\ T_2Q'_t(\xi) &= \frac{(\hat\nu)_t(-|\xi|)^s}{t!\,(|\nu|+s-1)_s}\, F^{1:1}_{0:1}\left(\begin{matrix} |\nu|+s-1 & : & -t \\ - & : & \hat\nu \end{matrix}\;\middle|\; \frac{\xi_1}{|\xi|},\dots,\frac{\xi_{n-1}}{|\xi|}\right), \\ T_3Q'_t(X) &= \frac{(\hat\nu)_t(\frac{|\nu|}{2}-i|X|)_s}{t!\,(|\nu|+s-1)_s}\, F^{1:2}_{1:1}\left(\begin{matrix} |\nu|+s-1 & : & -t,\frac12\hat\nu-i\hat X \\ \frac12|\nu|-i|X| & : & \hat\nu \end{matrix}\;\middle|\; 1\right). \end{aligned}$$

Note that the vector parameters in Karlsson's functions have dimension $n-1$ here but dimension $n$ in Theorem 6.1.

*Proof.* If $u=(u_1,\dots,u_n)$, $|u|=s$, we may write

$$\Pi_s z^u = \sum_v \langle z^u, Q_v\rangle_\nu\, Q'_v = \sum_v \overline{T_1Q_v(u)}\, Q'_v.$$

Thus

$$T_i\Pi_s z^u(x) = \sum_v \overline{T_1Q_v(u)}\, T_iQ'_v(x) = P_s^{1i}(u,x). \tag{35}$$

Now choose

$$u=(t,0)=(t_1,\dots,t_{n-1},0), \qquad |t|=s. \tag{36}$$

Then

$$\langle z^u, Q_v\rangle_\nu = \frac{t_1!\cdots t_{n-1}!}{(\nu_1)_{t_1}\cdots(\nu_{n-1})_{t_{n-1}}}\,\delta_{t,v},$$

so that in fact

$$Q'_t = \frac{(\nu_1)_{t_1}\cdots(\nu_{n-1})_{t_{n-1}}}{t_1!\cdots t_{n-1}!}\,\Pi_s z^u. \tag{37}$$

Combining (35) and (37) gives

$$T_iQ'_t(x) = \frac{(\nu_1)_{t_1}\cdots(\nu_{n-1})_{t_{n-1}}}{t_1!\cdots t_{n-1}!}\, P_s^{1i}(u,x), \qquad i=1,2,3.$$

It is now easy to check that the expressions for $P_s^{1i}(u,x)$ given in Theorem 6.1 simplify to those of the corollary when $u$ is of the form (36). ☐

## 7. Further properties of the coupling kernels

We continue to study the formal properties of the coupling kernels. Some formulas follow from Corollary 4.4 and Theorem 5.1 by choosing highest weight vectors $Q$ of the form

$$Q_{isx} = \sum_{k\in\Lambda_s} \overline{T_iQ_k(x)}\, Q'_k,$$

where, as above, $(Q_k)_{k\in\Lambda_s}$ is any basis of $V_s$ and $(Q'_k)_{k\in\Lambda_s}$ the dual basis. For such $Q$ one has

$$\langle Q, Q_{isx}\rangle_\nu = T_iQ(x), \qquad Q\in V_s,$$

so $Q_{isx}$ is the kernel of $T_i$, viewed as an operator on $V_s$. We will write down these formulas explicitly since we find them quite interesting.

Applying Corollary 4.4 with $\tilde Q = Q_{isx}$ gives the following identities.

**Proposition 7.1.** *Let $Q$ be a highest weight polynomial in $\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n}$ of degree $s$, and let $\mu=|\nu|+2s$. Then, for $i=1,2,3$, $k=0,1,2,\ldots$, $B\in\mathbb{R}_+$, and $C\in\mathbb{R}$, one has the equalities*

$$\begin{aligned}
T_iQ(x) &= \frac{k!}{(\mu)_k}\sum_{|m|=k+s}\frac{(\nu_1)_{m_1}\cdots(\nu_n)_{m_n}}{m_1!\cdots m_n!}\,T_1Q(m)\,P_s^{1i}(x,m)\\
&= \frac{\Gamma(\mu)B^{1-\mu}}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)}\int_{\xi\in\mathbb{R}_+^n:|\xi|=B} T_2Q(\xi)\,P_s^{2i}(x,\xi)\,\xi_1^{\nu_1-1}\cdots\xi_n^{\nu_n-1}\,d\xi\\
&= \frac{\Gamma(\mu)}{(2\pi)^{n-1}\left|\Gamma\left(\frac{\mu}{2}+iC\right)\right|^2}\int_{X\in\mathbb{R}^n:|X|=C} T_3Q(X)\,P_s^{3i}(x,X)\prod_{r=1}^n\frac{\left|\Gamma\left(\frac{\nu_r}{2}+iX_r\right)\right|^2}{\Gamma(\nu_r)}\,dX.
\end{aligned}$$

In particular, $P_s^{ii}$ is essentially the reproducing kernel for the space of coupling coefficients $T_iQ$ of degree $s$. If we recall the connection between the polynomials $T_2Q$ and spherical harmonics, we see that the reproducing kernels for Vilenkin's poly-spherical harmonics may be expressed in terms of $P_s^{22}$ and thus in terms of Karlsson's function $F^{2:0}_{0:1}$. This is an interesting fact which we have not found in the literature.

If we choose $\tilde{Q}=Q_{isx}$ and $Q=Q_{jty}$ in Corollary 4.4, we obtain addition formulas for the coupling kernels.

**Proposition 7.2.** *In the notation above, one has for $i,j=1,2,3$ the equalities*

$$\begin{aligned}
P_s^{ij}(x,y)\,\delta_{s,t} &= \frac{k!}{(\mu)_k}\sum_{|m|=k+s}\frac{(\nu_1)_{m_1}\cdots(\nu_n)_{m_n}}{m_1!\cdots m_n!}\,P_s^{i1}(x,m)P_t^{1j}(m,y)\\
&= \frac{\Gamma(\mu)B^{1-\mu}}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)}\int_{\xi\in\mathbb{R}_+^n:|\xi|=B} P_s^{i2}(x,\xi)P_t^{2j}(\xi,y)\,\xi_1^{\nu_1-1}\cdots\xi_n^{\nu_n-1}\,d\xi\\
&= \frac{\Gamma(\mu)}{(2\pi)^{n-1}\left|\Gamma\left(\frac{\mu}{2}+iC\right)\right|^2}\int_{X\in\mathbb{R}^n:|X|=C} P_s^{i3}(x,X)P_t^{3j}(X,y)\prod_{r=1}^n\frac{\left|\Gamma\left(\frac{\nu_r}{2}+iX_r\right)\right|^2}{\Gamma(\nu_r)}\,dX.
\end{aligned}$$

If we choose $Q=Q_{isx}$ in Theorem 5.1, we obtain eighteen convolution formulas.

**Proposition 7.3.** *In the notation above, one has for $i=1,2,3$ the equalities*

$$\begin{aligned}
&\frac{(\mu)_k}{k!}c^s\,{}_2F_1\left(\begin{matrix}-k,\,s-|l|\\ \mu\end{matrix}\,\middle|\,c\right)P_s^{i1}(x,l)\\
&= \sum_{|m|=k+s}\frac{(\nu)_m}{m!}P_s^{i1}(x,m)\prod_{r=1}^n{}_2F_1\left(\begin{matrix}-m_r,\,-l_r\\ \nu_r\end{matrix}\,\middle|\,c\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{(\mu)_k}{k!}\,{}_1F_1\left(\begin{matrix}-k\\ \mu\end{matrix}\,\middle|\,|\xi|\right)P_s^{i2}(x,\xi)\\
&= (-1)^s\sum_{|m|=k+s}\frac{(\nu)_m}{m!}\,P_s^{i1}(x,m)\prod_{r=1}^n{}_1F_1\left(\begin{matrix}-m_r\\ \nu_r\end{matrix}\,\middle|\,\xi_r\right),
\end{aligned}$$

$$\frac{\xi^{\mu-1}}{\Gamma(\mu)}\,{}_1F_1\left(\begin{matrix} s-|m| \\ \mu \end{matrix}\,\middle|\, |\xi|\right) P_s^{i1}(x,m)$$
$$= \frac{(-1)^s}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)}\int_{\eta\in\mathbb{R}_+^n\,:\,|\eta|=\xi} P_s^{i2}(x,\eta)\prod_{r=1}^n {}_1F_1\left(\begin{matrix} -m_r \\ \nu_r \end{matrix}\,\middle|\, \eta_r\right)\eta_1^{\nu_1-1}\cdots\eta_n^{\nu_n-1}\,d\eta,$$

$$\frac{\xi^{\mu-1}}{\Gamma(\mu)}\,{}_0F_1\left(\begin{matrix} - \\ \mu \end{matrix}\,\middle|\, -\xi|\zeta|\right) P_s^{i2}(x,\zeta)$$
$$= \frac{(-1)^s}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)}\int_{\eta\in\mathbb{R}_+^n\,:\,|\eta|=\xi} P_s^{i2}(x,\eta)\prod_{r=1}^n {}_0F_1\left(\begin{matrix} - \\ \nu_r \end{matrix}\,\middle|\, -\eta_r\zeta_r\right)\eta_1^{\nu_1-1}\cdots\eta_n^{\nu_n-1}\,d\eta,$$

$$\frac{\left|\Gamma\left(\frac{\mu}{2}+iX\right)\right|^2}{\Gamma(\mu)}c^s\,{}_2F_1\left(\begin{matrix} s-|m|,\frac{\mu}{2}+iX \\ \mu \end{matrix}\,\middle|\, c\right) P_s^{i1}(x,m)$$
$$= \frac{1}{(2\pi)^{n-1}}\int_{Y\in\mathbb{R}^n\,:\,|Y|=X} P_s^{i3}(x,Y)\prod_{r=1}^n \frac{\left|\Gamma\left(\frac{\nu_r}{2}+iY_r\right)\right|^2}{\Gamma(\nu_r)}\,{}_2F_1\left(\begin{matrix} -m_r,\frac{\nu_r}{2}+iY_r \\ \nu_r \end{matrix}\,\middle|\, c\right)dY,$$

$$\frac{\left|\Gamma\left(\frac{\mu}{2}+iX\right)\right|^2}{\Gamma(\mu)}\,{}_1F_1\left(\begin{matrix} \frac{\mu}{2}+iX \\ \mu \end{matrix}\,\middle|\, i|\xi|\right) P_s^{i2}(x,\xi)$$
$$= \frac{i^s}{(2\pi)^{n-1}}\int_{Y\in\mathbb{R}^n\,:\,|Y|=X} P_s^{i3}(x,Y)\prod_{r=1}^n \frac{\left|\Gamma\left(\frac{\nu_r}{2}+iY_r\right)\right|^2}{\Gamma(\nu_r)}\,{}_1F_1\left(\begin{matrix} \frac{\nu_r}{2}+iY_r \\ \nu_r \end{matrix}\,\middle|\, i\xi_r\right)dY.$$

Degenerate cases of these formulas are obtained by choosing $Q = Q_{isx}$ in Corollaries 5.2 and 5.3.

We will now give some expansions which are different in nature from those obtained so far. They are of a type known as *linearization formulas*. Let $\Pi_s$ be the orthogonal projection introduced in Section 6. Then the scalar product of $\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n}$ decomposes as

$$\langle f,g\rangle_\nu = \sum_{s=0}^\infty \langle \Pi_s f, g\rangle_\nu. \tag{38}$$

We insert in this equality $f = e^i_{x_1}\otimes\cdots\otimes e^i_{x_n}$, $g = (e^i_{y_1}\otimes\cdots\otimes e^i_{y_n})*\phi$. Then the left-hand side is a product of matrix elements for the spaces $\mathcal{A}^{\nu_i}$, while, as we have observed, each term on the right-hand side is the product of a coupling kernel $P_s^{ij}(x,y)$ and a matrix element for $\mathcal{A}^{|\nu|+2s}$. In view of (33) and (34) it suffices to consider the three cases

$$(i,j) = (1,1),\ (1,2),\ (2,2).$$

This gives the following linearization formulas.

**Proposition 7.4.** *In the notation above,*

$$\prod_{r=1}^n {}_2F_1\left(\begin{matrix} -k_r,-l_r \\ \nu_r \end{matrix}\,\middle|\, c\right) = \sum_{s=0}^{\min(|k|,|l|)} P_s^{11}(k,l)\,c^s\,{}_2F_1\left(\begin{matrix} s-|k|,s-|l| \\ |\nu|+2s \end{matrix}\,\middle|\, c\right), \tag{39}$$

$$\prod_{r=1}^n {}_1F_1\left(\begin{matrix} -k_r \\ \nu_r \end{matrix}\,\middle|\, \xi_r\right) = \sum_{s=0}^{|k|} P_s^{12}(k,\xi)(-1)^s\,{}_1F_1\left(\begin{matrix} s-|k| \\ |\nu|+2s \end{matrix}\,\middle|\, |\xi|\right),$$

$$\prod_{r=1}^{n} {}_0F_1\left(\begin{matrix} - \\ \nu_r \end{matrix} \,\middle|\, -\xi_r\eta_r\right) \quad = \quad \sum_{s=0}^{\infty} P_s^{22}(\xi,\eta)\, {}_0F_1\left(\begin{matrix} - \\ |\nu|+2s \end{matrix} \,\middle|\, -|\xi||\eta|\right).$$

For $n=2$, when the coupling kernels factor as a product of two Clebsch–Gordan coefficients, the first of these formulas is the Burchnall–Chaundy formula [BC], while the third one is attributed to Bateman in [VK]. The bilinear case is also treated in [KV2].

It is not hard to prove these formulas directly, or, in fact, to obtain them as special cases of more general expansion formulas. For instance, one may prove that (assuming convergence)

$$\begin{aligned} &\prod_{r=1}^{n} {}_2F_1\left(\begin{matrix} A_r, B_r \\ C_r \end{matrix} \,\middle|\, x\right) \\ &= \sum_{s=0}^{\infty} \frac{(a)_s(b)_s(-1)^s}{s!(c+s-1)_s} F^{2:2}_{2:1}\left(\begin{matrix} c+s-1, -s & : & A, B \\ a, b & : & C \end{matrix} \,\middle|\, 1\right) x^s\, {}_2F_1\left(\begin{matrix} a+s, b+s \\ c+2s \end{matrix} \,\middle|\, x\right), \end{aligned} \tag{40}$$

which reduces to (39) for special values of the parameters and to Chaundy's formula [C]

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix} \,\middle|\, x\right) \\ &= \sum_{s=0}^{\infty} \frac{(a)_s(b)_s(-1)^s}{s!(c+s-1)_s}\, {}_4F_3\left(\begin{matrix} c+s-1, -s, A, B \\ a, b, C \end{matrix} \,\middle|\, 1\right) x^s\, {}_2F_1\left(\begin{matrix} a+s, b+s \\ c+2s \end{matrix} \,\middle|\, x\right) \end{aligned} \tag{41}$$

for $n=1$. In fact, (40) is a special case of an even more general expansion formula due to H. M. Srivastava [S]; cf. also [SK], pp. 340–341. (I would like to thank Professor Srivastava for pointing this out to me.) In [R], we gave a group theoretic interpretation of (41), where the ${}_4F_3$-series occur as Racah coefficients (or Wigner 6-$j$-symbols). It would be interesting to have an interpretation of the more general formula (40).

We finally note some degenerate cases of (39). If we put $c=1$, we get

$$\frac{(\nu_1)_{k_1+l_1}\cdots(\nu_n)_{k_n+l_n}}{(\nu_1)_{k_1}\cdots(\nu_n)_{k_n}(\nu_1)_{l_1}\cdots(\nu_n)_{l_n}} = \sum_{s=0}^{\min(|k|,|l|)} \frac{(|\nu|+2s)_{|k|+|l|-2s}}{(|\nu|+2s)_{|k|-s}(|\nu|+2s)_{|l|-s}}\, P_s^{11}(k,l).$$

If we multiply (39) by $1/c^{|l|}$ and let $c\to\infty$, we get

$$\frac{(-k_1)_{l_1}\cdots(-k_n)_{l_n}}{(\nu_1)_{l_1}\cdots(\nu_n)_{l_n}} = (-1)^s \sum_{s=0}^{|l|} \frac{(s-|k|)_{|l|-s}}{(|\nu|+2s)_{|l|-s}}\, P_s^{11}(k,l).$$

The case $|k|=|l|$ of this equation reads

$$\frac{k_1!\cdots k_n!}{(\nu_1)_{k_1}\cdots(\nu_n)_{k_n}}\,\delta_{k,l} = \sum_{s=0}^{|k|} \frac{(|k|-s)!}{(|\nu|+2s)_{|k|-s}}\, P_s^{11}(k,l), \qquad |k|=|l|.$$

This may also be deduced by inserting $f=z_1^{k_1}\cdots z_n^{k_n}$, $g=z_1^{l_1}\cdots z_n^{l_n}$ in (38). For $n=2$, writing $P_s^{11}(k,l)$ as the product of two ${}_3F_2$-series, this gives the orthogonality relation for the dual Hahn polynomials (Eberlein polynomials).

## 8. Proof of Theorem 6.1

In this section we will prove the explicit expressions for coupling kernels given in Theorem 6.1. The proof will be based on equation (30). Thus we will first compute the matrix elements

$$\langle \Pi_s z^t, z^u \rangle_\nu, \qquad |t| = |u| = s \tag{42}$$

of the projection $\Pi_s$.

We will work with the action of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ derived from the group action. This associative algebra has three generators $E$, $F$, and $H$. The action is given on $\mathcal{A}^\nu$ by the densely defined operators (cf. [R])

$$E = -\frac{d}{dz}, \qquad F = z^2 \frac{d}{dz} + \nu z, \qquad H = -\left(2z\frac{d}{dz} + \nu\right)$$

and on $\mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n}$ by

$$E = -\sum_{i=1}^n \frac{\partial}{\partial z_i}, \qquad F = \sum_{i=1}^n \left(z_i^2 \frac{\partial}{\partial z_i} + \nu_i z_i\right), \qquad H = -\sum_{i=1}^n \left(2z_i \frac{\partial}{\partial z_i} + \nu_i\right).$$

These operators satisfy the structure equations

$$[EF] = H, \qquad [HE] = 2E, \qquad [HF] = -2F$$

and the reality conditions

$$E^* = -F, \qquad F^* = -E, \qquad H^* = H.$$

Let $\mathcal{H}_s$ be the subspace of $\mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n}$ consisting of homogeneous polynomials of degree $s$ and let, as above, $V_s$ be the subspace consisting of highest weight polynomials of degree $s$. There is then an orthogonal decomposition

$$\mathcal{H}_s = V_s \oplus F\mathcal{H}_{s-1},$$

and we are interested in the orthogonal projection $\Pi_s|_{\mathcal{H}_s}$ onto the first summand.

**Lemma 8.1.** *We have*

$$EF^k\big|_{\mathcal{H}_s} = F^k E - k(|\nu| + 2s + k - 1)F^{k-1}.$$

*Proof.* First we observe that

$$EF\big|_{\mathcal{H}_s} = (FE + H)\big|_{\mathcal{H}_s} = FE - (|\nu| + 2s)\mathrm{Id}.$$

Proceeding by induction on $k$, we find that

$$\begin{aligned} EF^{k+1}\big|_{\mathcal{H}_s} &= EF^k\big|_{\mathcal{H}_{s+1}} F\big|_{\mathcal{H}_s} = F^k EF - k(|\nu| + 2s + k + 1)F^k \\ &= F^k(FE - (|\nu| + 2s)) - k(|\nu| + 2s + k + 1)F^k \\ &= F^{k+1}E - (k+1)(|\nu| + 2s + k)F^k. \end{aligned}$$

$\square$

**Lemma 8.2.** *The orthogonal projection of $\mathcal{H}_s$ onto $V_s$ is given by*

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2-|\nu|-2s)_k} F^k E^k.$$

Note that since $E^{s+1}\big|_{\mathcal{H}_s} = 0$, the sum terminates.

*Proof.* We first show that the operator maps into $V_s$. Using the previous lemma, we find that in general

$$\begin{aligned}
E\sum_{k=0}^{\infty} c_k F^k E^k\big|_{\mathcal{H}_s} &= \sum_{k=0}^{\infty} c_k E F^k\big|_{\mathcal{H}_{s-k}} E^k \\
&= \sum_{k=0}^{\infty} c_k (F^k E - k(|\nu|+2s-k-1)F^{k-1})E^k \\
&= \sum_{k=0}^{\infty} (c_k - c_{k+1}(k+1)(|\nu|+2s-k-2))F^k E^{k+1}.
\end{aligned}$$

This vanishes if

$$c_{k+1} = \frac{1}{(k+1)(|\nu|+2s-k-2)} c_k,$$

which is indeed solved by

$$c_k = \frac{(-1)^k}{k!(2-|\nu|-2s)_k}.$$

We then have, for any $f$ in $\mathcal{H}_s$,

$$f = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2-|\nu|-2s)_k} F^k E^k f - \sum_{k=1}^{\infty} \frac{(-1)^k}{k!(2-|\nu|-2s)_k} F^k E^k f,$$

where the first term is in $V_s$. Since the second term is in the image of $F$, it is orthogonal to the first term. This completes the proof. $\square$

We use this lemma to compute the matrix elements (42).

**Lemma 8.3.** *For $|t| = |u| = s$,*

$$\langle \Pi_s z^t, z^u\rangle_\nu = \frac{(-1)^s s!}{(|\nu|+s-1)_s} F^{1:2}_{1:1}\left( \begin{matrix} |\nu|+s-1 & : & -t, -u \\ -s & : & \nu \end{matrix} \,\middle|\, 1 \right). \tag{43}$$

*Proof.* Since $F^* = -E$, the previous lemma gives

$$\langle \Pi_s z^t, z^u\rangle_\nu = \sum_{k=0}^{\infty} \frac{1}{k!(2-|\nu|-2s)_k} \langle E^k z^t, E^k z^u\rangle_\nu. \tag{44}$$

Since $E$ is the sum of the commuting operators $-\frac{\partial}{\partial z_j}$, it follows from the multinomial theorem that

$$E^k(z^t) = (-1)^s k! \sum_{|j|=s-k} \frac{(-t)_j}{j!} z^j$$

and thus that

$$\langle E^k z^t, E^k z^u\rangle_\nu = (k!)^2 \sum_{|j|=s-k} \frac{(-t)_j(-u)_j}{j!\,(\nu)_j}.$$

Inserting this in (44), we obtain

$$\begin{aligned}\langle \Pi_s z^t, z^u\rangle_\nu &= \sum_{k=0}^{\infty}\sum_{|j|=s-k}\frac{k!\,(-t)_j(-u)_j}{(2-|\nu|-2s)_k j!\,(\nu)_j}\\ &= \sum_{j_1,\dots,j_n=0}^{\infty}\frac{(s-|j|)!\,(-t)_j(-u)_j}{(2-|\nu|-2s)_{s-|j|}j!\,(\nu)_j}\\ &= \frac{(-1)^s s!}{(|\nu|+s-1)_s}\sum_{j_1,\dots,j_n=0}^{\infty}\frac{(|\nu|+s-1)_{|j|}(-t)_j(-u)_j}{(-s)_{|j|}j!\,(\nu)_j}\\ &= \frac{(-1)^s s!}{(|\nu|+s-1)_s}F^{1:2}_{1:1}\left(\begin{matrix}|\nu|+s-1 & : & -t,-u\\ -s & : & \nu\end{matrix}\,\middle|\, 1\right).\end{aligned}$$

□

We are now ready to prove Theorem 6.1.

*Proof of Theorem* 6.1. Insert the expression (43) in (30). In the case of $P_s^{12}$ this gives

$$\begin{aligned}P_s^{12}(k,\xi) &= \sum_{|t|=|u|=s}\frac{(\nu)_t(\nu)_u}{t!\,u!}\langle \Pi_s z^t, z^u\rangle_\nu(-1)^s\frac{(-k)_t\,\xi^u}{(\nu)_t(\nu)_u}\\ &= \sum_{|t|=|u|=s}\frac{s!}{(|\nu|+s-1)_s}F^{1:2}_{1:1}\left(\begin{matrix}|\nu|+s-1 & : & -t,-u\\ -s & : & \nu\end{matrix}\,\middle|\, 1\right)\frac{(-k)_t\,\xi^u}{t!\,u!}\\ &= \frac{s!}{(|\nu|+s-1)_s}\sum_{|t|=|u|=s}\sum_{j_1,\dots,j_n=0}^{\infty}\frac{(|\nu|+s-1)_{|j|}(-t)_j(-u)_j(-k)_t\,\xi^u}{(-s)_{|j|}(\nu)_j j!\,t!\,u!}.\end{aligned}$$

Since the terms for which $t_i \leq j_i$ or $u_i \leq j_i$ for some $i$ vanish, we put

$$t = j+p, \qquad u = j+q$$

and change the order of summation. Then

$$\frac{(-t)_j(-u)_j}{t!\,u!} = \frac{(-j-p)_j(-j-q)_j}{(j+p)!\,(j+q)!} = \frac{1}{p!\,q!}$$

so that we get

$$\begin{aligned}&P_s^{12}(k,\xi)\\ &\quad= \frac{s!}{(|\nu|+s-1)_s}\sum_{j\,:\,|j|\leq s}\frac{(|\nu|+s-1)_{|j|}}{(-s)_{|j|}(\nu)_j j!}\sum_{|p|=s-|j|}\frac{(-k)_{j+p}}{p!}\sum_{|q|=s-|j|}\frac{\xi^{j+q}}{q!}.\end{aligned}\tag{45}$$

Now, by the generalized Chu–Vandermonde formula,

$$\begin{aligned}\sum_{|p|=s-|j|}\frac{(-k)_{j+p}}{p!} &= (-k)_j\sum_{|p|=s-|j|}\frac{(j-k)_p}{p!} = (-k)_j\frac{(|j|-|k|)_{s-|j|}}{(s-|j|)!}\\ &= (-1)^{|j|}\frac{(-|k|)_s}{s!}\frac{(-s)_{|j|}(-k)_j}{(-|k|)_{|j|}}.\end{aligned}\tag{46}$$

Similarly, the multinomial theorem gives

$$\sum_{|q|=s-|j|} \frac{\xi^{j+q}}{q!} = \xi^j \frac{|\xi|^{s-|j|}}{(s-|j|)!} = (-1)^{|j|} \frac{|\xi|^s}{s!} \frac{(-s)_{|j|}\xi^j}{|\xi|^{|j|}}.$$

Inserting these expressions in (45) we finally obtain

$$\begin{aligned} P_s^{12}(k,\xi) &= \frac{(-|k|)_s|\xi|^s}{(|\nu|+s-1)_s s!} \sum_{j_1,\dots,j_n=0}^{\infty} \frac{(|\nu|+s-1)_{|j|}(-s)_{|j|}(-k)_j\xi^j}{(-|k|)_{|j|}(\nu)_j|\xi|^{|j|}} \\ &= \frac{(-|k|)_s|\xi|^s}{(|\nu|+s-1)_s s!} F^{2:1}_{1:1}\left( \begin{matrix} |\nu|+s-1, -s & : & -k \\ -|k| & : & \nu \end{matrix} \,\middle|\, \frac{\xi}{|\xi|} \right). \end{aligned}$$

The remaining two cases are similar, except that for $P_s^{11}$ one uses the generalized Chu–Vandermonde formula twice and for $P_s^{22}$ the multinomial theorem twice.

## 9. The Fock space

So far our discussion has been based on the unit disc. However, one may study analogous problems for other Hermitean symmetric spaces. In dimension 1 one has, apart from the disc, the Riemann sphere and the plane. For the Riemann sphere, the spaces $\mathcal{A}^\nu$ should be replaced by finite-dimensional spaces of polynomials. Since the group SU(2) has only one conjugacy class of one-parameter subgroups, there will only be one analogue of the transforms $T_i$. This transform is similar to $T_1$ with the parameters $\nu_i$ replaced by *negative* integers.

A more interesting case is the plane $\mathbb{C}$. We will write down the analogues of our results in this section. Since most proofs are similar to the case of the disc, we will not give the details. First we introduce the Fock space $\mathcal{F}_\alpha$ ($\alpha > 0$), consisting of entire functions for which

$$\|f\|^2 = \frac{\alpha}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-\alpha|z|^2}\, dxdy = \sum_{k=0}^{\infty} \frac{k!}{\alpha^k} |\hat{f}(k)|^2 < \infty.$$

This space may be viewed as a limit of $\mathcal{A}^\nu$ when $\nu \to \infty$. In fact, if $\nu > 1$, the norm of $\mathcal{A}^\nu$ is given by

$$\|f\|^2 = \frac{\nu-1}{\pi} \int_{\mathbf{D}} |f(z)|^2 (1-|z|^2)^{\nu-2}\, dxdy.$$

If we replace $\mathbf{D}$ with the disc $\{|z| < R\}$ and let $\nu$ and $R$ tend to infinity so that $\nu/R^2 \to \alpha$, the weight tends to

$$\lim_{\nu,\, R\to\infty} \left(1 - \frac{|z|^2}{R^2}\right)^{\nu-2} = e^{-\alpha|z|^2}.$$

At the level of Lie algebras, this corresponds to deforming $\mathfrak{su}(1,1)$ into the Heisenberg algebra.

On the space $\mathcal{F}_\alpha$ we introduce the Heisenberg operators

$$\mathcal{U}_h f(z) = e^{-\alpha|h|^2/2-\alpha\bar{h}z} f(z+h), \qquad h \in \mathbb{C}.$$

They satisfy the Heisenberg relation

$$\mathcal{U}_{h_1}\mathcal{U}_{h_2} = e^{i\alpha \mathrm{Im}(h_1\bar{h}_2)} \mathcal{U}_{h_1+h_2}$$

and are unitary operators.

As an analogue of the parabolic basis for $\mathcal{A}^\nu$ we introduce the Gauss–Weierstrass functions

$$e_x^2(z) = e^{-\alpha z^2/2+xz},$$

and write $\mathfrak{F}_2$ for the transform

$$\mathfrak{F}_2 f(x) = \langle f, e_x^2\rangle.$$

We also write $e_k^1(z) = z^k$ and

$$\mathfrak{F}_1 f(k) = \langle f, e_k^1\rangle.$$

In analogy with (13), and with the same interpretation, we have the Plancherel formulas

$$\begin{aligned}\langle f,g\rangle &= \sum_{k=0}^{\infty}\frac{\alpha^k}{k!}\,\mathfrak{F}_1 f(k)\,\overline{\mathfrak{F}_1 g(k)}\\ &= \frac{1}{\sqrt{2\pi\alpha}}\int_{-\infty}^{\infty}\mathfrak{F}_2 f(x)\,\overline{\mathfrak{F}_2 g(x)}\,e^{-x^2/2\alpha}\,dx.\end{aligned} \tag{47}$$

The matrix elements of $\mathcal{U}_h$ are given by

$$\langle \mathcal{U}_h e_k^1, e_l^1\rangle = (-1)^l h^k \bar{h}^l e^{-\alpha|h|^2/2}\,{}_2F_0\left(\left.\begin{matrix}-k,-l\\ -\end{matrix}\right|-\frac{1}{\alpha|h|^2}\right),$$

which are essentially Charlier polynomials, and by

$$\begin{aligned}\langle e_k^1, e_x^2\rangle &= \frac{x^k}{\alpha^k}\,{}_2F_0\left(\left.\begin{matrix}-\frac{k}{2},\frac12-\frac{k}{2}\\ -\end{matrix}\right|-\frac{2\alpha}{x^2}\right)\\ &= \sum_{0\leq j\leq\left[\frac{k}{2}\right]}\frac{(-1)^j k!\,x^{k-2j}}{(k-2j)!\,j!\,2^j\alpha^{k-j}},\end{aligned}$$

which are essentially Hermite polynomials. It will be convenient to denote the latter quantity by $H_k(x,\alpha)$, so that

$$e_x^2(z) = \sum_{k=0}^{\infty}\frac{\alpha^k}{k!}H_k(x,\alpha)z^k. \tag{48}$$

In the standard notation for Hermite polynomials, cf. [VK], one has

$$H_k(x,\alpha) = \frac{1}{(2\alpha)^{k/2}}H_k\left(\frac{x}{\sqrt{2\alpha}}\right),\qquad \alpha>0.$$

The operators $\mathcal{U}_h$ permute the functions $e_x^2$, and thus the matrix elements $\langle\mathcal{U}_h e_k^1, e_x^2\rangle$ are also given by Hermite polynomials. For the same reason, the matrix elements $\langle\mathcal{U}_h e_x^2, e_y^2\rangle$ are essentially Dirac measures. We will also consider the matrix elements $\langle\delta_c e_x^2, e_y^2\rangle$, where, as above, $\delta_c$ is the dilatation $(\delta_c f)(z) = f(cz)$. Expanding the scalar product as in (47), we get

$$\langle\delta_c e_x^2, e_y^2\rangle = \sum_{k=0}^{\infty}\frac{\alpha^k c^k}{k!}H_k(x,\alpha)H_k(y,\alpha). \tag{49}$$

Mehler's formula, cf. [VK], equation 9.6.8(5), then gives

$$\langle \delta_c e_x^2, e_y^2 \rangle = \frac{1}{\sqrt{1-c^2}} \exp\left(\frac{2cxy - c^2(x^2+y^2)}{2\alpha(1-c^2)}\right), \qquad |c| < 1 \tag{50}$$

(cf. [P1] for another computation of this scalar product).

Later we will need power series expansions of $1/e_x^2$ and $1/\langle \delta_c e_x^2, e_y^2 \rangle$. Since

$$\frac{1}{e_x^2(z)} = e^{\alpha z^2/2 - xz}$$

is obtained from $e_x^2(z)$ by replacing $\alpha$ with $-\alpha$ and $z$ with $-z$, it follows from (48) that

$$\frac{1}{e_x^2(z)} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} H_k(x, -\alpha) z^k. \tag{51}$$

Similarly, if in the right-hand side of (50) we formally replace $\alpha$ by $-\alpha$ and multiply by $1-c^2$, we obtain the reciprocal quantity. Manipulating (49) in the same way gives

$$\begin{aligned} \frac{1}{\langle \delta_c e_x^2, e_y^2 \rangle} &= (1-c^2) \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k c^k}{k!} H_k(x,-\alpha) H_k(y,-\alpha) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k c^k}{k!} G_k(x,y,\alpha), \end{aligned} \tag{52}$$

where

$$\begin{aligned} G_k(x,y,\alpha) &= H_k(x,-\alpha)H_k(y,-\alpha) - \frac{k(k-1)}{\alpha^2} H_{k-2}(x,-\alpha)H_{k-2}(y,-\alpha) \\ &= \sum_{0 \le i,j \le \left[\frac{k}{2}\right]} \frac{k!\,(k-2)!\,(k(k-1)-4ij)x^{k-2i}y^{k-2j}}{(k-2i)!\,(k-2j)!\,i!\,j!\,2^{i+j}\alpha^{2k-i-j}}. \end{aligned} \tag{53}$$

Let us now consider an arbitrary tensor product

$$\mathcal{F}_{\alpha_1} \otimes \cdots \otimes \mathcal{F}_{\alpha_n}$$

of Fock spaces. It decomposes under the group action into infinitely many copies of the space $\mathcal{F}_{|\alpha|}$. To get a finer decomposition one may add to the group auxiliary operators of the form

$$f(z) \mapsto e^{im\theta} f(e^{i\theta} z),$$

where $m$ is an integer; cf. [VK], Section 8.6.1. Temporarily writing $\mathcal{F}_{\alpha,m}$ for the corresponding representation space, we have a decomposition

$$\mathcal{F}_{\alpha_1,m_1} \otimes \cdots \otimes \mathcal{F}_{\alpha_n,m_n} = \bigoplus_{s=0}^{\infty} \binom{n+s-2}{n-2} \mathcal{F}_{|\alpha|,|m|+s}.$$

This means that the intertwining embeddings $\mathcal{F}_{|\alpha|,|m|+s} \to \mathcal{F}_{\alpha_1,m_1} \otimes \cdots \otimes \mathcal{F}_{\alpha_n,m_n}$ are required to increase homogeneity by $s$. If $Q$ is an image of 1 under such an embedding, then $Q$ must satisfy the same condition (3) as in the case of the disc. In the case at hand, there is a closed expression for the corresponding embedding $\mathcal{K}_Q$:

$$(\mathcal{K}_Q g)(z_1, \ldots, z_n) = Q(z_1, \ldots, z_n)\, g\left(\frac{(\alpha, z)}{|\alpha|}\right), \tag{54}$$

where

$$(\alpha, z) = \alpha_1 z_1 + \cdots + \alpha_n z_n.$$

¿From now on we write

$$\langle f, g\rangle_\alpha = \langle f, g\rangle_{\mathcal{F}_{\alpha_1}\otimes\cdots\otimes\mathcal{F}_{\alpha_n}}, \qquad \langle f, g\rangle_{|\alpha|} = \langle f, g\rangle_{\mathcal{F}_{|\alpha|}}.$$

To proceed in analogy with the case of the disc, we should now seek factorizations of the transforms

$$\mathfrak{F}_i^h = \mathfrak{F}_i\,\mathcal{U}_h, \qquad i = 1, 2.$$

However, we get a useful factorization only in the case $i = 1$. Then we may write

$$\mathfrak{F}_1^h = M_1\,\mathfrak{T}_1\,\delta_{-1/\bar{h}}\,\tau_h \qquad (h \neq 0),$$

where

$$M_1 f(k) = e^{-\alpha|h|^2/2}(-1)^k \bar{h}^k f(k)$$

and

$$\mathfrak{T}_1 z^k(m) = (-1)^k \frac{(-m)_k}{\alpha^k}.$$

In analogy with Theorem 4.2, we have

$$(\mathfrak{F}_1^h\,\mathcal{K}_Q g)(m) = \mathfrak{T}_1 Q(m)(\mathfrak{F}_1^h g)(|m| - s). \tag{55}$$

Thus if

$$Q(z) = \sum_{|t|=s} c_t\, z_1^{t_1}\cdots z_n^{t_n}$$

is a highest weight vector, we have a corresponding coupling coefficient

$$\mathfrak{T}_1 Q(m_1, \dots, m_n) = (-1)^s \sum_{|t|=s} c_t\, \frac{(-m_1)_{t_1}\cdots(-m_n)_{t_n}}{\alpha_1^{t_1}\cdots\alpha_n^{t_n}},$$

which is again a polynomial of degree $s$.

For $i = 2$ we define

$$\mathfrak{T}_2 Q(x_1, \dots, x_n) = \langle Q, e_{x_1}^2 \otimes \cdots \otimes e_{x_n}^2\rangle_\alpha,$$

where $Q$ is a highest weight vector. There is no very simple expression for $\mathfrak{T}_2 Q$; we must be content with writing

$$\mathfrak{T}_2 Q(x_1, \dots, x_n) = \sum_{|t|=s} c_t\, H_{t_1}(x_1, \alpha_1)\cdots H_{t_n}(x_n, \alpha_n),$$

which shows that $\mathfrak{T}_2 Q$ is a polynomial of degree $s$. One may also show that

$$\mathfrak{T}_2 Q(x_1 + b\alpha_1, \dots, x_n + b\alpha_n) = \mathfrak{T}_2 Q(x_1, \dots, x_n), \qquad b \in \mathbb{C}. \tag{56}$$

Note that in the case of the disc, the polynomial $T_2 Q$ is homogeneous. Thus in both cases something of the property (3) is preserved. As in Theorem 4.2, we have

$$(\mathfrak{F}_2^h\,\mathcal{K}_Q g)(x) = \mathfrak{T}_2 Q(x)(\mathfrak{F}_2^h g)(|x|) \tag{57}$$

and, for the dilatations,

$$(\mathfrak{F}_2 \delta_c \mathcal{K}_Q g)(x) = c^s \mathfrak{T}_2 Q(x)(\mathfrak{F}_2 \delta_c g)(|x|). \tag{58}$$

In the bilinear case, $n = 2$, we may take $Q(z) = (z_1 - z_2)^s$. Then $\mathfrak{T}_1 Q$ are essentially Krawtchouk polynomials; in the notation of [VK] we have

$$\mathfrak{T}_1 Q(m_1, m_2) = \frac{(-m_1 - m_2)_s}{\alpha_2^s} K_s(m_1; \tfrac{\alpha_1}{\alpha_1 + \alpha_2}; m_1 + m_2),$$

while the polynomials $\mathfrak{T}_2 Q$ may be expressed in terms of Hermite polynomials. Indeed, we may write

$$\begin{aligned} \mathfrak{T}_2 Q(x_1, x_2) &= \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} H_k(x_1, \alpha_1) H_{s-k}(x_2, \alpha_2) \\ &= \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)^s H_s\left(\frac{x_1}{\alpha_1} - \frac{x_2}{\alpha_2}, \frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right), \end{aligned}$$

where the last equality is a special case of the addition formula for Hermite polynomials, cf. [VK], equation 9.6.8(7), which is in turn easily derived from (48). Note that this expression is in agreement with (56). That Krawtchouk polynomials are Clebsch–Gordan coefficients for the Heisenberg algebra is well-known, cf. [VK], while for the Hermite polynomial this fact, as far as we know, first occurs in [VdJ].

We now write down some properties of the coupling coefficients. First we have the Clebsch–Gordan type formulas analogous to Corollary 4.3. In this case we may use (54) to write them more explicitly as

$$\begin{aligned} Q(z) \frac{(\alpha, z)^k}{k!} &= \sum_{|m| = k+s} \frac{\alpha_1^{m_1} \cdots \alpha_n^{m_n}}{m_1! \cdots m_n!} \mathfrak{T}_1 Q(m)\, z_1^{m_1} \cdots z_n^{m_n}, \\ Q(z) \exp\left(-\frac{(x - (\alpha, z))^2}{2|\alpha|}\right) &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \sqrt{\frac{|\alpha|}{\alpha_1 \cdots \alpha_n}} \\ \times \int_{y \in \mathbb{R}^n \,:\, |y| = x} & \mathfrak{T}_2 Q(y) \exp\left(-\left(\frac{(y_1 - \alpha_1 z_1)^2}{2\alpha_1} + \cdots + \frac{(y_n - \alpha_n z_n)^2}{2\alpha_n}\right)\right) dy, \end{aligned}$$

where in the last equality we express the product of the generalized basis elements and the weight of the Plancherel measures as

$$e_y^2(z)\, e^{-y^2/2\alpha} = e^{-(y - \alpha z)^2/2\alpha}.$$

Next we have orthogonality relations as in Corollary 4.4. If $Q$ and $\tilde{Q}$ are two highest weight vectors, then their scalar product $\langle Q, \tilde{Q} \rangle_\alpha$ equals both the sum

$$\frac{k!}{|\alpha|^k} \sum_{|m| = k+s} \frac{\alpha_1^{m_1} \cdots \alpha_n^{m_n}}{m_1! \cdots m_n!} \mathfrak{T}_1 Q(m)\, \overline{\mathfrak{T}_1 \tilde{Q}(m)}$$

for each $k = 0, 1, 2, \ldots$ and, for each $x \in \mathbb{R}$, the integral

$$\frac{1}{(2\pi)^{\frac{n-1}{2}}} \sqrt{\frac{|\alpha|}{\alpha_1 \cdots \alpha_n}} \int_{y \in \mathbb{R}^n \,:\, |y| = x} \mathfrak{T}_2 Q(y)\, \overline{\mathfrak{T}_2 \tilde{Q}(y)} \exp\left(-\left(\frac{y_1^2}{2\alpha_1} + \cdots + \frac{y_n^2}{2\alpha_n}\right)\right) dy.$$

There are also analogues of the convolution formulas of Theorem 5.1. If $Q$ is a highest weight polynomial of degree $s$, we have

$$\begin{aligned}
&\frac{|\alpha|^k}{k!} c^s {}_2F_0\left(\begin{matrix} -k, s-|l| \\ - \end{matrix}\,\middle|\, \frac{c}{|\alpha|}\right) \mathfrak{T}_1 Q(l) \\
&\quad = \sum_{|m|=k+s} \frac{\alpha_1^{m_1}\cdots\alpha_n^{m_n}}{m_1!\cdots m_n!} \mathfrak{T}_1 Q(m) \prod_{r=1}^n {}_2F_0\left(\begin{matrix} -m_r, -l_r \\ - \end{matrix}\,\middle|\, \frac{c}{\alpha_r}\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{|\alpha|^k}{k!} H_k(|x|,|\alpha|)\, \mathfrak{T}_2 Q(x) \\
&\quad = \sum_{|m|=k+s} \frac{\alpha_1^{m_1}\cdots\alpha_n^{m_n}}{m_1!\cdots m_n!} \mathfrak{T}_1 Q(m) \prod_{r=1}^n H_{m_r}(x_r,\alpha_r),
\end{aligned}$$

$$\begin{aligned}
\exp\left(-\frac{x^2}{2|\alpha|}\right) H_{|m|-s}(x,|\alpha|)\, \mathfrak{T}_1 Q(m) &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \sqrt{\frac{|\alpha|}{\alpha_1\cdots\alpha_n}} \\
&\times \int_{y\in\mathbb{R}^n\,:\,|y|=x} \mathfrak{T}_2 Q(y) \prod_{r=1}^n H_{m_r}(y_r,\alpha_r) \exp\left(-\left(\frac{y_1^2}{2\alpha_1}+\cdots+\frac{y_n^2}{2\alpha_n}\right)\right) dy,
\end{aligned}$$

$$\begin{aligned}
c^s \exp\left(\frac{2cx|z|-x^2-c^2|z|^2}{2|\alpha|(1-c^2)}\right) \mathfrak{T}_2 Q(z) &= \frac{1}{(2\pi(1-c^2))^{\frac{n-1}{2}}} \sqrt{\frac{|\alpha|}{\alpha_1\cdots\alpha_n}} \\
&\times \int_{y\in\mathbb{R}^n\,:\,|y|=x} \mathfrak{T}_2 Q(y) \exp\left(\sum_{j=1}^n \frac{2cy_jz_j - y_j^2 - c^2 z_j^2}{2\alpha_j(1-c^2)}\right) dy.
\end{aligned}$$

For $n=2$, the first of these formulas is the addition formula for Charlier polynomials, cf. [VK], Section 8.6.5, while the second one is the generalization of the addition formula for Hermite polynomials given in [VdJ] and [KV1]. As degenerate cases of the first of these formulas we obtain analogues of the equations (26) and (27):

$$\begin{aligned}
\frac{|\alpha|^{k+s}}{(k+s)!} \mathfrak{T}_1 Q(l) &= \sum_{|m|=k+s} \frac{\alpha_1^{m_1}\cdots\alpha_n^{m_n}}{m_1!\cdots m_n!} \mathfrak{T}_1 Q(m+l), \\
\frac{(s-|l|)_k}{k!} \mathfrak{T}_1 Q(l) &= (-1)^s \sum_{|m|=k+s} \frac{(-l_1)_{m_1}\cdots(-l_n)_{m_n}}{m_1!\cdots m_n!} \mathfrak{T}_1 Q(m).
\end{aligned}$$

They are proved similarly using the limit

$$\lim_{c\to\infty} \frac{1}{c^m} {}_2F_0\left(\begin{matrix} -m, -l \\ - \end{matrix}\,\middle|\, \frac{c}{a}\right) = (-1)^m \frac{(-l)_m}{a^m} \qquad (m,\ l = 0,1,2,\dots).$$

We now turn to the coupling kernels for the present situation. They are defined as in the case of the disc and have a similar group theoretic interpretation. We denote them by $\mathfrak{P}_s^{ij}$. We may reduce ourselves to the three cases

$$\mathfrak{P}_s^{11}(k,l), \qquad \mathfrak{P}_s^{12}(k,x), \qquad \mathfrak{P}_s^{22}(x,y).$$

We first give explicit expressions for the coupling kernels similar to those given in Theorem 6.1. It seems that the kernels $\mathfrak{P}_s^{12}$ and $\mathfrak{P}_s^{22}$ do not have simple expressions in terms of Karlsson's functions. Instead we give expansions involving Hermite polynomials. Our explicit expressions are

$$\mathfrak{P}_s^{11}(k,l) = \frac{(-1)^s(-|k|)_s(-|l|)_s}{|\alpha|^s s!} F^{1:2}_{2:0}\left(\begin{array}{cc} -s & : -k,-l \\ -|k|,-|l| & : - \end{array}\middle| \frac{|\alpha|}{\alpha}\right), \tag{59}$$

$$\mathfrak{P}_s^{12}(k,x) =$$

$$= \frac{(-|k|)_s}{|\alpha|^s} \sum_{m\,:\,|m|\le s} \frac{(-1)^{|m|}|\alpha|^{|m|}(-k)_m}{(-|k|)_{|m|}m!} \sum_{|p|=s-|m|} \frac{\alpha^p}{p!}\mathbf{H}_{p+m}(x,\alpha) \tag{60}$$

$$= \frac{(-1)^s(-|k|)_s}{s!} \sum_{m_1,\dots,m_n=0}^{\infty} (-1)^{|m|}\frac{(-s)_{|m|}(-k)_m}{(-|k|)_{|m|}m!} H_{s-|m|}(|x|,-|\alpha|)\mathbf{H}_m(x,\alpha), \tag{61}$$

$$\mathfrak{P}_s^{22}(x,y) =$$

$$= \frac{(-1)^s s!}{|\alpha|^s} \sum_{m\,:\,|m|\le s} \frac{|\alpha|^{|m|}\alpha^m}{(-s)_{|m|}m!} \sum_{|p|=s-|m|} \frac{\alpha^p}{p!}\mathbf{H}_{p+m}(x,\alpha) \sum_{|q|=s-|m|} \frac{\alpha^q}{q!}\mathbf{H}_{q+m}(y,\alpha) \tag{62}$$

$$= \frac{(-1)^s|\alpha|^s}{s!} \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(-s)_{|m|}\alpha^m}{|\alpha|^{|m|}m!} G_{s-|m|}(|x|,|y|,|\alpha|)\mathbf{H}_m(x,\alpha)\mathbf{H}_m(y,\alpha), \tag{63}$$

where in (59) we write for short

$$\frac{|\alpha|}{\alpha} = \left(\frac{\alpha_1+\cdots+\alpha_n}{\alpha_1},\dots,\frac{\alpha_1+\cdots+\alpha_n}{\alpha_n}\right),$$

where

$$\mathbf{H}_m(x,\alpha) = H_{m_1}(x_1,\alpha_1)\cdots H_{m_n}(x_n,\alpha_n),$$

and where $G$ is the polynomial defined in (53).

As for the proof, the expression (59) may be proved in a similar manner as in the case of the disc, using instead of Lemma 8.2 the formula

$$\Pi_s|_{\mathcal{H}_s} = \sum_{k=0}^{\infty} \frac{1}{k!|\alpha|^k} F^k E^k,$$

where $\Pi_s$ is defined as the orthogonal projection of $\mathcal{F}_{\alpha_1}\otimes\cdots\otimes\mathcal{F}_{\alpha_n}$ onto the subspace spanned by all $\mathcal{K}_Q g$, where $g \in \mathcal{F}_{|\alpha|}$ and $Q$ is a highest weight polynomial of degree $s$ and where

$$E = -\sum_{i=1}^n \frac{\partial}{\partial z_i}, \qquad F = \sum_{i=1}^n \alpha_i z_i$$

are Lie algebra operators derived from the group action. If one continues to imitate the proof of Theorem 6.1, one arrives at the expressions (60) and (62). However, in this case there is no "multinomial theorem" to sum the inner series. To prove (61) and (63), which we believe are the natural analogues of the expressions in Theorem 6.1, we use a different approach.

*Proof of* (61). Consider a tensor product $e^2_{x_1} \otimes \cdots \otimes e^2_{x_n}$. We try to decompose it as

$$e^2_{x_1} \otimes \cdots \otimes e^2_{x_n} = \sum_{s=0}^{\infty} \mathcal{K}_{Q_{sx}} g_{sx},$$

where $Q_{sx}$ is a highest weight polynomial of degree $s$. Because of (57) we expect that one must take $g_{sx} = e^2_{|x|}$ for all $s$, and thus, in view of (54), that

$$e^2_{x_1}(z_1) \cdots e^2_{x_n}(z_n) = e^2_{|x|}\left(\frac{(\alpha, z)}{|\alpha|}\right) \sum_{s=0}^{\infty} Q_{sx}.$$

That there is such an expansion may be verified easily by checking that the function

$$\frac{e^2_{x_1}(z_1) \cdots e^2_{x_n}(z_n)}{e^2_{|x|}\left(\frac{(\alpha,z)}{|\alpha|}\right)} = \exp\left(\sum_{i=1}^{n}\left(-\frac{\alpha_i}{2} z_i^2 + x_i z_i\right) + \frac{(\alpha, z)^2}{2|\alpha|} - \frac{|x|(\alpha, z)}{|\alpha|}\right) \tag{64}$$

is in the kernel of $E$ and thus that its homogeneous parts are highest weight polynomials. This may be made more transparent by rewriting the above expression as

$$\frac{e^2_{x_1}(z_1) \cdots e^2_{x_n}(z_n)}{e^2_{|x|}\left(\frac{(\alpha,z)}{|\alpha|}\right)} = \exp\left(-\frac{1}{2|\alpha|}\sum_{i<j} \alpha_i\alpha_j(z_i - z_j)^2 + 2(x_i\alpha_j - \alpha_i x_j)(z_i - z_j)\right).$$

Now, consider the matrix element

$$\langle \Pi_s e^2_x, e^1_k\rangle_\alpha = \langle \Pi_s\, e^2_{x_1} \otimes \cdots \otimes e^2_{x_n}, e^1_{k_1} \otimes \cdots \otimes e^1_{k_n}\rangle_\alpha$$

for $\Pi_s$. As for the disc, it factors as

$$\langle \Pi_s e^2_x, e^1_k\rangle_\alpha = \mathfrak{P}^{12}_s(k, x)\langle e^2_{|x|}, e^1_{|k|-s}\rangle_{|\alpha|}.$$

On the other hand, (55) gives

$$\langle \Pi_s e^2_x, e^1_k\rangle_\alpha = \langle \mathcal{K}_{Q^s_x} e^2_{|x|}, e^1_k\rangle_\alpha = \mathfrak{T}_1 Q_{sx}(k)\langle e^2_{|x|}, e^1_{|k|-s}\rangle_{|\alpha|}.$$

Thus we have found that

$$\mathfrak{P}^{12}_s(k, x) = \mathfrak{T}_1 Q_{sx}(k),$$

where $Q_{sx}$ is the term of degree $s$ in the homogeneous expansion of the function (64).

Now, by (48)

$$e^2_{x_1}(z_1) \cdots e^2_{x_n}(z_n) = \sum_{m_1,\dots,m_n=0}^{\infty} \frac{\alpha^m z^m}{m!} \mathbf{H}_m(x, \alpha),$$

and by (51)

$$\frac{1}{e^2_{|x|}\left(\frac{(\alpha,z)}{|\alpha|}\right)} = \sum_{l=0}^{\infty} \frac{(\alpha, z)^l}{l!} H_l(|x|, -|\alpha|).$$

Thus

$$Q_{sx}(z) = \sum_{m\,:\,|m|\leq s} \frac{\alpha^m z^m}{m!} \mathbf{H}_m(x, \alpha)\, \frac{(\alpha, z)^{s-|m|}}{(s - |m|)!} H_{s-|m|}(|x|, -|\alpha|).$$

Expanding the power $(\alpha, z)^{s-|m|}$ gives

$$Q_{sx}(z) = \sum_{|m|+|p|=s} \frac{\alpha^{m+p} z^{m+p}}{m!\, p!} \mathbf{H}_m(x,\alpha) H_{s-|m|}(|x|, -|\alpha|).$$

By the definition of $\mathfrak{T}_1$, it follows that

$$\mathfrak{T}_1 Q_{sx}(k) = (-1)^s \sum_{|m|+|p|=s} \frac{(-k)_{m+p}}{m!\, p!} \mathbf{H}_m(x,\alpha) H_{s-|m|}(|x|, -|\alpha|).$$

Summing in $p$, we have as in (46) that

$$\sum_{|p|=s-|m|} \frac{(-k)_{m+p}}{p!} = (-1)^{|m|} \frac{(-|k|)_s}{s!} \frac{(-s)_{|m|}(-k)_m}{(-|k|)_{|m|}}.$$

Inserting this in the previous formula gives the desired expression (61).

*Proof of* (63). We consider the matrix element

$$\langle \delta_c e_x^2, e_y^2 \rangle_\alpha = \langle (\delta_c)^{\otimes n} e_{x_1}^2 \otimes \cdots \otimes e_{x_n}^2, e_{y_1}^2 \otimes \cdots \otimes e_{y_n}^2 \rangle_\alpha$$

of $\delta_c$, $|c| < 1$, and expand it as

$$\langle \delta_c e_x^2, e_y^2 \rangle_\alpha = \sum_{s=0}^{\infty} \langle \Pi_s \delta_c e_x^2, e_y^2 \rangle_\alpha.$$

By (58)

$$\langle \Pi_s \delta_c e_x^2, e_y^2 \rangle_\alpha = c^s\, \mathfrak{P}_s^{22}(x,y) \langle \delta_c e_{|x|}^2, e_{|y|}^2 \rangle_{|\alpha|}.$$

It follows that $\mathfrak{P}_s^{22}(x,y)$ is the coefficient of $c^s$ in

$$\frac{\langle \delta_c e_x^2, e_y^2 \rangle_\alpha}{\langle \delta_c e_{|x|}^2, e_{|y|}^2 \rangle_{|\alpha|}}.$$

Using the expansions (48) and (52) one obtains the expression (63).

For $n = 2$, the coupling kernels factor as products of two Clebsch–Gordan coefficients. Choosing $Q(z) = (z_1 - z_2)^s$, we may write

$$\mathfrak{P}_s^{ij}((x_1,x_2),(y_1,y_2)) = \frac{1}{\|Q\|_\alpha^2} \mathfrak{T}_i Q(x_1,x_2)\, \mathfrak{T}_j Q(y_1,y_2), \qquad i,\, j = 1,2,$$

where the norm is easily computed:

$$\|Q\|_\alpha^2 = s! \left( \frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right)^s.$$

Comparing with the expressions for $\mathfrak{P}_s^{ij}$ given above leads to Watson-type formulas, which we have not found in the literature. The case $i = j = 1$ gives the identity

$$\begin{aligned}
&\frac{\alpha_1^s}{\alpha_2^s} K_s(k_1; \tfrac{\alpha_1}{\alpha_1+\alpha_2}; k_1+k_2)\, K_s(l_1; \tfrac{\alpha_1}{\alpha_1+\alpha_2}; l_1+l_2) \\
&\quad = (-1)^s F_{2:0}^{1:2} \left( \begin{matrix} -s \\ -k_1-k_2, -l_1-l_2 \end{matrix} \begin{matrix} : \\ : \end{matrix} \begin{matrix} -(k_1,k_2), -(l_1,l_2) \\ - \end{matrix} \middle| \frac{\alpha_1+\alpha_2}{\alpha_1}, \frac{\alpha_1+\alpha_2}{\alpha_2} \right).
\end{aligned}$$

Changing the names of the variables, this may be rewritten as

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix}-n,a\\ b\end{matrix}\,\middle|\,x\right){}_2F_1\left(\begin{matrix}-n,c\\ d\end{matrix}\,\middle|\,x\right)\\
&\quad=(1-x)^n\,F^{1:2}_{2:0}\left(\begin{matrix}-n & : & (a,b-a),(c,d-c)\\ b,d & : & -\end{matrix}\,\middle|\,x,\frac{x}{x-1}\right),
\end{aligned}$$

which is true for arbitrary $a,b,c,d$ since the series terminate. The case $i=1$, $j=2$ gives

$$\begin{aligned}
&\frac{1}{\alpha_2^s}\,K_s(k_1;\tfrac{\alpha_1}{\alpha_1+\alpha_2};k_1+k_2)H_s\left(\frac{x_1}{\alpha_1}-\frac{x_2}{\alpha_2},\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)\\
&\quad=\;(-1)^s\sum_{m_1,m_2=0}^{\infty}(-1)^{m_1+m_2}\frac{(-s)_{m_1+m_2}(-k_1)_{m_1}(-k_2)_{m_2}}{(-k_1-k_2)_{m_1+m_2}m_1!\,m_2!}\\
&\qquad\times H_{s-m_1-m_2}(x_1+x_2,-\alpha_1-\alpha_2)H_{m_1}(x_1,\alpha_1)H_{m_2}(x_2,\alpha_2).
\end{aligned}$$

Finally, the case $i=j=2$ gives

$$\begin{aligned}
&H_s\left(\frac{x_1}{\alpha_1}-\frac{x_2}{\alpha_2},\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)H_s\left(\frac{y_1}{\alpha_1}-\frac{y_2}{\alpha_2},\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)\\
&=\;(-\alpha_1\alpha_2)^s\sum_{m_1,m_2=0}^{\infty}\frac{(-s)_{m_1+m_2}\alpha_1^{m_1}\alpha_2^{m_2}}{(\alpha_1+\alpha_2)^{m_1+m_2}m_1!\,m_2!}\,G_{s-m_1-m_2}(x_1+x_2,y_1+y_2,\alpha_1+\alpha_2)\\
&\quad\times H_{m_1}(x_1,\alpha_1)H_{m_2}(x_2,\alpha_2)H_{m_1}(y_1,\alpha_1)H_{m_2}(y_2,\alpha_2).
\end{aligned}$$

We now discuss the biorthogonal polynomial systems analogous to the Appell and dual Appell polynomials of Sections 2.2 and 2.3. Thus we introduce the basis

$$Q_t(z)=(z_1-z_n)^{t_1}\cdots(z_{n-1}-z_n)^{t_{n-1}},\qquad t_1+\cdots+t_{n-1}=s$$

of the space of highest weight vectors of degree $s$ in $\mathcal{F}_{\alpha_1}\otimes\cdots\otimes\mathcal{F}_{\alpha_n}$ and let $(Q_t')$ denote the dual basis. As in the case of the disc, it is easy to find expressions for the polynomials $\mathfrak{T}_iQ_t$ using the definition of $\mathfrak{T}_i$ directly, while the polynomials $\mathfrak{T}_iQ_t'$ arise as special cases of the coupling kernels; in fact

$$\mathfrak{T}_iQ_t'(x)=\frac{\alpha_1^{t_1}\cdots\alpha_{n-1}^{t_{n-1}}}{t_1!\cdots t_{n-1}!}\,\mathfrak{P}_s^{1i}((t_1,\dots,t_{n-1},0),(x_1,\dots,x_n)).$$

Writing as above $\hat{x}=(x_1,\dots,x_{n-1})$, where $x=(x_1,\dots,x_n)$, we have

$$\begin{aligned}
\mathfrak{T}_1Q_t(m)&=\frac{(-m_n)_s}{\alpha_n^s}\,F^{0:2}_{1:0}\left(\begin{matrix}- & : & -t,-\hat{m}\\ 1+m_n-s & : & -\end{matrix}\,\middle|\,-\frac{\alpha_n}{\alpha_1},\dots,-\frac{\alpha_n}{\alpha_{n-1}}\right),\\
\mathfrak{T}_1Q_t'(m)&=\frac{\hat{\alpha}^t(-|m|)_s}{t!\,|\alpha|^s}\,F^{0:2}_{1:0}\left(\begin{matrix}- & : & -t,-\hat{m}\\ -|m| & : & -\end{matrix}\,\middle|\,\frac{|\alpha|}{\alpha_1},\dots,\frac{|\alpha|}{\alpha_{n-1}}\right),\\
\mathfrak{T}_2Q_t(x)&=(-1)^s\sum_{k_1,\dots,k_{n-1}=0}^{\infty}\frac{(-t)_k}{k!}\prod_{j=1}^{n-1}H_{k_j}(x_j,\alpha_j)H_{s-|k|}(x_n,\alpha_n),\\
\mathfrak{T}_2Q_t'(x)&=\frac{\hat{\alpha}^t}{t!}\sum_{k_1,\dots,k_{n-1}=0}^{\infty}(-1)^{|k|}\frac{(-t)_k}{k!}\prod_{j=1}^{n-1}H_{k_j}(x_j,\alpha_j)H_{s-|k|}(|x|,-|\alpha|)
\end{aligned}$$

(Karlsson's $F^{0:2}_{1:0}$-function equals Lauricella's $F_B$-function). The first of these bi-orthogonal systems is Tratnik's multivariable Krawtchouk polynomials [T3], while we have not found the second one in the literature.

Continuing this exposition of the Fock space analogues of our results, we should now proceed with Section 7. Special choices of $Q$ in the orthogonality and convolution formulas above lead to analogues of Propositions 7.1, 7.2, and 7.3. The details of this are left to the reader. There are also analogues of the linearization formulas of Proposition 7.4. For $\mathfrak{P}^{11}_s$ we have the linearization formula

$$\prod_{r=1}^{n} {}_2F_0\left(\begin{matrix}-k_r,-l_r\\-\end{matrix}\,\middle|\,\frac{c}{\alpha_r}\right)=\sum_{s=0}^{\min(|k|,|l|)}\mathfrak{P}^{11}_s(k,l)\,c^s\,{}_2F_0\left(\begin{matrix}s-|k|,s-|l|\\-\end{matrix}\,\middle|\,\frac{c}{|\alpha|}\right).$$

Degenerate cases of this formula are

$$\begin{aligned}\frac{(-k_1)_{l_1}\cdots(-k_n)_{l_n}}{\alpha_1^{l_1}\cdots\alpha_n^{l_n}} &= (-1)^s\sum_{s=0}^{|l|}\frac{(s-|k|)_{|l|-s}}{|\alpha|^{|l|-s}}\,\mathfrak{P}^{11}_s(k,l),\\ \frac{k_1!\cdots k_n!}{\alpha_1^{k_1}\cdots\alpha_n^{k_n}}\,\delta_{k,l} &= \sum_{s=0}^{|k|}\frac{(|k|-s)!}{|\alpha|^{|k|-s}}\,\mathfrak{P}^{11}_s(k,l),\qquad |k|=|l|.\end{aligned}$$

For $\mathfrak{P}^{12}_s$ and $\mathfrak{P}^{22}_s$ the linearization formulas seem to be of a more trivial nature. For $\mathfrak{P}^{12}_s$ we obtain

$$\prod_{r=1}^{n}H_{k_r}(x_r,\alpha_r)=\sum_{s=0}^{|k|}\mathfrak{P}^{12}_s(k,x)H_{|k|-s}(|x|,|\alpha|).$$

This is, however, very easy to deduce from (61). For $\mathfrak{P}^{22}_s$, the linearization formula is

$$\prod_{r=1}^{n}\langle\delta_c e_{x_r},e_{y_r}\rangle_{\alpha_r}=\sum_{s=0}^{\infty}c^s\,\mathfrak{P}^{22}_s(x,y)\langle\delta_c e_{|x|},e_{|y|}\rangle_{|\alpha|},$$

which we have already used in the proof of (63). As for the disc, when $n=2$ we may factor $\mathfrak{P}^{ij}_s$ as a product of two Clebsch–Gordan coefficients and obtain formulas of Burchnall–Chaundy-type. In the case of $\mathfrak{P}^{11}_s$ we obtain the multiplication formula for Charlier polynomials in this way; cf. [VK], Section 8.6.5.

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