Hej!

Här kommer några uppgifter du kan titta på som förberedelse inför nästa års matematiktävling, eller bara för att det är roligt att jobba med matematik. En del av problemen är relativt enkla, andra är något att bita i. Bli inte förvånad om du inte kan lösa alla, åtminstone inte med en gång. En del är riktigt svåra. Om du fått sådana problem tidigare kommer du att määrka att de till största delen upprepas från år till år, dock finns det varje gång flera nya uppgifter att jobba med. Din lärare kan kontakta oss för lösningsförslag. På matematiktävlingens hemsida hittar du bl.a. uppgifterna från de senaste årens kval- och finalomgångar (med lösningar).

Om du har några frågor är du välkommen att skriva till mig

jana@chalmers.se

Hoppas att du även i fortsättningen tycker att matematik är ett roligt ämne som är värt att satsa på!

Hälsningar

Jana Madjarova, Matematik, Chalmers,
Ordförande i Tävlingskommittén

Problem

1. Givet 16 fotbollslag och att varje lag möter varje annat lag, visa att det vid varje tidpunkt finns minst två lag som har spelat samma antal matcher.

2. Finn alla heltalslösningar till ekvationen $xy = 2x - y$.

3. Talet x bildas genom att man på ett godtyckligt sätt blandar siffrorna i 111 exemplar av talet 2000. Visa att x inte är kvadraten till något heltal.

4. Givet är sex kongruenta cirkelskivor i planet med icke-tomt snitt. Visa att minst en av dem innehåller medelpunkten till en av de andra.

5. Givet är en vinkel med spets i punkten O och ben p och q. Låt $A, B \in p$ och $C, D \in q$. Finn mängden av alla punkter M i vinkelns inre sådana att summan av areorna av trianglarna ABM och CDM är lika med S (konstant).

6. Finn alla heltalslösningar till ekvationen

$$x^y = y^x.$$

7. Det naturliga talet n är sådant att det finns en rätvinklig triangel med hypotenusa $2n$ och kateter naturliga tal. Visa att det även finns en rätvinklig triangel med hypotenusa n och kateter naturliga tal.

8. En biljardboll ligger vid kanten till ett runt biljardbord med radie R. Efter ett slag reflekteras bollen i kanten sex gånger, utan att på vägen ha kommit närmare bordets centrum än $\frac{9R}{10}$. Kan bollen vid den sjätte reflexionen ha fullbordat ett helt varv kring bordets centrum? Motivera!

9. I varje punkt med heltalskoordinater i planet har man placerat en säck med bollar så att (a) det finns inte fler än 2005 bollar i någon säck; (b) antalet bollar i varje säck är lika med (det aritmetiska) medelvärdet av antalet bollar i säckarna i punktens fyra närmsta grannpunkter. Visa att det finns lika många bollar i alla säckar.

10. Låt f vara en kongruensavbildning i planet som är sammansättning (i valfri ordning) av en translation och en rotation (ej ett helt antal varv). Visa att det finns en entydigt bestämd punkt P i planet sådan att $|PA| = |Pf(A)|$ för varje punkt A i samma plan.

Ur "Korrespondenskurs 2002/2003"

1. Let $a_1, a_2, \ldots, a_n$ be positive real numbers in arithmetic progression. Prove that

$$\sum_{k=1}^{n} \frac{1}{a_k a_{n-k+1}} > \frac{4n}{(a_1 + a_n)^2}\,.$$

2. A mouse eats his way through a $3 \times 3 \times 3$ cube of cheese by tunneling through all of the 27 $1 \times 1 \times 1$ sub-cubes. If he starts at one of the corner sub-cubes and always moves onto an uneaten adjacent sub-cube can he finish at the center of the cube? (Assume that he can tunnel through walls but not edges or corners.)

3. Find all positive real solutions of the system

$$x_1 + x_2 + \cdots + x_n = 9,$$
$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1.$$

4. Prove that for any real numbers a, b, c such that $0 < a, b, c < 1$, the following inequality holds

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1.$$

5. In the triangle ABC, the midpoint of BC is D. Given that $\angle ADB = 45°$ and $\angle ACB = 30°$, determine $\angle BAD$.

6. Five points are given in the plane such that each of the 10 triangles they define has area greater than 2. Prove that there exists a triangle of area greater than 3.

Ur "Korrespondenskurs 2003/2004"

1. Let $p(3) = 2$ where $p(x)$ is a polynomial with integer coefficients. Is it possible that $p(2003)$ is a perfect square?

2. The cat from the neighbouring village keeps coming to Dunce-village to irritate their dogs. Every night when all of them are sleeping, the cat sneaks up to Dunce-village, mews out loud and runs back home. When the cat mews, all dogs that are up to 90 m away from the cat, start barking. As Dunce-village is small, any two dogs in the village are up to 100 m from each other. Is it possible for the cat to take a position such that all dogs start barking at the same time?

3. Let D be the midpoint of the hypotenuse AB of the right triangle ABC. Denote by O_1 and O_2 the circumcenters of the triangles ADC and DBC, respectively. Prove that AB is tangent to the circle with diameter O_1O_2.

4. Prove the inequality

$$\frac{a^2+b^2}{c^2+ab} + \frac{b^2+c^2}{a^2+bc} + \frac{c^2+a^2}{b^2+ca} \geq 3$$

for all positive numbers a, b, c.

5. Solve the following system in real numbers

$$\begin{array}{ccccccc} x^2 & + & y^2 & - & z(x+y) & = & 2 \\ y^2 & + & z^2 & - & x(y+z) & = & 4 \\ z^2 & + & x^2 & - & y(z+x) & = & 8 \end{array}.$$

6. Given a triangle such that the sines of all three angles are rational numbers, prove that the cosines of all three angles are rational too.

Ur "Korrespondenskurs 2004/2005"

1. Each of the players in a tennis tournament played one match against each of the others. If every player won at least one match, show that there is a group A, B, C of three players for which A beat B, B beat C and C beat A.

2. Suppose p, q are distinct primes and S is a subset of $\{1, 2, \ldots, p-1\}$. Let $N(S)$ denote the number of solutions of the equation

$$\sum_{i=1}^{q} x_i \equiv 0 \pmod{p},$$

where $x_i \in S,\ i = 1, 2, \ldots, q$. Prove that $N(S)$ is a multiple of q.

3. The points A, B, C, D, E, F on a circle of radius R are such that $AB = CD = EF = R$. Show that the middle points of BC, DE, FA are vertices of an equilateral triangle.

4. Find all functions $f : \mathbb{N} \cup \{0\} \to \mathbb{R}$ such that

$$f(n+m) + f(n-m) = f(kn), \qquad \forall n, m \in \mathbb{N} \cup \{0\}, \quad m \leq n,$$

where k is a fixed nonnegative integer.

5. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$ and $\angle A = 20°$.The point D on AC is such that $AD = BC$. Determine the angle $\angle BDC$.

6. The numbers $a_1, a_2, \ldots, a_n$ are positive and such that $a_1 + a_2 + \ldots + a_n = 1$. Show that

$$\frac{a_1}{1 + a_1\sqrt{2}} + \frac{a_2}{1 + a_2\sqrt{2}} + \ldots + \frac{a_n}{1 + a_n\sqrt{2}} \leq \frac{n}{n + \sqrt{2}}.$$

When does equality occur?

Ur "Korrespondenskurs 2005/2006"

1. The circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at A and B. The tangent line to $\mathcal{C}_2$ at A meets $\mathcal{C}_1$ at the point C and the tangent line to $\mathcal{C}_1$ at A meets $\mathcal{C}_2$ at the point D. A ray from A, interior to the angle $\angle CAD$, intersects $\mathcal{C}_1$ at M, $\mathcal{C}_2$ at N and the circumcircle of the triangle $\triangle ACD$ at P. Prove that $AM = NP$.

2. Let h be a positive integer and let $\{a_n\}_{n \in \mathbb{N} \cup \{0\}}$ be the sequence defined by recursion as follows:

$$a_0 = 1; \qquad a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \text{ is even,} \\ a_n + h & \text{if } a_n \text{ is odd.} \end{cases}$$

(For instance, if $h = 27$ one has $a_1 = 28, a_2 = 14, a_3 = 7, a_4 = 34, a_5 = 17, a_6 = 44 \ldots$).

For which values of h does there exist $n > 0$ ($n \in \mathbb{N}$) such that $a_n = 1$?

3. Find all natural numbers n such that there exists a polynomial p with real coefficients for which

$$p\left(x - \frac{1}{x}\right) = x^n - \frac{1}{x^n} \qquad \forall x \in \mathbb{R},\ x \neq 0.$$

4. Let S be a set of n points in the plane such that any two points of S are at least 1 unit apart. Prove that there is a subset T of S with at least $n/7$ points such that any two points of T are at least $\sqrt{3}$ units apart.

5. A number of points on a circle of radius 1 are joined by chords. It is known that any diameter of the circle intersects at most 6 of the chords. Prove that the sum of the lengths of the chords is less than 19.

Ur "Korrespondenskurs 2006/2007"

1. How many five-digit palindromic numbers divisible by 37 are there?

2. How many points (x, y) are there on the circle $x^2 + y^2 = 1$ such that x and y have at most (a) two (b) m decimal places? (m is a non-negative integer)

3. Sixteen points in the plane form a 4×4 grid. Show that any seven of these points contain three which are the vertices of an isosceles triangle. Will the statement still hold if we replace seven by six?

4. An (a) $n \times n$ square ((b) $n \times p$ rectangle) is divided into (a) n^2 ((b) np) unit squares. Initially m of these squares are black and all the others are white. The following operation is allowed: If there exists a white square which is adjacent to at least two black squares, we can change the colour of this square from white to black. Find the smallest possible m such that there exists an initial position from which, by applying repeatedly this operation, all unit squares can be made black.

5. Show that the inequality

$$3(a+b+c) \geq 8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3+b^3+c^3}{3}}$$

holds for all positive reals a, b, c. Determine the cases of equality.

Ur "Korrespondenskurs 2007/2008"

1. For all real numbers a, let $\lfloor a \rfloor$ denote the largest integer less than or equal to a, and let $\{a\} = a - \lfloor a \rfloor$. Solve the system of equations

$$\begin{cases} x + \lfloor y \rfloor - \{z\} = 2,98 \\ \lfloor x \rfloor + \{y\} + z = 4,05 \\ -\{x\} + y + \lfloor z \rfloor = 5,01 \end{cases}.$$

2. Let A, B, C, D be points in the plane such that

$$0 < AB, AC, AD < 1 < BC, BD, CD.$$

Show that the four points do not lie on a circle.

3. The sum of n real numbers is positive and the sum of their squares is greater than n^2. Show that at least one of the numbers is greater than 1.

4. Between every pair of cities in a country there is a flight in each direction operated by one of several airlines. If an airline has direct flights between A and B and between B and C, then it has no direct flights between A and C. Show that one can travel through the country, passing all the cities exactly once, and in such a way that one changes airlines on each transfer occasion. (The travel begins and ends in different cities.)

5. The infinite sequence $a_1, a_2, \ldots, a_n, \ldots$, consists of natural numbers (i.e. positive integers) and is such that

$$a_1 = 1, \qquad a_n^2 > a_{n-1}a_{n+1} \quad \text{for all} \quad n \in \mathbb{N}.$$

Prove that $a_n \geq n$ for all $n \in \mathbb{N}$.

Ur "Korrespondenskurs 2008/2009"

1. Prove that in each triangle there are two sides of lengths a and b such that

$$\frac{\sqrt{5}-1}{2} < \frac{a}{b} < \frac{\sqrt{5}+1}{2}.$$

2. Let AD and AM be the altitude and the median from A in the triangle ABC and let E be a point on the side BC. Prove that AM bisects the segment DE if and only if $AB^2 + BE^2 = AC^2 + CE^2$.

3. Find all pairs x, y of real numbers such that

$$ax + by \neq c,$$

for any triple $a, b, c \in \mathbb{R}$ such that $a^2 + b^2 = c^2 \neq 0$.

4. Prove that among ten composite positive integers non greater than 840 there are at least two which are not relatively prime.

5. In a triangle each vertex is connected by a segment to an inner point of the opposite side. Show that the midpoints of the three segments are not collinear.

Ur "Korrespondenskurs 2009/2010"

1. Two robbers have stolen an amount of gold coins and have divided them into two heaps. The two heaps together weigh w (weight units), and the heaviest coin weighs c units. If w_1 and w_2 are the weights of the heaps, and $w_1 \leq w_2$, prove that
(a) $w_1 \leq w - c$;
(b) the coins can be divided into heaps so that $w_1 \geq \frac{w-c}{2}$.

2. For a, b positive integers, show that there exist positive integers c, d such that $a^2 + b^2 = c^2 - d^2$ if and only if ab is an even number.

3. A rectangular table contains integers in all its cells. The following operations on the table are permitted: choose exactly one number in each row and increase each chosen number by 1, or, choose exactly one number in each column and decrease each chosen number by one. Determine whether it is always possible to end up with a table containing only zeros after a finite number of operations of the two kinds if the dimensions of the table are

(a) 15×20; **(b)** 5×11.

4. Let a, b, c, d be the side lengths of a convex quadrilateral which lies (strictly) inside a unit square (it cannot coincide with the square). Show that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} > 4.$$

5. Let ABC be a triangle such that $BC < AC$. Let l be the straight line through C, orthogonal to the angle bisector BE of $\angle B$ (where $E \in AC$). The line l intersects BE at F and the median BD at G (where $D \in AC$). Prove that DF passes through the midpoint of GE.

Ur "Korrespondenskurs 2010/2011"

1. Let $\triangle ABC$ be an equilateral triangle. The point P lies on the circumcircle of $\triangle ABC$, on the smaller of the two arcs AB. If $PA = x$, and $PB = y$, find PC.

2. The integers x, y, z satisfy the equality $(x - y)(y - z)(z - x) = x + y + z$. Prove that $x + y + z$ is divisible by 27.

3. Find the minimal number of points in a set in the plane such that the number of different straight lines connecting pairs of points of the set is exactly 20.

4. Anders needs to buy three pencils and an eraser. He has 100 coins. He knows that he has to pay an integer number of coins for each item, that 175 pencils cost more than 125 erasers, but less than 126 erasers. Does Anders have enough money to buy what he needs?

5. The numbers a, b, c are rational, and such that

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

Prove that the number $\sqrt{a^2 + b^2 + c^2}$ is rational too.

6. Given a number a, a new one is selected by choosing randomly one of the numbers $2a+1$ and $\frac{a}{a+2}$. The process starts with a positive integer n. Prove that if at any moment the number 2010 appears, then the initial number n must have been equal to 2010.

7. Is it possible to colour every side and every diagonal of a regular 12-gon in 12 colours in such a way that for each three of the 12 colours there exists a triangle with vertices at the vertices of the original 12-gon and such that its sides are of these three colours?

8. Given an acute triangle, find all points in its interior such that their orthogonal projections on the sides of the triangle are vertices of a triangle similar to the original one.