ON THE TRIPLE SUM FORMULA FOR WIGNER $9j$-SYMBOLS

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Abstract. We give a new proof of the triple sum formula for Wigner $9j$-symbols due to Ališauskas and Jucys. The proof uses explicit expressions for the coupling kernels recently introduced by the author. Parts of our results generalize to general recoupling coefficients.

1. Introduction

The Wigner $9j$-symbols arise in quantum mechanics as recoupling coefficients of four angular momenta. They were first introduced by Eugene Wigner in 1940. For the general theory of Wigner symbols and related objects we refer to the two-volume treatise [BL1, BL2].

There are many known expressions for $9j$-symbols as multiple hypergeometric sums. Generally speaking, the larger the number of summation variables, the more symmetries do these sums exhibit. The most compact known expression is the triple sum formula of Ališauskas and Jucys [AJ], which does not exhibit any of the 72 symmetries of the $9j$-symbol. Because of the low number of summation variables, this formula is useful in numerical computations [SRC], and it has also been used to study the zeroes of $9j$-symbols [SPV, SR1]. On the other hand, since the 72 symmetries are not exhibited in the triple sum formula, they give rise to nontrivial symmetries of this type of sums. This has been used to find several new hypergeometric summation and transformation formulas, cf. [PV, VPS1, VPS2].

The original proof in [AJ] uses a more complicated interpretation of $9j$-symbols than the standard one, involving an embedding $\mathrm{SU}(2) \times \mathrm{SU}(2) \subseteq \mathrm{SO}(5)$. An elementary proof was given in the book [JB]. There the starting point is a formula expressing the $9j$-symbol as a double sum, each term being the product of three $3j$-symbols (Clebsch–Gordan coefficients). The triple sum formula is then derived using only elementary series manipulations. The proof is however quite tricky, and it seems very hard to find without knowledge of the end result.

In a previous paper [R2], we introduced some polynomials in $2n$ variables which were called coupling kernels. They are the reproducing kernels of certain spaces of coupling coefficients for n angular momenta, and may be expressed as n-variable hypergeometric sums. Using this explicit formula (for $n = 4$) we will derive an expression for Wigner $9j$-symbols as a double sum, each term being the product of only two $3j$-symbols. After inserting appropriate single sum expressions for the $3j$-symbols it is possible to perform the summation in one variable and arrive at a

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triple sum, which is equivalent to the one given in [AJ]. Thus there are no series manipulations involved and one can easily derive the formula without knowing it in advance.

We will follow the approach to coupling and recoupling coefficients of [R1, R2]. In particular, we will work with the Lie algebra $\mathfrak{su}(1,1)$ rather than with $\mathfrak{su}(2)$, which is more common in the literature. It is well-known that the Wigner symbols of $\mathfrak{su}(2)$ may be obtained from those of $\mathfrak{su}(1,1)$ by continuation. One may also generalize the relevant results of [R1, R2] to $\mathfrak{su}(2)$, and then give a direct proof, analogous to the one below, of the triple sum formula in this case.

In Section 2 we survey parts of our previous papers [R1, R2]. Most of this is reformulation of well-known facts in slightly different terms. The only new result that we need is the explicit expression for coupling kernels from [R2] (equation (16) below). In Section 3 we prove an expansion formula, given in Theorem 3.1, of which our double sum mentioned above is a special case. In Section 4 we derive the Endlišauskas–Jucys triple sum formula. Finally, in Section 5 we generalize Theorem 3.1 to arbitrary recoupling coefficients. We hope that this result, Theorem 5.2, may be useful also in the study of Wigner $3nj$-symbols with $n > 3$.

2. Preliminaries

We will work with the holomorphic realization of highest weight representations of $\mathfrak{su}(1,1)$. Thus we let $\mathcal{A}^\nu$ ($\nu > 0$) be the Hilbert space of analytic functions in the complex unit disc with reproducing kernel $(1 - z\bar{w})^{-\nu}$. The norm of a function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

is then given by

$$\|f\|^2 = \sum_{k=0}^{\infty} \frac{k!}{(\nu)_k} |a_k|^2,$$

where

$$(\nu)_k = \nu(\nu+1)\cdots(\nu+k-1), \quad (\nu)_0 = 1$$

is the Pochhammer symbol. This norm is invariant under the transformations

$$f \mapsto f(\phi(z))\phi'(z)^{\frac{\nu}{2}},$$

where ϕ is an analytic automorphism of the disc. The derived representation is a unitary representation of $\mathfrak{su}(1,1)$ on $\mathcal{A}^\nu$ with the highest weight vector 1 of weight $-\nu$. Passing to the complexified Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ yields the densely defined operators

$$E = -\frac{d}{dz}, \quad F = z^2\frac{d}{dz} + \nu z, \quad H = -\left(2z\frac{d}{dz} + \nu\right),$$

which satisfy the structure equations

$$[EF] = H, \quad [HE] = 2E, \quad [HF] = -2F,$$

and the $\mathfrak{su}(1,1)$-reality conditions

$$E^* = -F, \quad F^* = -E, \quad H^* = H.$$

Consider a Hilbert tensor product $\mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n}$ of such spaces. We will use $\| \ \|$ and $\langle \ , \ \rangle$ to denote its norm and scalar product. It decomposes under the Lie algebra action as

$$\mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n} \simeq \bigoplus_{s=0}^{\infty} \binom{n+s-2}{n-2} \mathcal{A}^{|\nu|+2s} \tag{1}$$

(where $|\nu| = \sum \nu_i$), that is, the space $\mathcal{A}^{|\nu|+2s}$ occurs with multiplicity $\binom{n+s-2}{n-2}$. A highest weight vector Q in $\mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n}$ of weight $-(|\nu|+2s)$ is the image of the function 1 under an intertwining embedding

$$\mathcal{A}^{|\nu|+2s} \to \mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n}.$$

In our realization this means that Q is a homogeneous polynomial of degree s, which can be expressed in terms of the differences $z_i - z_j$ of the coordinates. Equivalently, Q satisfies the identity

$$Q(az_1 + b, \dots, az_n + b) = a^s Q(z_1, \dots, z_n), \qquad a, b \in \mathbb{C}. \tag{2}$$

In agreement with (1), the space of such polynomials has dimension $\binom{n+s-2}{n-2}$.

Let $\mathcal{K}_Q$ be the embedding defined by $\mathcal{K}_Q(1) = Q$. In [R1] it was proved that if

$$Q(z) = \sum_{|t|=s} c_t \, z_1^{t_1} \cdots z_n^{t_n}, \tag{3}$$

then the adjoint map $\mathcal{K}_Q^*$ is the differential operator given by

$$\mathcal{K}_Q^*(f_1 \otimes \cdots \otimes f_n)(z) = \sum_{|t|=s} \bar{c}_t \, \frac{f_1^{(t_1)}(z) \cdots f_n^{(t_n)}(z)}{(\nu_1)_{t_1} \cdots (\nu_n)_{t_n}}. \tag{4}$$

In particular, writing $e_m(z) = z^m$,

$$\begin{aligned} \mathcal{K}_Q^*(e_{m_1} \otimes \cdots \otimes e_{m_n}) &= \sum_{|t|=s} \bar{c}_t \, \frac{(-m_1)_{t_1} \cdots (-m_n)_{t_n}}{(\nu_1)_{t_1} \cdots (\nu_n)_{t_n}} (-1)^s e_{|m|-s} \\ &= \overline{TQ(m_1, \dots, m_n)} e_{|m|-s}, \end{aligned}$$

where TQ is the polynomial of degree s given by

$$TQ(m) = (-1)^s \sum_{|t|=s} c_t \, \frac{(-m_1)_{t_1} \cdots (-m_n)_{t_n}}{(\nu_1)_{t_1} \cdots (\nu_n)_{t_n}}.$$

This exhibits TQ as a coupling coefficient for our representation. If Q and Q' are two highest weight vectors, the corresponding coupling coefficients satisfy

$$\sum_{|m|=k+s} \frac{(\nu_1)_{m_1} \cdots (\nu_n)_{m_n}}{m_1! \cdots m_n!} \, TQ(m) \, \overline{TQ'(m)} = \frac{(|\nu|+2s)_k}{k!} \langle Q, Q' \rangle, \tag{5}$$

for $k = 0, 1, 2, \dots$. Here s is the degree of Q or Q'; if their degrees are different both sides vanish.

For $n = 2$, the space of highest weight vectors of degree s is one-dimensional, and spanned by $(z_1 - z_2)^s$. The norm of this vector is given by

$$\|(z_1 - z_2)^s\|^2 = \frac{s!(\nu_1 + \nu_2 + s - 1)_s}{(\nu_1)_s (\nu_2)_s}. \tag{6}$$

It will be convenient to write

$$
\begin{aligned}
P_s^{(\nu_1,\nu_2)}(m_1,m_2) &= (\nu_1)_s(\nu_2)_s(T(z_1-z_2)^s)(m_1,m_2)\\
&= \sum_{k=\max(0,\,s-m_2)}^{\min(s,\,m_1)} \binom{s}{k}(-1)^k\frac{(\nu_1)_s(\nu_2)_s}{(\nu_1)_k(\nu_2)_{s-k}}(-m_1)_k(-m_2)_{s-k}. \qquad (7)
\end{aligned}
$$

These polynomials are essentially Clebsch–Gordan coefficients, or Wigner $3j$-symbols, of our representation. As is well-known, they can be expressed in terms of Hahn polynomials; in standard notation [KS]

$$P_s^{(\nu_1,\nu_2)}(m_1,m_2) = (\nu_1)_s(-m_1-m_2)_s\, Q_s(m_1;\nu_1-1,\nu_2-1,m_1+m_2).$$

We will use some well-known properties of Clebsch–Gordan coefficients. First, it is clear from (7) that

$$P_s^{(\nu_1,\nu_2)}(m_1,m_2) = \begin{cases} (-1)^{m_2}s!\,\frac{(\nu_1)_s(\nu_2)_s}{(\nu_1)_{m_1}(\nu_2)_{m_2}}, & m_1+m_2=s,\\ 0, & m_1+m_2<s. \end{cases} \qquad (8)$$

¿From (5) and (6) one obtains the orthogonality relations

$$
\begin{aligned}
&\sum_{m_1+m_2=s+k}\frac{(\nu_1)_{m_1}(\nu_2)_{m_2}}{m_1!\,m_2!}P_s^{(\nu_1,\nu_2)}(m_1,m_2)P_t^{(\nu_1,\nu_2)}(m_1,m_2) \qquad (9)\\
&\qquad\qquad = \frac{(\nu_1+\nu_2+2s)_k}{k!}C_s^{\nu_1\nu_2}\,\delta_{st},
\end{aligned}
$$

where

$$C_s^{\nu_1\nu_2} = s!(\nu_1)_s(\nu_2)_s(\nu_1+\nu_2+s-1)_s. \qquad (10)$$

The expression (7) may be rewritten in terms of terminating ${}_3F_2(1)$-series

$${}_3F_2\left(\begin{matrix}-n,a,b\\ c,d\end{matrix}\,\middle|\,1\right) = \sum_{k=0}^n\frac{(-n)_k(a)_k(b)_k}{(c)_k(d)_kk!}.$$

Because of the known symmetries of such series, cf. [GR], p. 59, there is a large number of similar expressions, such as

$$
\begin{aligned}
&P_s^{(\nu_1,\nu_2)}(m_1,m_2)\\
&= \sum_{k=0}^s\binom{s}{k}(-1)^{s-k}\frac{(\nu_1)_s(\nu_2)_s}{(\nu_1)_k(\nu_2)_{s-k}}(\nu_1+m_1)_k(\nu_2+m_2)_{s-k} \qquad (11)\\
&= (-1)^s\frac{m_2!}{(m_1+m_2-s)!}(\nu_1)_s(\nu_2+m_2)_{m_1}\,{}_3F_2\left(\begin{matrix}-m_1,\nu_1+s,1-\nu_2-s\\ \nu_1,1-\nu_2-m_1-m_2\end{matrix}\,\middle|\,1\right) \qquad (12)\\
&= (\nu_1)_s(-m_1-m_2)_s\,{}_3F_2\left(\begin{matrix}-m_1,-s,\nu_1+\nu_2+s-1\\ \nu_1,-m_1-m_2\end{matrix}\,\middle|\,1\right). \qquad (13)
\end{aligned}
$$

From the transformation formulas used to derive (12) it follows that

$${}_3F_2\left(\begin{matrix}-m_1,\nu_1+s,1-\nu_2-s\\ \nu_1,1-\nu_2-m_1-m_2\end{matrix}\,\middle|\,1\right) = 0 \qquad (14)$$

if m_1, m_2, and s are integers with

$$m_1, s \geq 0, \qquad m_1 + m_2 \geq s, \qquad m_2 < 0.$$

This is a special case of Karlsson's summation formula [K].

We will now discuss the coupling kernels introduced in [R2]. Let us consider the Hilbert subspace of highest weight vectors of degree s in $\mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n}$. It has a reproducing kernel

$$K_s(z, \bar{w}) = \sum_{|t|=|u|=s} c_{t,u}\, z_1^{t_1} \cdots z_n^{t_n} \bar{w}_1^{u_1} \cdots \bar{w}_n^{u_n}.$$

The coupling kernels are obtained by applying the transform T in each of the two variables z and w; thus we define

$$\mathbf{P}_s(i, j) = \sum_{|t|=|u|=s} c_{t,u} \frac{(-i)_t (-j)_u}{(\nu)_t (\nu)_u},$$

where we use multi-index notation such as

$$(-i)_t = (-i_1)_{t_1} \cdots (-i_n)_{t_n}.$$

This is the reproducing kernel for the space of coupling coefficients of degree s, in the sense that

$$\frac{(|\nu| + 2s)_k}{k!} TQ(l) = \sum_{|m|=k+s} \frac{(\nu_1)_{m_1} \cdots (\nu_n)_{m_n}}{m_1! \cdots m_n!} TQ(m) \mathbf{P}_s(l, m)$$

for each $k = 0, 1, 2, \dots$ and each highest weight vector Q of degree s. If (Q_k) and $(\tilde{Q}_l)$ are any two orthogonal bases of this space, then

$$\mathbf{P}_s(i, j) = \sum_{k,l} \frac{\langle Q_k, \tilde{Q}_l \rangle}{\|Q_k\|^2 \|\tilde{Q}_l\|^2} \overline{TQ_k(i)}\, T\tilde{Q}_l(j). \tag{15}$$

In [R2] an explicit expression for the coupling kernels was found, namely,

$$\mathbf{P}_s(i, j) = \frac{(-1)^s (-|i|)_s (-|j|)_s}{(|\nu| + s - 1)_s s!} \sum_{m_1, \dots, m_n = 0}^{\infty} \frac{(|\nu| + s - 1)_{|m|} (-s)_{|m|} (-i)_m (-j)_m}{(-|i|)_{|m|} (-|j|)_{|m|} m! (\nu)_m} \tag{16}$$

(with the convention $0/0 = 0$, so that the sum is actually finite). Our proof of the triple sum formula will use this expression, though in fact we only need the case $|i| = |j| = s$, which is slightly easier to prove.

Let us now see how to introduce recoupling coefficients, and in particular Wigner 9j-symbols, in the present approach. First we define the binary coupling of two highest weight vectors. Suppose that we decompose the tensor product as

$$\mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n} \simeq (\mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_k}) \otimes (\mathcal{A}^{\nu_{k+1}} \otimes \cdots \otimes \mathcal{A}^{\nu_n}),$$

where $1 \leq k \leq n-1$. Let Q_1 be a highest weight vector in $\mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_k}$ of weight $-\mu_1$ and Q_2 one in $\mathcal{A}^{\nu_{k+1}} \otimes \cdots \otimes \mathcal{A}^{\nu_n}$ of weight $-\mu_2$. Thus

$$\mu_1 = \nu_1 + \cdots + \nu_k + 2t_1, \qquad \mu_2 = \nu_{k+1} + \cdots + \nu_n + 2t_2,$$

where t_1 and t_2 are the degrees of Q_1 and Q_2. For s a non-negative integer we define

$$(Q_1, Q_2)_s = (\mu_1)_s (\mu_2)_s (\mathcal{K}_{Q_1} \otimes \mathcal{K}_{Q_2})(z_1 - z_2)^s,$$

which is a highest weight vector in $\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n}$ of weight $-(\mu_1+\mu_2+2s)$ and degree t_1+t_2+s. Equivalently, $(Q_1,Q_2)_s$ is determined by

$$\mathcal{K}^*_{(Q_1,Q_2)_s}(f_1\otimes\cdots\otimes f_n)=(\mu_1)_s(\mu_2)_s\mathcal{K}^*_{(z_1-z_2)^s}(\mathcal{K}^*_{Q_1}(f_1\otimes\cdots\otimes f_k)\otimes\mathcal{K}^*_{Q_2}(f_{k+1}\otimes\cdots\otimes f_n)).$$

Choosing $f_i(z)=z^{m_i}$ gives the important formula

$$T(Q_1,Q_2)_s(m)=TQ_1(m')\,TQ_2(m'')\,P_s^{(\mu_1,\mu_2)}(|m'|-t_1,|m''|-t_2), \tag{17}$$

where

$$m'=(m_1,\dots,m_k),\quad m''=(m_{k+1},\dots,m_n).$$

Moreover,

$$\|(Q_1,Q_2)_s\|^2=C_s^{\mu_1\mu_2}\|Q_1\|^2\|Q_2\|^2,$$

with the constant as in (10).

We also need a more general type of binary coupling, corresponding to a decomposition

$$\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n}\simeq(\mathcal{A}^{\nu_{\sigma_1}}\otimes\cdots\otimes\mathcal{A}^{\nu_{\sigma_k}})\otimes(\mathcal{A}^{\nu_{\sigma_{k+1}}}\otimes\cdots\otimes\mathcal{A}^{\nu_{\sigma_n}}),$$

where σ is a permutation of $(1,2,\dots,n)$. If Q_1 and Q_2 are highest weight vectors in $\mathcal{A}^{\nu_{\sigma_1}}\otimes\cdots\otimes\mathcal{A}^{\nu_{\sigma_k}}$ and $\mathcal{A}^{\nu_{\sigma_{k+1}}}\otimes\cdots\otimes\mathcal{A}^{\nu_{\sigma_n}}$, respectively, we define

$$(Q_1,Q_2)_s^\sigma(z_1,\dots,z_n)=(Q_1,Q_2)_s(z_{\sigma_1},\dots,z_{\sigma_n}).$$

Then $(Q_1,Q_2)_s^\sigma$ is a highest weight vector in $\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n}$.

Now let $n=4$ and $\sigma=(1,3,2,4)$. We consider the two orthogonal systems of highest weight vectors in $\mathcal{A}^{\nu_1}\otimes\mathcal{A}^{\nu_2}\otimes\mathcal{A}^{\nu_3}\otimes\mathcal{A}^{\nu_4}$ defined by

$$\begin{aligned} Q_{pqr}&=((1,1)_p,(1,1)_q)_r,\\ \tilde Q_{tuv}&=((1,1)_t,(1,1)_u)_v^\sigma. \end{aligned} \tag{18}$$

Graphically, these iterated couplings may be represented as in Figure 1.

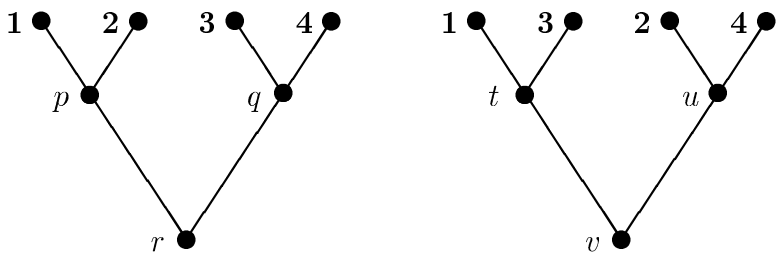

FIGURE 1

Let

$$W_{pqr}^{tuv}=\langle Q_{pqr},\tilde Q_{tuv}\rangle,$$

where the parameters ν_i are suppressed from the notation. These are essentially the Wigner $9j$-coefficients of our representation. Since W_{pqr}^{tuv} vanishes if $p+q+r\neq t+u+v$, we may assume that

$$p+q+r=t+u+v=s.$$

In more standard notation for $9j$-symbols, we have

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} = \frac{1}{\|Q_{pqr}\|\|\tilde{Q}_{tuv}\|}\, W^{tuv}_{pqr},$$

where the "angular momenta" are given by

$$\begin{aligned} j_1 &= -\frac{1}{2}\nu_1, \quad j_2 = -\frac{1}{2}\nu_2, \quad j_3 = -\frac{1}{2}\nu_3, \quad j_4 = -\frac{1}{2}\nu_4, \\ j_{12} &= -\frac{1}{2}(\nu_1+\nu_2+2p), \quad j_{34} = -\frac{1}{2}(\nu_3+\nu_4+2q), \\ j_{13} &= -\frac{1}{2}(\nu_1+\nu_3+2t), \quad j_{24} = -\frac{1}{2}(\nu_2+\nu_4+2u), \\ j &= -\frac{1}{2}(\nu_1+\nu_2+\nu_3+\nu_4+2s), \end{aligned}$$

and the normalizing factor by

$$\|Q_{pqr}\|\|\tilde{Q}_{tuv}\| = \big(C_p^{\nu_1\nu_2} C_q^{\nu_3\nu_4} C_r^{\nu_1+\nu_2+2p,\,\nu_3+\nu_4+2q} C_t^{\nu_1\nu_3} C_u^{\nu_2\nu_4} C_v^{\nu_1+\nu_3+2t,\,\nu_2+\nu_4+2u}\big)^{\frac{1}{2}}.$$

We conclude this section with a remark on transvectants and Wigner symbols. Let us consider the case $n=2$ of (4). It will be convenient to write

$$\begin{aligned} \mathcal{J}_s^{\nu_1\nu_2}(f_1,f_2)(z) &= (\nu_1)_s(\nu_2)_s\,\mathcal{K}^*_{(z_1-z_2)^s}(f_1\otimes f_2)(z) \\ &= \sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k}\frac{(\nu_1)_s(\nu_2)_s}{(\nu_1)_k(\nu_2)_{s-k}} f_1^{(k)}(z) f_2^{(s-k)}(z), \end{aligned}$$

so that, again with $e_m(z) = z^m$,

$$\mathcal{J}_s^{\nu_1\nu_2}(e_{m_1}, e_{m_2}) = P_s^{(\nu_1,\nu_2)}(m_1,m_2)\, e_{m_1+m_2-s}.$$

This operator occurs in classical invariant theory and is known as the transvectant. It has also been called Cohen's operator and the Rankin–Cohen bracket [Z]. It is clear from the above that the Wigner $9j$-symbols occur in the expansion

$$\begin{aligned} &\mathcal{J}_r^{\nu_1+\nu_2+2p,\,\nu_3+\nu_4+2q}(\mathcal{J}_p^{\nu_1\nu_2}(f_1,f_2), \mathcal{J}_q^{\nu_3\nu_4}(f_3,f_4)) \\ &\qquad = \sum_{t+u+v=p+q+r} \frac{W^{tuv}_{pqr}}{\|\tilde{Q}_{tuv}\|^2}\, \mathcal{J}_v^{\nu_1+\nu_3+2t,\,\nu_2+\nu_4+2u}(\mathcal{J}_t^{\nu_1\nu_3}(f_1,f_3), \mathcal{J}_u^{\nu_2\nu_4}(f_2,f_4)). \end{aligned}$$

It is interesting that in a recent paper [CMZ] the coefficients occurring in

$$\mathcal{J}_r^{\nu_1+\nu_2+2q,\,\nu_3}(\mathcal{J}_q^{\nu_1\nu_2}(f_1,f_2),f_3) = \sum_{t+u=q+r} R^{tu}_{qr}\, \mathcal{J}_u^{\nu_1,\,\nu_2+\nu_3+2t}(f_1, \mathcal{J}_t^{\nu_2\nu_3}(f_2,f_3))$$

arise in a seemingly different context. These are, up to a normalizing factor, the Racah coefficients, or Wigner $6j$-symbols, of our representation.

3. An expansion formula

In this section we will derive a formula expressing the product of one $9j$- and two $3j$-symbols as a quadruple sum, each term being a product of four $3j$-symbols. As a special case we get an expression for the $9j$-symbols as a double sum of products of two $3j$-symbols, of which the triple sum formula is a rather easy consequence.

Let us choose Q and $\tilde{Q}$ in (15) as the bases given by (18). This gives

$$\mathbf{P}_s(i,j) = \sum_{\substack{p+q+r=s\\ t+u+v=s}} \frac{W_{pqr}^{tuv}}{\|Q_{pqr}\|^2\|\tilde{Q}_{tuv}\|^2}\,\overline{TQ_{pqr}(i)}\,T\tilde{Q}_{tuv}(j). \tag{19}$$

Moreover, by (17),

$$\begin{aligned}
TQ_{pqr}(i) &= P_p^{(\nu_1,\nu_2)}(i_1,i_2)\,P_q^{(\nu_3,\nu_4)}(i_3,i_4)\,P_r^{(\nu_1+\nu_2+2p,\nu_3+\nu_4+2q)}(i_1+i_2-p,i_3+i_4-q),\\
T\tilde{Q}_{tuv}(j) &= P_t^{(\nu_1,\nu_3)}(j_1,j_3)\,P_u^{(\nu_2,\nu_4)}(j_2,j_4)\,P_v^{(\nu_1+\nu_3+2t,\nu_2+\nu_4+2u)}(j_1+j_3-t,j_2+j_4-u).
\end{aligned}$$

Since these are real polynomials we may ignore the conjugate bar in (19). Let us temporarily write $P_p = P_p^{(\nu_1,\nu_2)}$ and so on. We fix eight non-negative integers

$$k_{12},\quad k_{34},\quad k_{13},\quad k_{24},\quad p',\quad q',\quad t',\quad u',$$

and apply, to both sides of (19), the summation operator

$$\sum_{\substack{i_1+i_2=p'+k_{12}\\ i_3+i_4=q'+k_{34}}} \frac{(\nu)_i}{i!} P_{p'}(i_1,i_2)P_{q'}(i_3,i_4) \sum_{\substack{j_1+j_3=t'+k_{13}\\ j_2+j_4=u'+k_{24}}} \frac{(\nu)_j}{j!} P_{t'}(j_1,j_3)P_{u'}(j_2,j_4).$$

By the orthogonality of Clebsch–Gordan coefficients (9), the right hand side vanishes unless $(p',q',t',u')=(p,q,t,u)$, and we obtain

$$\begin{aligned}
&\sum_{\substack{i_1+i_2=p+k_{12}\\ i_3+i_4=q+k_{34}}} \sum_{\substack{j_1+j_3=t+k_{13}\\ j_2+j_4=u+k_{24}}} \frac{(\nu)_i(\nu)_j}{i!\,j!}\,\mathbf{P}_s(i,j)P_p(i_1,i_2)P_q(i_3,i_4)P_t(j_1,j_3)P_u(j_2,j_4)\\
&\qquad = \frac{(\nu_1+\nu_2+2p)_{k_{12}}(\nu_3+\nu_4+2q)_{k_{34}}(\nu_1+\nu_3+2t)_{k_{13}}(\nu_2+\nu_4+2u)_{k_{24}}}{k_{12}!\,k_{34}!\,k_{13}!\,k_{24}!\,C_r^{\nu_1+\nu_2+2p,\,\nu_3+\nu_4+2q}C_v^{\nu_1+\nu_3+2t,\,\nu_2+\nu_4+2u}}\\
&\qquad\quad \times W_{pqr}^{tuv}\,P_r(k_{12},k_{34})P_v(k_{13},k_{24}).
\end{aligned} \tag{20}$$

Now we expand $\mathbf{P}_s$ according to (16). Then the left hand side of (20) takes the form

$$(21)\qquad (-1)^s \frac{(-p-q-k_{12}-k_{34})_s(-t-u-k_{13}-k_{24})_s}{(|\nu|+s-1)_s s!}$$
$$\times \sum_{m_1,m_2,m_3,m_4=0}^{\infty} \frac{(|\nu|+s-1)_{|m|}(-s)_{|m|}}{(-p-q-k_{12}-k_{34})_{|m|}(-t-u-k_{13}-k_{24})_{|m|}m!(\nu)_m}$$
$$\times \sum_{i_1+i_2=p+k_{12}} \frac{(\nu_1)_{i_1}(\nu_2)_{i_2}}{i_1!\,i_2!}(-i_1)_{m_1}(-i_2)_{m_2}P_p(i_1,i_2)$$
$$\times \sum_{i_3+i_4=q+k_{34}} \frac{(\nu_3)_{i_3}(\nu_4)_{i_4}}{i_3!\,i_4!}(-i_3)_{m_3}(-i_4)_{m_4}P_q(i_3,i_4)$$
$$\times \sum_{j_1+j_3=t+k_{13}} \frac{(\nu_1)_{j_1}(\nu_3)_{j_3}}{j_1!\,j_3!}(-j_1)_{m_1}(-j_3)_{m_3}P_t(j_1,j_3)$$
$$\times \sum_{j_2+j_4=u+k_{24}} \frac{(\nu_2)_{j_2}(\nu_4)_{j_4}}{j_2!\,j_4!}(-j_2)_{m_2}(-j_4)_{m_4}P_u(j_2,j_4).$$

The four inner sums may be computed using the probably well-known formula

$$(22)\qquad \sum_{i_1+i_2=s+k} \frac{(\nu_1)_{i_1}(\nu_2)_{i_2}}{i_1!\,i_2!}(-i_1)_{m_1}(-i_2)_{m_2}P_s^{(\nu_1,\nu_2)}(i_1,i_2)$$
$$= (-1)^{m_1+m_2}\frac{(\nu_1+\nu_2+s+m_1+m_2)_{k+s-m_1-m_2}}{(k+s-m_1-m_2)!}(\nu_1)_{m_1}(\nu_2)_{m_2}P_s^{(\nu_1,\nu_2)}(m_1,m_2).$$

Note that, by (8), the left hand side vanishes unless $m_1+m_2\le s+k$, and the right hand side unless $m_1+m_2\ge s$. One may prove (22) by inserting expression (11) for the Clebsch–Gordan coefficients (see the proof of Lemma 5.1 below).

It follows that the expression (21) equals

$$(-1)^s \frac{(-p-q-k_{12}-k_{34})_s(-t-u-k_{13}-k_{24})_s}{(|\nu|+s-1)_s s!}$$
$$\times \sum_{m_1,m_2,m_3,m_4=0}^{\infty} \frac{(|\nu|+s-1)_{|m|}(-s)_{|m|}(\nu)_m}{(-p-q-k_{12}-k_{34})_{|m|}(-t-u-k_{13}-k_{24})_{|m|}m!}$$
$$\times \frac{(\nu_1+\nu_2+p+m_1+m_2)_{k_{12}+p-m_1-m_2}(\nu_3+\nu_4+q+m_3+m_4)_{k_{34}+q-m_3-m_4}}{(k_{12}+p-m_1-m_2)!(k_{34}+q-m_3-m_4)!}$$
$$\times \frac{(\nu_1+\nu_3+t+m_1+m_3)_{k_{13}+t-m_1-m_3}(\nu_2+\nu_4+u+m_2+m_4)_{k_{24}+u-m_2-m_4}}{(k_{13}+t-m_1-m_3)!(k_{24}+u-m_2-m_4)!}$$
$$\times P_p(m_1,m_2)P_q(m_3,m_4)P_t(m_1,m_3)P_u(m_2,m_4).$$

On the other hand, this is equal to the right hand side of (20). After some cancellations, this gives the following formula.

Theorem 3.1. *Let k_{12}, k_{34}, k_{13}, k_{24}, p, q, r, t, u, and v be non-negative integers with $p+q+r=t+u+v=s$. Then, in the notation above, the equality*

$$\begin{aligned}
&\frac{1}{k_{12}!\,k_{34}!\,k_{13}!\,k_{24}!\,C_r^{\nu_1+\nu_2+2p,\,\nu_3+\nu_4+2q}C_v^{\nu_1+\nu_3+2t,\,\nu_2+\nu_4+2u}}\\
&\qquad\qquad \times W_{pqr}^{tuv}\,P_r^{(\nu_1+\nu_2+2p,\nu_3+\nu_4+2q)}(k_{12},k_{34})P_v^{(\nu_1+\nu_3+2t,\nu_2+\nu_4+2u)}(k_{13},k_{24})\\
&\qquad = (-1)^s\frac{(-p-q-k_{12}-k_{34})_s(-t-u-k_{13}-k_{24})_s}{(|\nu|+s-1)_s s!}\\
&\qquad\qquad \times \sum_{m_1,m_2,m_3,m_4}\frac{(|\nu|+s-1)_{|m|}(-s)_{|m|}(\nu)_m}{(-p-q-k_{12}-k_{34})_{|m|}(-t-u-k_{13}-k_{24})_{|m|}m!}\\
&\qquad\qquad \times \frac{1}{(k_{12}+p-m_1-m_2)!(k_{34}+q-m_3-m_4)!(k_{13}+t-m_1-m_3)!(k_{24}+u-m_2-m_4)!}\\
&\qquad\qquad \times \frac{1}{(\nu_1+\nu_2+2p)_{m_1+m_2-p}(\nu_3+\nu_4+2q)_{m_3+m_4-q}(\nu_1+\nu_3+2t)_{m_1+m_3-t}(\nu_2+\nu_4+2u)_{m_2+m_4-u}}\\
&\qquad\qquad \times P_p^{(\nu_1,\nu_2)}(m_1,m_2)P_q^{(\nu_3,\nu_4)}(m_3,m_4)P_t^{(\nu_1,\nu_3)}(m_1,m_3)P_u^{(\nu_2,\nu_4)}(m_2,m_4)
\end{aligned}$$

holds, where the sum is over all non-negative integers such that

$$\begin{cases}p\leq m_1+m_2\leq p+k_{12},\\ q\leq m_3+m_4\leq q+k_{34},\\ t\leq m_1+m_3\leq t+k_{13},\\ u\leq m_2+m_4\leq u+k_{24},\\ m_1+m_2+m_3+m_4\leq s.\end{cases}$$

To prove the triple sum formula we only need the case

$$k_{12}=k_{13}=0,\quad k_{34}=r,\quad k_{24}=v.$$

Then the two conditions

$$m_1+m_2=p,\quad m_1+m_3=t$$

hold, so that the sum in the theorem reduces to a double sum. Moreover, the factors P_r, P_v, P_p, and P_t become trivial by (8), so we obtain a formula expressing the Wigner $9j$-symbol as a double sum of products of two Clebsch–Gordan coefficients. After further simplifications, such as writing

$$k_{34}+q-m_3-m_4=s-|m|,$$

we obtain the following formula.

Corollary 3.2. *For p, q, r, t, u, and v non-negative integers with*

$$p+q+r=t+u+v=s,$$

the equality

$$
\begin{aligned}
W_{pqr}^{tuv} &= (-1)^{r+v+s}\frac{p!\,r!\,t!\,v!(\nu_1)_p(\nu_2)_p(\nu_1)_t(\nu_3)_t}{(|\nu|+s-1)_s} \\
&\quad\times (|\nu|+2p+2q+r-1)_r(|\nu|+2t+2u+v-1)_v \\
&\quad\times \sum_{m_1,m_2,m_3,m_4} \frac{(|\nu|+s-1)_{|m|}(\nu_4)_{m_4}}{m!(s-|m|)!(\nu_1)_{m_1}}(-1)^{m_1+m_4}P_q^{(\nu_3,\nu_4)}(m_3,m_4)P_u^{(\nu_2,\nu_4)}(m_2,m_4) \\
&\quad\times (\nu_2+\nu_4+u+m_2+m_4)_{s-|m|}(\nu_3+\nu_4+q+m_3+m_4)_{s-|m|}
\end{aligned}
$$

holds, where the sum is over all non-negative integers such that

$$
\begin{cases}
m_1+m_2=p, \\
q\le m_3+m_4\le q+r, \\
m_1+m_3=t, \\
u\le m_2+m_4\le u+v.
\end{cases}
$$

4. The triple sum formula

We will now prove the triple sum formula. Let us rewrite the sum in Corollary 3.2 using $j=s-|m|$ and m_1 as summation variables, thus replacing m_2, m_3, and m_4 by

$$
p-m_1, \qquad t-m_1, \qquad s+m_1-p-t-j,
$$

respectively. Using the identities

$$
(a)_{s-k}=(-1)^k\frac{(a)_s}{(1-a-s)_k},
$$

$$
(a)_k=(-1)^k(1-a-k)_k,
$$

we obtain

$$
\begin{aligned}
W_{pqr}^{tuv} &= (-1)^{q+u}r!\,v!(|\nu|+2p+2q+r-1)_r(|\nu|+2t+2u+v-1)_v \\
&\quad\times (\nu_1)_p(\nu_2)_p(\nu_1)_t(\nu_3)_t\sum_{j,m_1}\frac{(1-\nu_2-\nu_4-2u-v)_j(1-\nu_3-\nu_4-2q-r)_j}{j!(2-|\nu|-2s)_j} \\
&\quad\times \frac{(-p)_{m_1}(-t)_{m_1}}{m_1!(\nu_1)_{m_1}}\frac{(\nu_4)_{s+m_1-p-t-j}}{(s+m_1-p-t-j)!} \\
&\quad\times P_q^{(\nu_3,\nu_4)}(t-m_1,s+m_1-p-t-j)\,P_u^{(\nu_2,\nu_4)}(p-m_1,s+m_1-p-t-j),
\end{aligned}
$$

where the sum is over j and m_1 such that

$$
\begin{cases}
0\le m_1\le \min(p,t), \\
0\le j\le \min(r,v), \\
p+t+j\le s+m_1.
\end{cases}
$$

Now let us expand P_q and P_u using formulas (12) and (13), respectively. Thus we write

$$\begin{aligned}P_q &= (-1)^q \frac{(s+m_1-p-t-j)!}{(r-j)!}(\nu_3)_q(\nu_4+s+m_1-p-t-j)_{t-m_1}\\ &\quad\times\sum_{i=0}^{t-m_1}\frac{(m_1-t)_i(\nu_3+q)_i(1-\nu_4-q)_i}{i!(\nu_3)_i(1-\nu_4-q-r+j)_i},\\ P_u &= (\nu_2)_u(j-u-v)_u\sum_{k=0}^{\min(u,p-m_1)}\frac{(-u)_k(m_1-p)_k(\nu_2+\nu_4+u-1)_k}{k!(j-u-v)_k(\nu_2)_k}.\end{aligned}$$

We then change the order of summation, pulling m_1 inside. We also manipulate the Pochhammer symbols to make i, j, k appear only as subscripts. This gives

$$\begin{aligned}W_{pqr}^{tuv} &= (u+v)!(\nu_1)_p(\nu_1)_t(\nu_2)_p(\nu_2)_u(\nu_3)_q(\nu_3)_t(\nu_4)_{q+r}\\ &\quad\times(|\nu|+2p+2q+r-1)_r(|\nu|+2t+2u+v-1)_v\\ &\quad\times\sum_{i=0}^{t}\sum_{j=0}^{\min(r,v)}\sum_{k=0}^{\min(p,u)}\frac{1}{i!\,j!\,k!}\frac{1}{(1-\nu_4-q-r)_{i+j}(-u-v)_{j+k}}\\ &\quad\times\frac{(-r)_j(-v)_j(1-\nu_3-\nu_4-2q-r)_j(1-\nu_2-\nu_4-2u-v)_j}{(2-|\nu|-2s)_j}\\ &\quad\times\frac{(-t)_i(\nu_3+q)_i(1-\nu_4-q)_i}{(\nu_3)_i}\frac{(-p)_k(-u)_k(\nu_2+\nu_4+u-1)_k}{(\nu_2)_k}\\ &\quad\times\sum_{m_1=\max(0,p+t+j-s)}^{\min(t-i,p-k)}\frac{(i-t)_{m_1}(k-p)_{m_1}}{m_1!(\nu_1)_{m_1}}.\end{aligned}$$

However, it follows from (14), that if we instead first sum over i, then the sums for which $m_1 \leq p+t+j-s$ (if there are any such) vanish. Thus we can replace the lower limit in the inner sum by 0, and then compute it using the Chu–Vandermonde formula:

$$\sum_{m_1}\frac{(i-t)_{m_1}(k-p)_{m_1}}{m_1!(\nu_1)_{m_1}} = \frac{(\nu_1)_{p+t-i-k}}{(\nu_1)_{t-i}(\nu_1)_{p-k}} = \frac{(\nu_1)_{p+t}}{(\nu_1)_p(\nu_1)_t}\frac{(1-\nu_1-t)_i(1-\nu_1-p)_k}{(1-\nu_1-p-t)_{i+k}}.$$

This gives the following result, which is equivalent to the註Ališauskas–Jucys triple sum formula. As remarked in [SR2], the sum may be expressed in terms of generalized hypergeometric functions introduced by Srivastava [S].

Theorem 4.1. *The Wigner 9j-symbols for $\mathfrak{su}(1,1)$ are given by*

$$\begin{aligned}
W^{tuv}_{pqr} &= (u+v)!(\nu_1)_{p+t}(\nu_2)_p(\nu_2)_u(\nu_3)_q(\nu_3)_t(\nu_4)_{q+r} \\
&\quad \times (|\nu|+2p+2q+r-1)_r(|\nu|+2t+2u+v-1)_v \\
&\quad \times \sum_{i=0}^{t}\sum_{j=0}^{\min(r,v)}\sum_{k=0}^{\min(p,u)} \frac{1}{i!\,j!\,k!}\frac{1}{(1-\nu_4-q-r)_{i+j}(1-\nu_1-p-t)_{i+k}(-u-v)_{j+k}} \\
&\quad \times \frac{(-t)_i(1-\nu_1-t)_i(\nu_3+q)_i(1-\nu_4-q)_i}{(\nu_3)_i} \\
&\quad \times \frac{(-r)_j(-v)_j(1-\nu_3-\nu_4-2q-r)_j(1-\nu_2-\nu_4-2u-v)_j}{(2-|\nu|-2s)_j} \\
&\quad \times \frac{(-p)_k(-u)_k(1-\nu_1-p)_k(\nu_2+\nu_4+u-1)_k}{(\nu_2)_k},
\end{aligned}$$

where $p+q+r=t+u+v=s$.

This formula might look awkward, but we stress that it is the most compact known expression for Wigner $9j$-symbols. Note that the parameter occurring in the Pochhammer symbol $(\ )_{i+j}$ may be written in two ways:

$$1-\nu_4-q-r=(1-\nu_4-q)+(-r)=(\nu_3+q)+(1-\nu_3-\nu_4-2q-r),$$

as a sum of those occurring in $(\ )_i$ and $(\ )_j$ in the numerator. This is also true for $(\ )_{i+k}$ and $(\ )_{j+k}$, and each of the twelve numerator parameters occur once in the corresponding six sums.

5. General recoupling coefficients

In this section we will extend Theorem 3.1 to arbitrary recoupling coefficients. Thus we consider two decompositions:

$$\begin{aligned}
\mathcal{A}^{\nu_1}\otimes\cdots\otimes\mathcal{A}^{\nu_n} &\simeq (\mathcal{A}^{\nu_{\sigma_1}}\otimes\cdots\otimes\mathcal{A}^{\nu_{\sigma_l}})\otimes(\mathcal{A}^{\nu_{\sigma_{l+1}}}\otimes\cdots\otimes\mathcal{A}^{\nu_{\sigma_n}}) = B_1\otimes B_2 \\
&\simeq (\mathcal{A}^{\nu_{\tau_1}}\otimes\cdots\otimes\mathcal{A}^{\nu_{\tau_{l'}}})\otimes(\mathcal{A}^{\nu_{\tau_{l'+1}}}\otimes\cdots\otimes\mathcal{A}^{\nu_{\tau_n}}) = B_1'\otimes B_2',
\end{aligned}$$

where σ and τ are permutations and $1\leq l,\, l'\leq n-1$. We choose arbitrary highest weight vectors

$$Q_1\in B_1,\qquad Q_2\in B_2,\qquad Q_1'\in B_1',\qquad Q_2'\in B_2',$$

of respective degree

$$s_1,\qquad s_2,\qquad s_1',\qquad s_2',$$

and weight

$$-\mu_1,\qquad -\mu_2,\qquad -\mu_1',\qquad -\mu_2'.$$

For a multi-index $(i_1,\dots,i_n)$ we write

$$\begin{aligned}
i_1^\sigma &= (i_{\sigma_1},\dots,i_{\sigma_l}), & i_2^\sigma &= (i_{\sigma_{l+1}},\dots,i_{\sigma_n}),\\
i_1^\tau &= (i_{\tau_1},\dots,i_{\tau_{l'}}), & i_2^\tau &= (i_{\tau_{l'+1}},\dots,i_{\tau_n}),
\end{aligned}$$

so that

$$\begin{aligned}
\mu_1 &= |\nu_1^\sigma|+2s_1, & \mu_2 &= |\nu_2^\sigma|+2s_2,\\
\mu_1' &= |\nu_1^\tau|+2s_1', & \mu_2' &= |\nu_2^\tau|+2s_2'.
\end{aligned}$$

We will study the recoupling coefficient

$$\langle (Q_1, Q_2)^{\sigma}_{s_3}, (Q_1', Q_2')^{\tau}_{s_3'} \rangle,$$

where we may assume that

$$s_1 + s_2 + s_3 = s_1' + s_2' + s_3' = s.$$

In (19) we replace Q_{pqr} and $\tilde{Q}_{tuv}$ by suitable orthogonal systems of highest weight vectors containing $(Q_1, Q_2)^{\sigma}_{s_3}$ and $(Q_1', Q_2')^{\tau}_{s_3'}$. Using (5) instead of (9) we obtain

$$\begin{aligned}
&\sum_{\substack{|i_1^\sigma| = s_1 + k_1,\ |i_2^\sigma| = s_2 + k_2 \\ |j_1^\tau| = s_1' + k_1',\ |j_2^\tau| = s_2' + k_2'}} \frac{(\nu)_i (\nu)_j}{i!\, j!} \, \mathbf{P}_s(i,j)\, TQ_1(i_1^\sigma)\, TQ_2(i_2^\sigma)\, \overline{TQ_1'(j_1^\tau)\, TQ_2'(j_2^\tau)} \\
&= \frac{(\mu_1)_{k_1} (\mu_2)_{k_2} (\mu_1')_{k_1'} (\mu_2')_{k_2'}}{k_1!\, k_2!\, k_1'!\, k_2'!\, C_{s_3}^{\mu_1 \mu_2} C_{s_3'}^{\mu_1' \mu_2'}} \langle (Q_1, Q_2)^{\sigma}_{s_3}, (Q_1', Q_2')^{\tau}_{s_3'} \rangle \, P_{s_3}^{(\mu_1, \mu_2)}(k_1, k_2) P_{s_3'}^{(\mu_1', \mu_2')}(k_1', k_2'),
\end{aligned}$$

where k_1, k_2, k_1', k_2' are arbitrary non-negative integers. By (16) the left hand side equals

$$\begin{aligned}
&(-1)^s \frac{(-s_1 - s_2 - k_1 - k_2)_s (-s_1' - s_2' - k_1' - k_2')_s}{(|\nu| + s - 1)_s s!} \\
&\qquad \times \sum_m \frac{(|\nu| + s - 1)_{|m|} (-s)_{|m|}}{(-s_1 - s_2 - k_1 - k_2)_{|m|} (-s_1' - s_2' - k_1' - k_2')_{|m|} m! (\nu)_m} \\
&\qquad \times \sum_{|i_1^\sigma| = s_1 + k_1} \frac{(\nu_1^\sigma)_{i_1^\sigma}}{i_1^\sigma!} (-i_1^\sigma)_{m_1^\sigma}\, TQ_1(i_1^\sigma) \sum_{|i_2^\sigma| = s_2 + k_2} \frac{(\nu_2^\sigma)_{i_2^\sigma}}{i_2^\sigma!} (-i_2^\sigma)_{m_2^\sigma}\, TQ_2(i_2^\sigma) \\
&\qquad \times \sum_{|j_1^\tau| = s_1' + k_1'} \frac{(\nu_1^\tau)_{j_1^\tau}}{j_1^\tau!} (-j_1^\tau)_{m_1^\tau}\, \overline{TQ_1'(j_1^\tau)} \sum_{|j_2^\tau| = s_2' + k_2'} \frac{(\nu_2^\tau)_{j_2^\tau}}{j_2^\tau!} (-j_2^\tau)_{m_2^\tau}\, \overline{TQ_2'(j_2^\tau)}.
\end{aligned}$$

To compute the inner sums we need instead of (22) the following lemma.

Lemma 5.1. *If Q is a highest weight vector of degree s in $\mathcal{A}^{\nu_1} \otimes \cdots \otimes \mathcal{A}^{\nu_n}$, then*

$$\sum_{|i|=s+k} \frac{(\nu)_i}{i!}(-i)_m \, TQ(i) = (-1)^{|m|} \frac{(|\nu| + s + |m|)_{k+s-|m|}}{(k+s-|m|)!}(\nu)_m \, TQ(m).$$

In [R2] this is derived as a limit case of a more general formula. We will give an independent proof here.

Proof. Let Q be given by (3). Then, by (2),

$$\begin{aligned}\sum_{|t|=s} c_t \, z^t &= \sum_{|t|=s} c_t (1+z_1)^{t_1} \cdots (1+z_n)^{t_n} \\ &= \sum_{|t|=s} c_t \sum_{k_1,\ldots,k_n \geq 0} \frac{(-t)_k}{k!}(-1)^{|k|} z^k.\end{aligned}$$

Applying T on both sides gives

$$TQ(m) = \sum_{|t|=s} c_t \sum_{k_1,\ldots,k_n \geq 0} \frac{(-t)_k(-m)_k}{k!(\nu)_k} = \sum_{|t|=s} c_t \frac{(\nu+m)_t}{(\nu)_t}, \tag{23}$$

by the Chu–Vandermonde formula; that is $TQ(m) = (-1)^s TQ(-\nu - m)$. This generalizes expression (11). Thus we have

$$\begin{aligned}\sum_{|i|=s+k} \frac{(\nu)_i}{i!}(-i)_m \, TQ(i) &= \sum_{|i|=s+k} \frac{(\nu)_i}{i!}(-i)_m \sum_{|t|=s} c_t \frac{(\nu+i)_t}{(\nu)_t} \\ &= \sum_{|t|=s} c_t \sum_{|i|=s+k} \frac{(\nu+t)_i}{i!}(-i)_m.\end{aligned}$$

Replacing i by $i+m$ in the inner sum gives

$$\begin{aligned}&\sum_{|i|=s+k} \frac{(\nu+t)_i}{i!}(-i)_m = \sum_{|i|=s+k-|m|} \frac{(\nu+t)_{i+m}}{(i+m)!}(-i-m)_m \\ &= (-1)^{|m|}(\nu+t)_m \sum_{|i|=s+k-|m|} \frac{(\nu+t+m)_i}{i!} = (-1)^{|m|}(\nu+t)_m \frac{(|\nu|+|t|+|m|)_{s+k-|m|}}{(s+k-|m|)!},\end{aligned}$$

where the last step is the multivariable Chu–Vandermonde formula. This gives

$$\sum_{|i|=s+k} \frac{(\nu)_i}{i!}(-i)_m \, TQ(i) = (-1)^{|m|}(\nu)_m \frac{(|\nu|+s+|m|)_{s+k-|m|}}{(s+k-|m|)!} \sum_{|t|=s} c_t \frac{(\nu+m)_t}{(\nu)_t},$$

which, again by (23), proves the lemma. $\square$

Applying this lemma gives the following generalization of Theorem 3.1.

Theorem 5.2. *In the notation above, the equality*

$$\begin{aligned}
&\frac{1}{k_1!\,k_2!\,k_1'!\,k_2'!\,C_{s_3}^{\mu_1\mu_2}C_{s_3'}^{\mu_1'\mu_2'}}\,\langle (Q_1,Q_2)^\sigma_{s_3},(Q_1',Q_2')^\tau_{s_3'}\rangle\,P_{s_3}^{(\mu_1,\mu_2)}(k_1,k_2)P_{s_3'}^{(\mu_1',\mu_2')}(k_1',k_2')\\
&\qquad=(-1)^s\frac{(-s_1-s_2-k_1-k_2)_s(-s_1'-s_2'-k_1'-k_2')_s}{(|\nu|+s-1)_s s!}\\
&\qquad\times\sum_{m_1,\dots,m_n}\frac{(|\nu|+s-1)_{|m|}(-s)_{|m|}(\nu)_m}{(-s_1-s_2-k_1-k_2)_{|m|}(-s_1'-s_2'-k_1'-k_2')_{|m|}m!}\\
&\qquad\times\frac{1}{(k_1+s_1-|m_1^\sigma|)!(k_2+s_2-|m_2^\sigma|)!(k_1'+s_1'-|m_1^\tau|)!(k_2'+s_2'-|m_2^\tau|)!}\\
&\qquad\times\frac{1}{(\mu_1)_{|m_1^\sigma|-s_1}(\mu_2)_{|m_2^\sigma|-s_2}(\mu_1')_{|m_1^\tau|-s_1'}(\mu_2')_{|m_2^\tau|-s_2'}}\\
&\qquad\times TQ_1(m_1^\sigma)\,TQ_2(m_2^\sigma)\,\overline{TQ_1'(m_1^\tau)\,TQ_2'(m_2^\tau)}
\end{aligned}$$

holds, where the sum is over all non-negative integers such that

$$\begin{cases} s_1\leq |m_1^\sigma|\leq s_1+k_1,\\ s_2\leq |m_2^\sigma|\leq s_2+k_2,\\ s_1'\leq |m_1^\tau|\leq s_1'+k_1',\\ s_2'\leq |m_2^\tau|\leq s_2'+k_2'.\end{cases}$$

We remark that for Wigner $6j$-symbols, that is, for the recoupling coefficient

$$\langle(((1,1)_q,1)_r,(1,(1,1)_t)_u\rangle,$$

then in certain cases such as

$$k_1=k_2'=0,\qquad k_2=r,\qquad k_1'=u,$$

the sum in the theorem reduces to a single hypergeometric sum, which is in fact a terminating balanced ${}_4F_3(1)$-series. This gives a new proof of Racah's single sum formula for $6j$-symbols.

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