Prisoner's dilemma may or may not appear in large random games

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Abstract

Consider a two-person general-sum game on $n \times n$ payoff random matrices A and B with iid continuous entries, for large n. It is shown that the probability that there exists a pure strategy Nash equilibrium that is not pure Pareto optimal remains bounded away from 0 and 1 as n increases.

We also consider the number of mixed strategy Nash equilibria: It is shown that for a mixed strategy Nash equilibrium the number of rows that are given nonzero probability by player I must equal the number of columns given nonzero probability by player II. We further investigate the expected number of $k \times k$ mixed strategy Nash equilibria when the entries are normally distributed and prove it to be of order $(\log n)^{k-1}/(2^k(k!)^2)$. As a consequence we derive that with high probability no $k \times k$ mixed equlibria will exist when $K > e^2$ and $k \geq K(\log n)^{1/2}$.

1 Introduction

The question in focus of this paper is: Is the prisoner's dilemma a phenomenon that appears in a "typical" two-person non-cooperative game or does one have to deliberately set things right? The classical prisoner's dilemma is described by the pay-off bimatrix

$$\begin{bmatrix} (3,3) & (0,4) \\ (4,0) & (1,1) \end{bmatrix}.$$

Here the position $(2,2)$ is a so called *pure strategy Nash equilibrium*, i.e. if player I has decided to play row 2 and player II has decided to play column 2, then none of them can change his/her mind without losing from it, unless the other player also changes. The dilemma arises from the fact that position (1,1) is clearly better than position (2,2) for both players, so provided that they can trust each other they would both prefer position $(1,1)$ instead. In more formal terms, letting $m \times n$-matrices A and B denote the payoff matrices for player I and player II respectively, a position (i,j) is a pure strategy Nash equilibrium (PNE) if the entry a_{ij} in A is a largest element in its column and the entry b_{ij} in B is a largest element in its row. We say the a PNE is a *prisoner's dilemma position* (PDP) if it is not *pure Pareto optimal*, i.e. if there is a position (k,l) such that $a_{kl} \geq a_{ij}$ and $b_{kl} \geq b_{ij}$.

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In general one allows the two players to use randomness to decide what row/column to choose. This is represented by two probability vectors $\mathbf{p} = [p_1, \ldots, p_m]^T$ and $\mathbf{q} = [q_1, \ldots, q_n]^T$, called *mixed strategies*, where p_i is the probability that player I picks row i and q_j is the probability that player II picks column j. A mixed strategy Nash equilibrium (MNE) is a pair $(\mathbf{p}, \mathbf{q})$ of such probability vectors such that if player I plays $\mathbf{p}$ then player II in order to maximize her expected payoff can do nothing better than using $\mathbf{q}$ and vice versa. By Nash's famous result there always exists at least one Nash equilibrium for any matrices A and B, see e.g. [3, Section VIII.1]. A Nash equilibrium $(\mathbf{p}, \mathbf{q})$ is said to be Pareto optimal if there exists no other pair of mixed strategies giving both players at least as good and at least one player strictly better expected payoff.

When saying that prisoner's dilemma appears or not, it is a matter of taste if one does this in terms of pure or mixed strategy Nash equilibria and pure or general Pareto optimality. Since it is in a given situation usually far from easy to find all mixed Nash equlibria, it would take very sophisticated players to tell if prisoner's dilemma in the most general sense does or does not appear. Therefore we feel that it is reasonable to say that prisoner's dilemma appears if there exists at least one *pure* Nash equilibrium that is not *pure* Pareto optimal.

So now, what shall we mean with a "typical game"? It is natural to say that something that is typical is something that could appear from a random choice. Therefore one usually models a typical two-person game by letting the entries in the two matrices A and B be independent random variables chosen from a common continuous probability distribution. We will also assume that A is independent of B.[1] For simplicity we also assume that the number of actions for the two players are equal, so that A and B are $n \times n$-matrices, The reader will observe that all that we do can easily be generalized to when $m \neq n$ as long as m and n are both large. Our results can also equally easily be generalized to situations with more than two players. In the next section we will prove that the probability that there exists at least one PDP stays bounded away from 0 as well as 1 as n increases. (See Cohen [1] for some results related to this.)

In the third section we consider mixed strategy Nash equilibria for a random game. First it is shown that with probability 1, the number of rows given a positive probability by player I in an MNE must equal the number of columns given a positive probability by player II. Then we consider the expected number of $k \times k$ MNE's for given k in the case where the entries in the payoff matrices are standard normal. We show that the expected number of $k \times k$ MNE's is of order $(\log n)^{k-1}/2^k(k!)^2$. As a consequence, there will with high probability be no $k \times k$ MNE's when $K > e^2$ and $k \geq K(\log n)^{1/2}$.

[1]Note however that in many situations it is natural to consider cases where the entries at the same position in the two matrices are dependent. If one for example wants to study random zero-sum games (see [2]) one has $a_{ij} = -b_{ij}$. On the other hand, for random common-payoff games one must have $a_{ij} = b_{ij}$. For a general discussion about and some results on the number of PNE's in situations with dependence between A and B, see Rinott and Scarsini [5]. It should also be noted that what questions that turn out to be interesting relies heavily on the level of dependence between A and B. For example in the independent case, the number of PNE's has a nontrivial distribution (which is approximately Poisson(1)) but in the zero-sum case there will with probability one by a unique Nash equlibrium and the probability that this is pure is exponentially small.

2 Prisoner's dilemma positions

Let N denote the number of PNE's in the random game described above. We will need to know the distribution of N. The limiting distribution was first calculated by Powers [4] and later Stanford [6] found the exact distribution:

THEOREM 2.1 *The probability distribution of N is the following.*

$$P(N = k) = \frac{1}{n^2}\binom{n}{k}^2 k!.$$

In particular the distribution approaches a Poisson(1) distribution as $n \to \infty$ in the sense that

$$\lim_{n\to\infty} P(N = k) = e^{-1}\frac{1}{k!}$$

for all $k = 0, 1, 2, 3, \ldots$.

THEOREM 2.2 *Let Z be the number of PDP's in the random game. Then*

$$(1 + o(1))\frac{e^{-1/2}}{16} < P(Z > 0) < e^{-1}.$$

Proof. The upper bound follows from Theorem 2.1, so we can focus on the lower bound. Recall that Z is the number of PNE's (i, j) such that there exists a position (k, l), with $k \neq i$ and $l \neq j$, such that $a_{kl} \geq a_{ij}$ and $b_{kl} \geq b_{ij}$. We say that such a position (k, l) *dominates* (i, j).

Let U and D be the two disjoint square sub-arrays of $[n] \times [n]$ given by

$$U = \{1, 2, \ldots, \lfloor n/2\rfloor\} \times \{1, 2, \ldots, \lfloor n/2\rfloor\}$$

and

$$D = \{\lceil (n + 1)/2\rceil, \ldots, n - 1, n\} \times \{\lceil (n + 1)/2\rceil, \ldots, n - 1, n\}.$$

The event that there exists a PDP clearly contains the event that there exists a PNE in U that is dominated by a position in D. This event in turn contains the event that that there is exactly one PNE in U that is dominated by a position in D. Therefore

$$P(Z > 0) \geq P(\text{exactly one PNE in } U)$$

$$\cdot P(\exists (k, l) \in D : (k, l) \text{ dominates the PNE in } U|\text{exactly one PNE in } U)$$

$$\geq (1 + o(1))\frac{e^{-1/2}}{2} P(W > 0|(1, 1) \text{ is a PNE}) \qquad (2.1)$$

where W is the number of positions in D that dominate $(1, 1)$. The inequality follows from the approxaimate Poisson(1/2) distribution of the number of PNE's in U and the independence between all entries in A and B. Now

$$W = \sum_{(i,j)\in D} W_{ij}$$

where W_{ij} is the indicator that (i,j) dominates $(1,1)$. Since $\mathbb{E}[W_{ij}|(1,1) \text{ is a PNE}]$ is the probability that a_{ij} beats the maximum in column 1 of A and b_{ij} beats the maximum of row 1 of B, we get

$$\mathbb{E}[W_{ij}|(1,1) \text{ is a PNE}] = \frac{1}{(n+1)^2}$$

and so

$$\mathbb{E}[W|(1,1) \text{ is a PNE}] = (1+o(1))\frac{1}{4}.$$

In the same way we find that if (i,j) and (k,l) are two different positions in D, then $\mathbb{E}[W_{ij}W_{kl}|(1,1) \text{ is a PNE}]$ is the probability that a_{ij} and a_{kl} both beat the maximum in column 1 of A and b_{ij} and b_{kl} both beat the maximum in row 1 of B. Thus

$$\mathbb{E}[W_{ij}W_{kl}|(1,1) \text{ is a PNE}] = \frac{1}{\binom{n+2}{2}^2} = (1+o(1))\frac{4}{n^4}$$

and so

$$\mathbb{E}[W^2|(1,1) \text{ is a PNE}] = (1+o(1))\binom{\lfloor n/2 \rfloor^2}{2}\frac{4}{n^4} = (1+o(1)\frac{1}{2}.$$

Now for any nonnegative integer-valued random variable X it follows from Schwarz' inequality that $\mathbb{E}[X]^2 = \mathbb{E}[XI_{X>0}]^2 \le \mathbb{E}[X^2]P(X>0)$ so that $P(X>0) \ge \mathbb{E}[X]^2/\mathbb{E}[X^2]$. Hence

$$P(W>0|(1,1) \text{ is a PNE}) \ge \frac{\mathbb{E}[W|(1,1) \text{ is a PNE}]^2}{\mathbb{E}[W^2|(1,1) \text{ is a PNE}]} = (1+o(1))\frac{(1/4)^2}{1/2}$$

$$= (1+o(1))\frac{1}{8}.$$

Inserting into (2.1) yields

$$P(Z>0) \ge (1+o(1))\frac{e^{-1/2}}{16}.$$

□

3 Mixed strategy Nash equilibria

In this section we investigate the expected number of MNE's with support on a given number of rows and columns respectively. Define the support of a probability vector $\mathbf{x}$ on $[n]$ as $S(\mathbf{x}) = \{i \in [n] : x_i > 0\}$. First we show that for an MNE $(\mathbf{p}, \mathbf{q})$ the number of rows in the support of $\mathbf{p}$ must equal the number of colums in the support of $\mathbf{q}$.

LEMMA 3.1 *With probability 1 it is the case that for any MNE* $(\mathbf{p}, \mathbf{q})$ *one has* $|S(\mathbf{p})| = |S(\mathbf{q})|$.

Proof. We show that with probability 1, $|S(\mathbf{p})| \le |S(\mathbf{q})|$; the result then follows from symmetry.

Since the entries of the payoff matrices are independent and chosen from a continuous distribution, all sub-matrices are with probability 1 non-singular. Now suppose that $|S(\mathbf{q})| = k$. Since the sub-matrix of A on the corresponding k columns is non-singular, no more than k elements of the vector $A\mathbf{q}$ can be equal. Now since $(\mathbf{p}, \mathbf{q})$ is an MNE, the support of $\mathbf{p}$ must be contained in the set of positions corresponding to maximal elements of $A\mathbf{q}$, i.e. a set with no more than k elements. □

Assume that the entries of the payoff matrices are standard normal. For $k = 1, 2, 3, \ldots$ let N_k denote the number of MNE's $(\mathbf{p}, \mathbf{q})$ with $|S(\mathbf{p})| = |S(\mathbf{q})| = k$. For simplicity and clarity we will concentrate on estimating $\mathbb{E}[N_2]$ and leave the straightforward generalization to arbitrary k to the reader. Now $\mathbb{E}[N_2]$ is $\binom{n}{2}^2$ times the probability of the event E that there is an MNE $(\mathbf{p}, \mathbf{q})$ with $S(\mathbf{p}) = S(\mathbf{q}) = \{1, 2\}$, i.e. a 2×2 MNE in the upper left corner of the payoff matrices. This happens if and only if there exist numbers $p, q \in (0, 1)$ such that

(a) $qa_{11} + (1 - q)a_{12} = qa_{21} + (1 - q)a_{22}$,

(b) $qa_{11} + (1 - q)a_{12} \ge qa_{i1} + (1 - q)a_{i2}$ for all $i = 3, 4, \ldots, n$,

(c) $pb_{11} + (1 - p)b_{21} = pb_{12} + (1 - p)b_{22}$,

(d) $pb_{11} + (1 - p)b_{21} \ge pb_{1j} + (1 - p)b_{2j}$ for all $j = 3, 4, \ldots, n$.

Put $E(a)$, $E(b)$, $E(c)$ and $E(d)$ for the events that (a), (b), (c) and (d) happen respectively. Clearly (a) and (b) are independent of (c) and (d) and so $P(E) = P(E(a) \cap E(b))^2$.

Now if M is a $k \times k$-matrix whose entries are iid random variables from a continuous distribution symmetric about the origin, the probability that there exists a probability vector $\mathbf{q}$ such all elements of $M\mathbf{q}$ are equal, is $1/2^{k-1}$. This is so because such a probability vector exists if and only if the vector $M^{-1}\mathbf{1}$ contains only positive or only negative elements. Since M is invariant under diagonal orthogonal transformations, M^{-1} also exhibits that invariance. Therefore, of the 2^k such transformations there is always exactly one that transforms M so that $M^{-1}\mathbf{1}$ gets only positive entries and one that gives M^{-1} only negative entries. (For a more extended argument, see [2,Section 2].) As a special case of this it follows that $P(E(a)) = 1/2$ so that $P(E(a) \cap E(b)) = \frac{1}{2}P(E(b)|E(a))$. Put $f(x)$ for the probability density function of the q that solves (a). We have

$$P(E(b)|E(a)) = \int_0^1 P(E(b)|q = q_0, E(a)) f(q_0|E(a)) dq_0.$$

However

$$P(E(b)|q = q_0, E(a)) = P(E(b)|q = q_0)$$

$$= P(E(b)|q_0 a_{11} + (1 - q_0)a_{12} = q_0 a_{21} + (1 - q_0)a_{22})$$
$$= P(X_1 > \max(X_3, X_4, \ldots, X_n)|X_1 = X_2)$$

where $X_1, X_2, \ldots, X_n$ are iid and normal (with expectation 0 and variance $q_0^2 + (1 - q_0)^2$ but for the last probability we may as well assume that the X_i's are standard normal).

Now if X and Y are two independent random variables with common density $f(x)$, the conditional density of X given that $X = Y$ is proportional to $f(x)^2$. When X and Y are standard normal this means that the distribution of X given $X = Y$ is normal with variance $1/2$. Thus the question is how probable it is that a normal(0,1/2) variable X is greater than $\max(X_3, X_4, \ldots, X_n)$. To calculate this, let as usual Φ denote the distribution function of the standard normal distribution and recall that as $x \to \infty$,

$$1 - \Phi(x) = (1 + o(1))\frac{1}{x}e^{-x^2/2}. \tag{3.1}$$

Put M for $\max(X_3, X_4, \ldots, X_n)$. Pick a so that $e^{-a^2} = \log n/n^2$ and note that $a = (1 + o(1))(2\log n)^{1/2}$. Since the density function of X is $\pi^{-1/2}e^{-x^2}$, conditioning on X and integrating yields

$$P(M < X) = \pi^{-1/2}\int_{-\infty}^{\infty}\Phi(x)^{n-2}e^{-x^2}dx.$$

Using (3.1) we see that there are constants $c, C > 0$ such that on $(a - (2\log n)^{-1/2}, a + (2\log n)^{-1/2})$

$$c\frac{\log n}{n^2} \leq \Phi(n)^{n-2} \leq C\frac{\log n}{n^2}.$$

Integrating only from $a - (2\log n)^{-1/2}$ to $a + (2\log n)^{-1/2}$ it follows that this part of the integral is of order $(\log n)^{1/2}/n^2$. Thus $P(M < X)$ is at least of order $(\log n)^{1/2}/n^2$. For an upper bound of the same order we also need to bound the other two parts of the integral. However for $j = 1, 2, 3, \ldots$,

$$e^{-(a+j(2\log n)^{-1/2})^2} \leq e^{-2j}\frac{\log n}{n^2}$$

so that

$$\int_{a+(2\log n)^{-1/2}}^{\infty}\Phi(x)^{n-2}e^{-x^2}dx \leq 2\sqrt{2}\frac{(\log n)^{1/2}}{n^2}\sum_{j=1}^{\infty}e^{-2j}$$

as desired. The left part remains: For $j = 1, 2, 3, \ldots$

$$\Phi(n)^{n-2}e^{-(a-j(2\log n)^{-1/2})^2} = \frac{\log n}{n^2}e^{(1+o(1))(2j-e^j)}$$

from which the desired bound on the left tail follows in the same way as for the right part. Thus we have found that for some $D = D(n) = \Theta(1)$,

$$P(E(b)|E(a)) = D\frac{(\log n)^{1/2}}{n^2}$$

so that

$$P(E(a) \cap E(b)) = \frac{D}{2}\frac{(\log n)^{1/2}}{n^2}$$

and thus

$$P(E) = \frac{D^2}{2^2}\frac{\log n}{n^4}.$$

Therefore

$$\mathbb{E}[N_2] = \binom{n}{2}^2 \frac{D^2}{2^2}\frac{\log n}{n^4} = D^2 \frac{\log n}{2^2 \cdot (2!)^2}$$

as claimed. Copying the argument for N_3, N_4, etc yields

THEOREM 3.2 *There exist constants with $0 < c < C < \infty$ independent of n and k such that for $k = 2, 3, 4, \ldots$,*

$$c\frac{(\log n)^{k-1}}{2^k (k!)^2} \leq \mathbb{E}[N_k] \leq C\frac{(\log n)^{k-1}}{2^k (k!)^2}.$$

Applying Theorem 3.2 with $k = K(\log n)^{1/2}$ for a constant K and using Stirling's formula yields

$$\mathbb{E}[N_k] \leq (\frac{(1 + o(1))e^2}{K})^k$$

which tends to 0 as $k \to \infty$ at exponential speed as soon as $K > e^2$. Thus, by Borel-Cantelli's Lemma,

COROLLARY 3.3 *If $K > e^2$, then with probability tending to 1 as $n \to \infty$, $N_k = 0$ for all $k \geq K(\log n)^{1/2}$.*

REFERENCES

1. J. E. COHEN, Cooperation and self-interest: Pareto-inefficiency of Nash equilibria in finite random games, *Proc. Natl. Acad. Sci. USA* **95** (1998), 9724-9731.

2. J. JONASSON, On the optimal strategy in a random game, *Electronic Comm. Probab.*, to appear. Can be found at http://www.math.chalmers.se/homepages/jonasson/recent.html.

3. G. OWEN, "Game Theory," 3rd edition, Academic Press, San Diego, 2001.

4. I. Y. POWERS, Limiting distributions of the number of pure strategy Nash equilibria in N-person games, *Internat. J. Game Theory* **19** (1990), 277-286.

5. Y, RINOTT AND M. SCARSINI, On the number of pure strategy Nash equilibria in random games, *Games Econom. Behav.* **33** (2000), 274-293.

6. W. STANFORD, A note on the probability of k pure Nash equilibria in matrix games, *Games Econom. Behav.* **9** (1995), 238-246.

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