Visibility to infinity in the hyperbolic plane, despite obstacles

Itai Benjamini* Johan Jonasson† Oded Schramm‡
Johan Tykesson§

Abstract

Suppose that $\mathcal{Z}$ is a random closed subset of the hyperbolic plane $\mathbb{H}^2$, whose law is invariant under isometries of $\mathbb{H}^2$. We prove that if the probability that $\mathcal{Z}$ contains a fixed ball of radius 1 is larger than some universal constant $p_0 < 1$, then there is positive probability that $\mathcal{Z}$ contains (bi-infinite) lines.

We then consider a family of random sets in $\mathbb{H}^2$ that satisfy some additional natural assumptions. An example of such a set is the covered region in the Poisson Boolean model. Let $f(r)$ be the probability that a line segment of length r is contained in such a set $\mathcal{Z}$. We show that if $f(r)$ decays fast enough, then there are a.s. no lines in $\mathcal{Z}$. We also show that if the decay of $f(r)$ is not too fast, then there are a.s. lines in $\mathcal{Z}$. In the case of the Poisson Boolean model with balls of fixed radius R we characterize the critical intensity for the a.s. existence of lines in the covered region by an integral equation.

We also determine when there are lines in the complement of a Poisson process on the Grassmannian of lines in $\mathbb{H}^2$.

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1 Introduction and main results

In this paper, we are interested in the existence of hyperbolic half-lines and lines (that is, infinite geodesic rays and bi-infinite geodesics respectively) contained

*Departments of Mathematics, The Weizmann Institute, Rehovot, Israel 76100. E-mail: itai.benjamini@weizmann.ac.il

†Department of Mathematical Sciences, Division of Mathematics, Chalmers University of Technology, S-41296 Göteborg, Sweden. E-mail: jonasson@math.chalmers.se. Research partially supported by the Swedish Natural Science Research Council.

‡Microsoft Research, One Microsoft Way, Redmond WA 98052, USA.

§Department of Mathematical Sciences, Division of Mathematical Statistics, Chalmers University of Technology, S-41296 Göteborg, Sweden. E-mail: johant@math.chalmers.se.
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in unbounded connected components of some continuum percolation models in the hyperbolic plane. Our first result is quite general:

Theorem 1.1. *Let $\mathcal{Z}$ be a random closed subset of $\mathbb{H}^2$, whose law is invariant under isometries of $\mathbb{H}^2$, and let B denote some fixed ball of radius 1 in $\mathbb{H}^2$. There is a universal constant $p_o < 1$ such that if $\mathbf{P}\big[B \subset \mathcal{Z}\big] > p_o$, then with positive probability $\mathcal{Z}$ contains hyperbolic lines.*

Of course, there is nothing special with taking B to be of radius 1 in 1.1; for any radius r there is a universal constant $p_o(r) < 1$ with the claimed property. The first result of this type was proven by O. Häggström [6] for regular trees of degree at least 3. That paper shows that for automorphism invariant site percolation on such trees, when the probability that a site is open is sufficiently close to 1, there are infinite open clusters with positive probability. This was subsequently generalized to transitive nonamenable graphs [2]. The proof of Theorem 1.1 is not too difficult, and is based on a reduction to the tree case.

We conjecture that Theorem 1.1 may be strengthened by taking $\mathcal{Z}$ to be open and replacing the assumption $\mathbf{P}\big[B \subset \mathcal{Z}\big] > p_o$ with $\mathbf{E}\big[\mathrm{length}(B \setminus \mathcal{Z})\big] < \delta$; see Conjecture 7.1 and the discussion which follows. Here $\mathrm{length}(A)$ stands for the length of the boundary of the set A.

We also obtain more refined results for random sets that satisfy a number of additional conditions. One example of such a set is the following. Consider a Poisson point process with intensity λ on a manifold M. In the *Poisson Boolean model of continuum percolation* with parameters λ and R, balls of radius R are centered around the points of the Poisson process. One then studies the geometry of the connected components of the union of balls, or the connected components of the complement. In particular, one asks for which values of the parameters there are unbounded connected components or a unique unbounded component.

In the setting of the Poisson Boolean model in the hyperbolic plane, Kahane [10, 11] showed that if $\lambda < 1/(2\sinh R)$, then the set of rays from a fixed point $o \in \mathbb{H}^2$ that are contained in the complement of the balls is non-empty with positive probability, while if $\lambda \geq 1/(2\sinh R)$ this set is empty a.s. Lyons [13] generalized the result of Kahane to d-dimensional complete simply-connected manifolds of negative curvature, and in the case of constant negative curvature also found the exact value of the critical intensity for the a.s. existence of rays.

In this note, we find not only rays but lines in the union of balls and/or its complement. We work mostly in the hyperbolic plane, but raise questions for other spaces as well. Our proofs cover the results of Kahane as well, but are also valid for a larger class of random sets. We remark that it is easy to see that in $\mathbb{R}^n$, there can never be rays in the union of balls or in the complement.

Other aspects of the Poisson Boolean model in $\mathbb{H}^2$ have previously been studied in [17]. For further studies of percolation in the hyperbolic plane, the reader may consult the papers [3, 12]. In [5], an introduction to hyperbolic geometry is found, and for an introduction to the theory of percolation on infinite graphs see, for example, [4, 14, 7].

Let $X = X_\lambda$ be the set of points in a Poisson process of intensity λ in $\mathbb{H}^2$. Let

$$\mathcal{B} := \bigcup_{x \in X} \overline{B(x, R)}$$

denote the *occupied set*, where $B(x, r)$ denotes the open ball of radius r centered at x. The closure of the complement

$$\mathcal{W} := \overline{\mathbb{H}^2 \setminus \mathcal{B}}$$

will be reffered to as the *vacant set*.

Let $\lambda_{\text{gc}} = \lambda_{\text{gc}}(R)$ denote the supremum of the set of $\lambda \geq 0$ such that for the parameter values (R, λ) a.s. $\mathcal{B}$ does not contain a hyperbolic line ("gc" stands for geodesic covered). Let $\bar{\lambda}_{\text{gc}}$ denote the supremum of the set of $\lambda \geq 0$ such that the probability that a fixed point $x \in \mathbb{H}^2$ belongs to a half-line contained in $\mathcal{B}$ is 0. Similarly let $\lambda_{\text{gv}} = \lambda_{\text{gv}}(R)$ denote the infimum of the set of $\lambda \geq 0$ such that for the parameter values (R, λ) a.s. $\mathcal{W}$ does not contain a hyperbolic line ("gv" stands for geodesic vacant). Finally, let $\bar{\lambda}_{\text{gv}}$ denote the infimum of the set of $\lambda \geq 0$ such that the probability that a fixed point $x \in \mathbb{H}^2$ belongs to a half-line contained in $\mathcal{W}$ is 0. Later, we shall see that $\lambda_{\text{gv}} = \bar{\lambda}_{\text{gv}}$ and $\lambda_{\text{gc}} = \bar{\lambda}_{\text{gc}}$. Clearly, if $\lambda > \lambda_{\text{gv}}$, there are a.s. no hyperbolic lines in $\mathcal{W}$ and if $\lambda < \lambda_{\text{gc}}$ there are a.s. no hyperbolic lines in $\mathcal{B}$. Let $f(r) = f_{R,\lambda}(r)$ denote the probability that a fixed line segment of length r in $\mathbb{H}^2$ is contained in $\mathcal{B}$.

Theorem 1.2. *For every $R > 0$, we have $0 < \lambda_{\text{gc}}(R) = \bar{\lambda}_{\text{gc}}(R) < \infty$, and the following statements hold at $\lambda_{\text{gc}}(R)$.*

1. *A.s. there are no hyperbolic lines within $\mathcal{B}$.*

2. *Moreover, $\mathcal{B}$ a.s. does not contain any hyperbolic ray (half-line).*

3. *There is a constant $c = c_R > 0$, depending only on R, such that*

$$c\, e^{-r} \leq f(r) \leq e^{-r}, \qquad \forall r > 0\,. \tag{1.1}$$

Furthermore, the analogous statements hold with $\mathcal{W}$ in place of $\mathcal{B}$ (with possibly a different critical intensity).

An equation characterizing λ_{gc} follows from our results (i.e., (4.1) with $\alpha = 1$).

The key geometric property allowing for geodesic percolation to occur for some λ is the exponential divergence of geodesics. This does not hold in Euclidean space. It is of interest to determine which homogeneous spaces admit a regime of intensities with geodesics percolating.

With regards to higher dimensions, we show that in hyperbolic space of any dimension $d \geq 3$ and for any $(R, \lambda) \in (0, \infty)^2$, there can never be planes contained in the covered or vacant region of the Poisson Boolean model.

We also consider a Poisson process Y on the Grassmannian of lines on $\mathbb{H}^2$. We show that if the intensity of Y is sufficiently small, then there are lines in

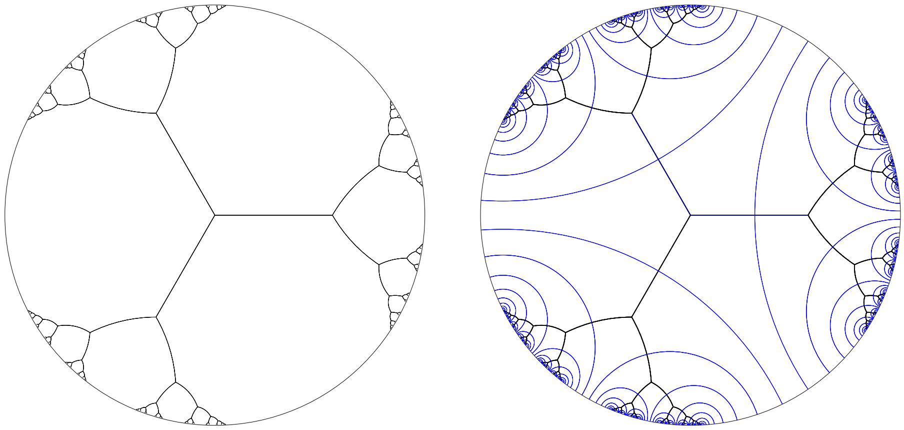

Figure 2.1: A tree embedded in the hyperbolic plane, in the Poincaré disk model. On the right appears the tree together with some of its lines of symmetry.

the complement of Y (when Y is viewed as a subset of $\mathbb{H}^2$), which means that Y is not connected. On the other hand, if the intensity is large enough, then the complement of Y contains no lines, which means that Y is connected. At the critical intensity, Y is connected.

Our paper ends with a list of open problems.

2 Lines appearing when the marginal is large

The proof of Theorem 1.1 is based on a reduction to the tree case. We will need the following construction of a tree embedded in $\mathbb{H}^2$, which is illustrated in Figure 2.1. (This construction should be rather obvious to the readers who are proficient in hyperbolic geometry.) Consider the hyperbolic plane in the Poincaré disk model. Let $o \in \mathbb{H}^2$ correspond to the center of the disk. Let A_0 be an arc on the unit circle of length smaller than $2\pi/3$. Let A_j denote the rotation of A_0 by $2\pi j/3$; that is $A_j := e^{2\pi j/3}A_0$, $j = 1, 2$. Let L_j, $j = 0, 1, 2$, denote the hyperbolic line whose endpoints on the ideal boundary $\partial\mathbb{H}^2$ are the endpoints of A_j. Let Γ denote the group of hyperbolic isometries that is generated by the reflections γ_0, γ_1 and γ_2 in the lines L_0, L_1 and L_2, respectively. If $w = (w_1, w_2, \dots, w_n) \in \{0, 1, 2\}^n$, then let γ_w denote the composition $\gamma_{w_1} \circ \gamma_{w_2} \circ \cdots \circ \gamma_{w_n}$. We will say that w is *reduced* if $w_{j+1} \neq w_j$ for $j = 1, 2, \dots, n-1$. A simple induction on n then shows that $\gamma_w(o)$ is separated from o by L_{w_1} when w is reduced and $n > 0$. In particular, for reduced $w \neq ()$, we have $\gamma_w(o) \neq o$ and $\gamma_w \neq \gamma_{()}$. Clearly, every γ_w where w has $w_j = w_{j+1}$ for some j is equal to $\gamma_{w'}$ where w' has these two consecutive elements of w dropped. It follows that Γ acts simply and transitively on the orbit Γo. ("Simply" means that $\gamma v = v$ where $\gamma \in \Gamma$ and $v \in \Gamma o$ implies that γ is the identity.) Now define a graph T on the vertex set Γo by letting each $\gamma(o)$ be connected by edges to the three points $\gamma \circ \gamma_j(o)$, $j = 0, 1, 2$. Then T is just the 3-regular tree embedded in the hyperbolic plane. In fact, this is a Cayley graph of the group Γ, since we may identify Γ with the orbit Γo. (One easily verifies that Γ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$.)

We will need a few simple properties of this embedding of the 3-regular tree in $\mathbb{H}^2$. It is easy to see that every simple path $v_0, v_1, \dots$ in T has a unique limit point on the ideal boundary $\partial\mathbb{H}^2$. (Figure 2.1 does not lie.) Moreover, if $v_0 = o$ and $v_1 = \gamma_j(o)$, then the limit point will be in the arc A_j. If $(v_j : j \in \mathbb{Z})$ is a bi-infinite simple path in T with $v_0 = o$, then its two limit points on the ideal boundary will be in two different arcs A_j. Hence, the distance from o to the line in $\mathbb{H}^2$ with the same pair of limit points on $\partial\mathbb{H}^2$ is bounded by some constant R, which does not depend on the path $(v_j : j \in \mathbb{Z})$. Invariance under the group Γ now shows that for every bi-infinite simple path β in T, the hyperbolic line L_β joining its limit points passes within distance R from each of the vertices of β. It follows that there is some constant $R' > 0$ such that L_β is contained in the R'-neighborhood of the set of vertices of β.

We are now ready to prove our first theorem.

Proof of Theorem 1.1. We use the above construction of T, Γ and the constant R'. Given $\mathcal{Z}$, let $\omega \subset V(T)$ denote the set of vertices $v \in V(T)$ such that the ball $B(v, R')$ is contained in $\mathcal{Z}$. Then ω is a (generally dependent) site percolation on T and its law is invariant under Γ. Set $q := \mathbf{P}\big[o \in \omega\big]$. By [2], there is some $p_0 \in (0, 1)$ such that if $q \geq p_0$, then ω has infinite connected components with positive probability. (We need to use [2], rather than [6], since the group Γ is not the full automorphism group of T.) Let N be the number of balls of radius 1 that are sufficient to cover $B(o, R')$. Now suppose that $\mathbf{P}\big[B(o, 1) \subset \mathcal{Z}\big] > 1 - (1 - p_0)/(2N)$. Then a sum bound implies that $q > (p_0 + 1)/2$. Therefore, if we intersect ω with an independent Bernoulli site percolation with marginal $p > (p_0+1)/2$, the resulting percolation will still have infinite components with positive probability, by the same argument as above. Thus, we conclude that with positive probability ω has infinite components with more than one end and therefore also bi-infinite simple paths. The line determined by the endpoints on $\partial\mathbb{H}^2$ of such a path will be contained in $\mathcal{Z}$, by the definition of R'. The proof is thus complete. $\square$

3 Lines in well-behaved percolation

The proofs of the statements in Theorem 1.2 concerning $\mathcal{B}$ are essentially the same as the proofs concerning $\mathcal{W}$. We therefore find it worthwhile to employ an axiomatic approach, which will cover both cases.

Definition 3.1. In the following, we fix a closed disk $B \subset \mathbb{H}^2$ of radius 1. A *well-behaved percolation* on $\mathbb{H}^2$ is a random closed subset $\mathcal{Z} \subset \mathbb{H}^2$ satisfying the following assumptions.

1. The law of $\mathcal{Z}$ is invariant under isometries of $\mathbb{H}^2$.

2. The set $\mathcal{Z}$ satisfies positive correlations; that is, for every pair g and h of bounded increasing measurable functions of $\mathcal{Z}$, we have

$$\mathbf{E}\big[g(\mathcal{Z})\, h(\mathcal{Z})\big] \geq \mathbf{E}\big[g(\mathcal{Z})\big]\, \mathbf{E}\big[h(\mathcal{Z})\big].$$

3. There is some $R_0 < \infty$ such that $\mathcal{Z}$ satisfies independence at distance R_0, namely, for every pair of closed subsets $A, A' \subset \mathbb{H}^2$ satisfying $\inf\{d(a,a') : a \in A, a' \in A'\} \geq R_0$, the intersections $\mathcal{Z} \cap A$ and $\mathcal{Z} \cap A'$ are independent.

4. The expected number m of connected components of $B \setminus \mathcal{Z}$ is finite.

5. The expected length ℓ of $B \cap \partial\mathcal{Z}$ is finite.

6. $p_0 := \mathbf{P}\big[B \subset \mathcal{Z}\big] > 0$.

Invariance under isometries implies that m, ℓ and p_0 do not depend on the position of B. We say that $\mathcal{Z}$ is Λ-well behaved, if it is well-behaved and $p_0, m^{-1}, \ell^{-1}, R_0^{-1} > \Lambda$. Many of our estimates below can be made to depend only on Λ. In the following, we assume that $\mathcal{Z}$ is Λ-well behaved, where $\Lambda > 0$, and use $O(g)$ to denote any quantity bounded by $c\, g$, where c is an arbitrary constant that may depend only on Λ. We remark that in condition 6 above it is not of any importance that the radius of B is 1. If for some $r > 0$ we have $\mathbf{P}\big[\overline{B(o,r)} \subset \mathcal{Z}\big] > 0$, then by positive correlations we have $\mathbf{P}\big[\overline{B(o,\tilde{r})} \subset \mathcal{Z}\big] > 0$ for every $\tilde{r} > 0$.

Measurability remark. It is not à priori obvious that the events mentioned in Theorems 1.1 and 1.2, condition 1-6 above and elsewhere are measurable. (Although all of these events are so natural in their formulation that the "should" be measurable.) For the random set $\mathcal{Z}$, most of the events are measurable either by their formulation or follow from the definition of the Fell topology for random closed sets in a fairly straightforward way. E.g. let L be a line segment and let $\{a_k\}$ be a countable dense set of points of L. Then

$$\{L \subseteq \mathcal{Z}\} = \cap_k \cap_n \{\mathcal{Z} \cap B(a_k, 1/n) \neq \emptyset\}.$$

However, conditions involving the length of $\partial\mathcal{Z}$ implicitly assume that the boundary of $\mathcal{Z}$ a.s. has a well-defined length. For e.g. the covered region or the closure of the vacant region of a Poisson process, this is obvious. Also, often measurability, although not immediatley obvious, follows from the proofs in that when is is shown that a set is contained in an (obviously measurable) event of probability 0, then the set itself is also measurable, at least after a completion of the probability space.

If $x, y \in \mathbb{H}^2$, let $[x,y]_s$ denote the union of all line segments $[x', y']$ where $d(x,x') < s$ and $d(y,y') < s$. Let $A(x,y,s)$ be the event that there is some connected component of $\mathcal{Z} \cap [x,y]_s$ that intersects $B(x,s)$ as well as $B(y,s)$, and let $Q(x,y,s)$ be the event that $[x,y]_s \subset \mathcal{Z}$. If $d(x,y)$ is large, the set $[x,y]_s$ becomes very thin, as is seen in the following lemma.

Lemma 3.2. *Let $0 < \epsilon < \epsilon_0 < \infty$ and $0 < t < \infty$. Let $\gamma : \mathbb{R} \to \mathbb{H}^2$ denote a hyperbolic line parameterized by arclength. There is a constant $c < \infty$ which depends only on ϵ_0 such that for $s \in (0, t)$ we have*

$$d(\gamma(s), \partial[\gamma(0), \gamma(t)]_\epsilon) \leq c\,\epsilon\, e^{-s\wedge(t-s)}. \tag{3.1}$$

Proof. In this proof we use the Poincaré unit disc model of $\mathbb{H}^2$. Suppose $s \in (0, t/2]$. Without loss of generality we assume that γ runs along the real line and that $\gamma(s) = (0, 0)$, $\gamma(0) = (x_1, 0)$ and $\gamma(t) = (x_2, 0)$ where $-1 < x_1 < 0$ and $0 < x_2 < 1$. Recall that hyperbolic balls are also Euclidean balls. Since $d(\gamma(0), (0, 0)) = s$, the Euclidean radius of $B(\gamma(0), \epsilon)$ is given by

$$r_\epsilon(\gamma(0)) = \frac{k_\epsilon(1 - \tanh(s/2)^2)}{1 - k_\epsilon^2 \tanh(s/2)^2}$$

where $k_\epsilon = \tanh(\epsilon/2)$ and the Euclidean radius of $B(\gamma(t), \epsilon)$ is given by

$$r_\epsilon(\gamma(t)) = \frac{k_\epsilon(1 - \tanh((t-s)/2)^2)}{1 - k_\epsilon^2 \tanh((t-s)/2)^2}.$$

Since $\tanh(x)$ increases to 1, $\tanh(x) \leq x$ for $x \geq 0$ and $1 - \tanh(x/2)^2 \leq 4\,e^{-x}$ we get

$$r_\epsilon(\gamma(0)) \leq \frac{4\,\epsilon\, e^{-s}}{1 - k_{\epsilon_0}^2}. \tag{3.2}$$

Let L be a geodesic line segment with one endpoint in $B(\gamma(0), \epsilon)$ and the other in $B(\gamma(t), \epsilon)$. In the Poincaré disc model, any such L belongs to a circle $C = C(L)$ which is perpendicular to the boundary of the unit disc. Consequently, the Euclidean distance between L and the real line is at most $\max(r_\epsilon(\gamma(0)), r_\epsilon(\gamma(t))$. Since $s \leq t - s$ we have $r_\epsilon(\gamma(0)) \geq r_\epsilon(\gamma(t))$ and therefore the Euclidean distance between $\gamma(s)$ and $\partial[\gamma(0), \gamma(t)]_\epsilon$ is at most $r_\epsilon(\gamma(0))$. Thus

$$d(\gamma(s), \partial[\gamma(0), \gamma(t)]_\epsilon) \leq \log \frac{1 + r_\epsilon(\gamma(0))}{1 - r_\epsilon(\gamma(0))} \leq C_1 r_\epsilon(\gamma(0)) \tag{3.3}$$

for some $C_1 = C_1(\epsilon_0) \in (1, \infty)$ where the first inequality comes from the relation between Euclidean and hyperbolic distance from the origin and the second inequality comes from the fact that $\epsilon \in (0, \epsilon_0]$. Now (3.1) follows from (3.2) and (3.3). $\square$

Lemma 3.3. *There is a constant $c = c(\Lambda) < \infty$, which depends only on Λ, such that for all $x, y \in \mathbb{H}^2$ satisfying $d(x, y) \geq 4$ and for all $\epsilon > 0$*

$$\mathbf{P}\big[Q(x, y, \epsilon)\big] > (1 - c\,\epsilon)\,\mathbf{P}\big[A(x, y, \epsilon)\big]. \tag{3.4}$$

Proof. Observe that the expected minimal number of disks of small radius ϵ that are needed to cover $\partial\mathcal{Z} \cap B$ is $O(\ell/\epsilon)$. It follows by invariance that

$$\mathbf{P}\big[B(x, \epsilon) \cap \partial\mathcal{Z} \neq \emptyset\big] = O(\epsilon)\,\ell = O(\epsilon) \tag{3.5}$$

holds for $x \in \mathbb{H}^2$.

Let $\gamma : \mathbb{R} \to \mathbb{H}^2$ denote a hyperbolic line parameterized by arclength, and let L_t denote the hyperbolic line through $\gamma(t)$ which is orthogonal to γ. Set

$$g(r,s) := \mathbf{P}\big[A\big(\gamma(0),\gamma(r),s\big) \setminus Q\big(\gamma(0),\gamma(r),s\big)\big].$$

By invariance, we have $\mathbf{P}\big[A(x,y,s) \setminus Q(x,y,s)\big] = g\big(d(x,y),s\big)$.

Set $B := B\big(\gamma(0),1\big)$. Fix some $\epsilon \in (0,1/10)$. Let S_j denote the intersection of B with the open strip between $L_{2j\epsilon}$ and $L_{2(j+1)\epsilon}$, where $j \in J := \mathbb{N} \cap [0,\epsilon^{-1}/10]$. Let x_j and y_j denote the two points in $L_{(2j+1)\epsilon} \cap \partial B$. Let J_1 denote the set of $j \in J$ such that S_j is not contained in $\mathcal{Z}$ but there is a connected component of $\mathcal{Z} \cap S_j$ that joins the two connected components of $S_j \cap \partial B$. Observe that the number of connected components of $B \setminus \mathcal{Z}$ is at least $|J_1| - 1$. Hence $\mathbf{E}\big[|J_1|\big] \le m+1$. Let J_2 denote the set of $j \in J$ such that $A(x_j,y_j,\epsilon) \setminus Q(x_j,y_j,\epsilon)$ holds. Note that if $j \in J_2 \setminus J_1$, then $\partial\mathcal{Z}$ is within distance $O(\epsilon)$ from $x_j \cup y_j$. Therefore, $\mathbf{P}\big[j \in J_2 \setminus J_1\big] = O(\epsilon)\,\ell$ holds for every $j \in J$, by (3.5). Consequently,

$$\mathbf{E}\big[|J_2|\big] \le \mathbf{E}\big[|J_2 \setminus J_1|\big] + \mathbf{E}\big[|J_1|\big] \le O(\epsilon)\,\ell\,|J| + m + 1 = O(1)\,.$$

Thus, there is at least one $j = j_\epsilon \in J$ satisfying

$$\begin{aligned} \mathbf{P}\big[A(x_j,y_j,\epsilon) \setminus Q(x_j,y_j,\epsilon)\big] &= \mathbf{P}\big[j \in J_2\big] \\ &\le O(1)/|J| = O(\epsilon)\,. \end{aligned} \tag{3.6}$$

Set $r_\epsilon := d(x_{j_\epsilon},y_{j_\epsilon})$, and note that $r_\epsilon \in (1,2]$. Now suppose that $x,y \in \mathbb{H}^2$ satisfy $d(x,y) = 2$. Let x_0 be the point in $[x,y]$ at distance r_ϵ from y, and let y_0 be the point in $[x,y]$ at distance r_ϵ from x. Observe that $A(x,y,\epsilon) \subset A(x_0,y,\epsilon) \cap A(x,y_0,\epsilon)$. Moreover, since $[x,y]_\epsilon \subset [x,y_0]_\epsilon \cup [x_0,y]_\epsilon$, we have $Q(x,y,\epsilon) \supset Q(x,y_0,\epsilon) \cap Q(x_0,y,\epsilon)$. Thus,

$$A(x,y,\epsilon) \setminus Q(x,y,\epsilon) \subset \big(A(x_0,y,\epsilon) \setminus Q(x_0,y,\epsilon)\big) \cup \big(A(x,y_0,\epsilon) \setminus Q(x,y_0,\epsilon)\big)$$

and therefore (3.6) and invariance gives

$$g(2,\epsilon) \le 2\,\mathbf{P}\big[A(x_{j_\epsilon},y_{j_\epsilon},\epsilon) \setminus Q(x_{j_\epsilon},y_{j_\epsilon},\epsilon)\big] = O(\epsilon)\,. \tag{3.7}$$

The same argument shows that

$$g(r',\epsilon) \le 2\,g(r,\epsilon), \qquad \text{if } 2 \le r < r' \le 2\,r\,. \tag{3.8}$$

We will now get a bound on $g(2\,k,\epsilon)$ for large $k \in \mathbb{N}$. For $j \in [k] := \mathbb{N} \cap [0,k]$, let r_j be the distance from $\gamma(2\,j)$ to the complement of $[\gamma(0),\gamma(2\,k)]_\epsilon$. Let $A_j := A\big(\gamma(2\,j),\gamma(2\,j+2),r_j \vee r_{j+1}\big)$, $Q_j := Q\big(\gamma(2\,j),\gamma(2\,j+2),r_j \vee r_{j+1}\big)$, where $j \in [k-1]$. Also set $\bar{A} := A\big(\gamma(0),\gamma(2\,k),\epsilon\big)$. Then

$$Q\big(\gamma(0),\gamma(2\,k),\epsilon\big) \supset \bigcap_{j=0}^{k-1} Q_j\,.$$

Hence,

$$g(2\,k,\epsilon) \leq \sum_{j=0}^{k-1} \mathbf{P}\big[\bar{A} \setminus Q_j\big]. \tag{3.9}$$

We now claim that

$$\mathbf{P}\big[\bar{A} \setminus Q_j\big] = O(1)\,\mathbf{P}\big[\bar{A}\big]\,\mathbf{P}\big[A_j \setminus Q_j\big], \tag{3.10}$$

where the implied constant depends only on p_0 and R_0. Let $j' := \lfloor j - R_0/2 - 2 \rfloor$ and $j'' := \lceil j + R_0/2 + 3 \rceil$. Suppose first that $j' > 0$ and $j'' < k$. Let $\bar{A}'(j)$ denote the event that $\mathcal{Z} \cap [\gamma(0), \gamma(2\,k)]_\epsilon$ contains a connected component that intersects both $B\big(\gamma(0),\epsilon\big)$ and $B\big(\gamma(2\,j'),\epsilon\big)$, and let $\bar{A}''(j)$ denote the event that $\mathcal{Z} \cap [\gamma(0), \gamma(2\,k)]_\epsilon$ contains a connected component that intersects both $B\big(\gamma(2\,j''),\epsilon\big)$ and $B\big(\gamma(2\,k),\epsilon\big)$. Then $\bar{A} \subset \bar{A}'(j) \cap \bar{A}''(j) \cap A_j$. Independence at distance R_0 therefore gives

$$\mathbf{P}\big[\bar{A} \setminus Q_j\big] \leq \mathbf{P}\big[\bar{A}'(j) \cap \bar{A}''(j)\big]\,\mathbf{P}\big[A_j \setminus Q_j\big].$$

Now note that $|j' - j''| = O(1)$. Consequently, $d(\gamma(2j'), \gamma(2j'')) < C$ where $C = O(1)$. Let Q be the event that the ball of radius C centered at $\gamma(2j')$ is contained in $\mathcal{Z}$. Positive correlations and condition 6 imply that $\mathbf{P}\big[Q\big] \geq 1/O(1)$ where $O(1)$ depends only on p_0. By positive correlations, we have

$$P[\bar{A}'(j) \cap \bar{A}''(j) \cap Q] \geq P[\bar{A}'(j) \cap \bar{A}''(j)]P[Q].$$

Since $\bar{A} \supset \bar{A}'(j) \cap \bar{A}''(j) \cap Q$, we get

$$\mathbf{P}\big[\bar{A}'(j) \cap \bar{A}''(j)\big] \leq O(1)\,\mathbf{P}\big[\bar{A}\big].$$

Thus, we get (3.10) in the case that $j' > 0$ and $j'' < k$. The general case is easy to obtain (one just needs to drop $\bar{A}'(j)$ or $\bar{A}''(j)$ from consideration). Now, (3.9) and (3.10) give

$$g(2\,k,\epsilon) \leq O(1)\,\mathbf{P}\big[\bar{A}\big] \sum_{j=0}^{k-1} g(2, r_j \vee r_{j+1})\,. \tag{3.11}$$

By Lemma 3.2, there is a universal constant $a \in (0,1)$ such that $r_j \leq a^{|j| \wedge |k-j|}\,O(\epsilon)$. Hence, we get by (3.7) and (3.11) that $g(2\,k,\epsilon) \leq O(1)\mathbf{P}\big[\bar{A}\big]\epsilon$, where the implied constant may depend on ℓ, m, R_0 and p_0. This proves (3.4) in the case where $d(x,y)$ is divisible by 2. The general case follows using (3.8) with $r' = d(x,y)$ and $r = 2\,\lfloor r'/2 \rfloor$. □

Let $f(r)$ denote the probability that a fixed line segment of length r is contained in $\mathcal{Z}$. Clearly,

$$\mathbf{P}\big[Q(x,y,s)\big] \leq f(\mathrm{length}[x,y]) \leq \mathbf{P}\big[A(x,y,s)\big],$$

and Lemma 3.3 shows that for s sufficiently small the upper and lower bounds are comparable.

Lemma 3.4. *There is a unique $\alpha \geq 0$ (which depends on the law of $\mathcal{Z}$) and some $c(\Lambda) > 0$ (depending only on Λ) such that*

$$c\, e^{-\alpha r} \leq f(r) \leq e^{-\alpha r} \tag{3.12}$$

holds for every $r \geq 0$.

Proof. Since the uniqueness statement is clear, we proceed to prove existence. Positive correlations imply that

$$f(r_1 + r_2) \geq f(r_1)\, f(r_2)\,, \tag{3.13}$$

that is, f is supermultiplicative. Therefore, $-\log f(r)$ is subadditive, and Fekete's Lemma says that we must have

$$\alpha := \lim_{r\to\infty} \frac{-\log f(r)}{r} = \inf_{r>0} \frac{-\log f(r)}{r}\,.$$

Since for every r we have $\alpha \leq -\log\big(f(r)\big)/r$, the right inequality in (3.12) follows.

On the other hand, if we fix some $R > R_0$, then independence at distance larger than R_0 gives

$$f(r_1)\, f(r_2) \geq f(r_1 + R + r_2) \overset{(3.13)}{\geq} f(r_1 + r_2)\, f(R)\,.$$

Dividing by $f(R)^2$, we find that the function $r \mapsto f(r)/f(R)$ is submultiplicative. Thus, by Fekete's lemma again,

$$\lim_{r\to\infty} \frac{\log\big(f(r)/f(R)\big)}{r} = \inf_{r>0} \frac{\log\big(f(r)/f(R)\big)}{r}\,.$$

The left hand side is equal to $-\alpha$, and we get for every $r > 0$

$$-\alpha \leq \frac{\log\big(f(r)/f(R)\big)}{r}\,.$$

By positive correlations, there is some $c = c(\Lambda) > 0$ such that $f(R) \geq c$, which implies the left inequality in (3.12). $\square$

Lemma 3.5. *If $\alpha \geq 1$ (where α is defined in Lemma 3.4), then a.s. there are no half-lines contained in $\mathcal{Z}$.*

Proof. Fix a basepoint $o \in \mathbb{H}^2$. Let $s = (2c)^{-1}$, where c is the constant in (3.4). Then

$$\mathbf{P}\big[A(x,y,s)\big]/2 \leq \mathbf{P}\big[Q(x,y,s)\big] \leq f\big(d(x,y)\big) \leq e^{-d(x,y)} \tag{3.14}$$

holds for every $x, y \in \mathbb{H}^2$ satisfying $d(x,y) \geq 4$. For every integer $r \geq 4$ let $V(r)$ be a minimal collection of points on the circle $\partial B(o,r)$ such that the disks

$B(z,s)$ with $z \in V$ cover that circle. Let X_r be the set of points $z \in V(r)$ such that $A(o,z,s)$ holds. By (3.14)

$$\mathbf{E}\big[|X_r|\big] \le 2\,|V(r)|\,f(r) = O(1)\,s^{-1}\,\mathrm{length}\big(\partial B(o,r)\big)\,e^{-r} = O(1)\,, \qquad (3.15)$$

since we are treating s as a constant and the length of $\partial B(o,r)$ is $O(e^r)$.

The rest of the argument is quite standard, and so we will be brief. By (3.15) and Fatou's lemma, we have $\liminf_{r\to\infty}|X_r| < \infty$ a.s. Now fix some large r and let $r' \in \mathbb{N}$ satisfy $r' > r + R_0 + 2$. Since $\mathcal{Z} \setminus B(o, r+R_0+1)$ is independent from $\mathcal{Z} \cap B(o,r)$, positive correlations implies that

$$\mathbf{P}\big[X_{r'} = \emptyset \;\big|\; \mathcal{Z} \cap B(o,r)\big] \ge p^{|X_r|}, \qquad (3.16)$$

where $p > 0$ is a constant (which we allow to depend on the law of $\mathcal{Z}$). Since $\liminf_{r\to\infty}|X_r| < \infty$ a.s., it follows by (3.16) that $\inf_r |X_r| = 0$ a.s., which means that $\max\{r : X_r \ne \emptyset\} < \infty$ a.s. Therefore, a.s. there is no half-line that intersects $B(o,s)$. Since $\mathbb{H}^2$ can be covered by a countable collection of balls of radius s, the lemma follows. $\square$

Lemma 3.6. *Suppose that $\alpha < 1$. Then (i) a.s. $\mathcal{Z}$ contains hyperbolic lines, (ii) for every fixed $x \in \mathbb{H}^2$, there is a positive probability that $\mathcal{Z}$ contains a half-line containing x, and (iii) for every fixed point x in the ideal boundary $\partial\mathbb{H}^2$ there is a.s. a geodesic line passing through x whose intersection with $\mathcal{Z}$ contains a half-line.*

Proof. We first prove (ii) using the second moment method. Fix some point $o \in \mathbb{H}^2$. Let A denote a closed half-plane with $o \in \partial A$, and let $I := A \cap \partial B(o,1)$. For $r > 1$ and $x \in \partial B(o,1)$, let $L_r(x)$ denote the line segment which contains x, has length r and has o as an endpoint. Set $Y_r := \{x \in I : L_r(x) \subset \mathcal{Z}\}$, and let y_r denote the length of Y_r. Then we have

$$\mathbf{E}\big[y_r\big] = \mathrm{length}(I)\,f(r)\,.$$

The second moment is given by

$$\mathbf{E}\big[y_r^2\big] = \int_I \int_I \mathbf{P}\big[x, x' \in Y_r\big]\,dx\,dx'\,.$$

Now note that if $r_2 > r_1 > 0$, then the distance from $L_{r_2}(x') \setminus L_{r_1}(x')$ to $L_{r_2}(x)$ is at least $(d(x,x')\,e^{r_1} \wedge r_1)/O(1)$. Consequently, by independence on sets at distance larger than R_0, we have

$$\mathbf{P}\big[x, x' \in Y_r\big] \le f(r)\,f\big((r + \log d(x,x') + O(1)) \vee 0\big).$$

Now applying the above and (3.12) gives

$$\begin{aligned}\frac{\mathbf{E}\big[y_r^2\big]}{\mathbf{E}\big[y_r\big]^2} &\le O(1)\int_I \int_I \exp\big(-\alpha\,\log d(x,x')\big)\,dx\,dx' \\ &= O(1)\int_I \int_I d(x,x')^{-\alpha}\,dx\,dx' = O(1)\,,\end{aligned}$$

since $\alpha < 1$. Therefore, the second moment method implies that

$$\inf_{r>1} \mathbf{P}\big[y_r > 0\big] > 0\,.$$

Since y_r is monotone non-increasing, it follows that

$$\mathbf{P}\big[\forall_{r>1}\ \ y_r > 0\big] > 0\,.$$

By compactness, on the event that $y_r > 0$ for all $r > 1$ we have $\bigcap_{r>1} Y_r \neq \emptyset$. If $x \in \bigcap Y_r$, then the half-line with endpoint o passing through x is contained in $\mathcal{Z} \cap A$. This proves (ii).

We now prove (i). Fix $s = 1/(2\,c)$, where c is given by Lemma 3.3. For $x \in \partial B(o,1)$ let $z_r(x)$ denote the endpoint of $L_r(x)$ that is different from o and let Y_r' be the set of points $x \in I$ such that $[z, z_r(x)] \subset \mathcal{Z}$ holds for every $z \in B(o,s)$. Let y_r' denote the length of Y_r'. Then $Y_r' \subset Y_r$ and therefore $y_r' \leq y_r$. By the choice of s, we have $\mathbf{E}\big[y_r'\big] \geq \mathbf{E}\big[y_r\big]/2$. On the other hand, $\mathbf{E}\big[(y_r')^2\big] \leq \mathbf{E}\big[y_r^2\big] = O(1)\,\mathbf{E}\big[y_r\big]^2$. As above, this implies that with positive probability $Y_\infty' := \bigcap_{r>1} Y_r' \neq \emptyset$. Suppose that $x \in Y_\infty'$. Let $\tilde{x}$ denote the endpoint on the ideal boundary $\partial\mathbb{H}^2$ of the half-line starting at o and passing through x. Then for every $z \in B(o,s)$ the half-line $[z, \tilde{x})$ is contained in $\mathcal{Z}$. By invariance and positive correlations, for every $\epsilon > 0$ there is positive probability that Y_∞' is within distance ϵ from each of the two points in $\partial A \cap I$. If x' and x'' are two points in Y_∞' that are sufficiently close to the two points in $\partial A \cap I$, then the hyperbolic line joining the two endpoints at infinity of the corresponding half-lines through o intersects $B(o,s)$. In such a case, this line will be contained in $\mathcal{Z}$. Thus, we see that for every line L (in this case ∂A) for every point $o \in L$ and for every $\epsilon > 0$, there is positive probability that $\mathcal{Z}$ contains a line passing within distance ϵ of the two points in $\partial B(o,1) \cap L$. Now (i) follows by invariance and by independence at a distance.

The proof of (iii) is similar to the above, and will be omitted. □

Remark 3.7. Let $o \in \mathbb{H}^2$. Let Y denote the set of points z in in the ideal boundary $\partial\mathbb{H}^2$ such that the half-line $[o, z)$ is contained in $\mathcal{Z}$. It can be concluded from the first and second moments computed in the proof of Lemma 3.6 and a standard Frostman measure argument that the essential supremum of the Hausdorff dimension of Y is given by

$$\|\dim_H(Y)\|_\infty = 1 - \alpha\,.$$

We conjecture that $\dim_H(Y) = 1 - \alpha$ a.s. on the event that $Y \neq \emptyset$.

A modification of the above arguments shows that there is positive probability that $\mathcal{Z}$ contains a line through o if and only if $\alpha < 1/2$. In case $\alpha < 1/2$, the essential supremum of the Hausdorff dimension of the set of lines in $\mathcal{Z}$ through o is $1 - 2\,\alpha$.

We believe that the Hausdorff dimension of the union of the lines in $\mathcal{Z}$ is a.s. $3 - 2\,\alpha$ when $\alpha \in [1/2, 1)$.

4 Boolean occupied and vacant percolation

Recall the definition of $\mathcal{B}$ and $\mathcal{W}$. First, we show that $\mathcal{B}$ and $\mathcal{W}$ are well-behaved.

Proposition 4.1. *Fix a compact interval $I \subset (0, \infty)$. Then there is some $\Lambda = \Lambda(I) > 0$ such that if $\lambda, R \in I$, then $\mathcal{B}$ and $\mathcal{W}$ are Λ-well behaved.*

Proof. It is well known that $\mathcal{B}$ and $\mathcal{W}$ satisfy positive correlations. For $\mathcal{W}$, m is bounded by the expected number of points in X that fall in the R-neighborhood of B. Observe that each connected component of $\mathcal{W} \cap B$, with the possible exception of one, has on its boundary an intersection point of two circles of radius R centered at points in X. Since the second moment of the number of points in X that fall inside the R-neighborhood of B is finite, it follows that m is also bounded for $\mathcal{B}$. The remaining conditions are easily verified and left to the reader. $\square$

We are now ready to prove one of our main theorems.

Proof of Theorem 1.2. We start by considering $\mathcal{B}$. Fix some $R \in (0, \infty)$. If we let $\lambda \nearrow \infty$, then $f(1) \nearrow 1$ and by (3.12) $\alpha \searrow 0$. Thus, Lemma 3.6 implies that $\lambda_{\mathrm{gc}} < \infty$. (We could alternatively prove this from Theorem 1.1.) It is also clear that $\lambda_{\mathrm{gc}} > 0$, since for λ sufficiently small a.s. $\mathcal{B}$ has no unbounded connected component.

Since the constant c in Lemma 3.4 depends only on Λ, that lemma implies that α is continuous in $(\lambda, R) \in (0, \infty)^2$. In particular, Lemmas 3.5 and 3.6 show that when $\lambda = \lambda_{\mathrm{gc}}(R)$, we have $\alpha = 1$ and that there are a.s. no half-lines in $\mathcal{B}$. Also, we get (1.1) from (3.12). Finally, it follows from Lemma 3.5 and Lemma 3.6 (ii) that $\lambda_{\mathrm{gc}} = \bar{\lambda}_{\mathrm{gc}}$. The proof for $\mathcal{W}$ is similar. $\square$

Next, we calculate α for $\mathcal{B}$ and $\mathcal{W}$.

Lemma 4.2. *The value of α for line percolation in $\mathcal{W}$ is given by*

$$\alpha = 2\,\lambda \sinh R\,.$$

Proof. Consider a line $\gamma : \mathbb{R} \to \mathbb{H}^2$, parameterized by arclength, and let $r > 0$. A.s. the interval $\gamma[0, r]$ is contained in $\mathcal{W}$ if and only if the R-neighborhood of the interval does not contain any points of X. Let N denote this neighborhood, and let A denote its area. Then $f(r) = e^{-\lambda A}$. For each point $z \in \mathbb{H}^2$, let t_z denote the t minimizing the distance from z to $\gamma(t)$. Then $N = N_0 \cup N_1 \cup N_2$, where $N_0 := \{z \in \mathbb{H}^2 : d(z, \gamma(t_z)) < R, t_z \in [0, r]\}$, $N_1 := \{z \in B(\gamma(0), R) : t_z < 0\}$ and $N_2 := \{z \in B(\gamma(r), R) : t_z > r\}$. Observe that N_1 and N_2 are two half-disks of radius R, so that their areas are independent of r. We can conveniently calculate the area of N_0 explicitly in the upper half-plane model for $\mathbb{H}^2$, for which the hyperbolic length element is given by $|ds|/y$, where $|ds|$ is the Euclidean length element. We choose $\gamma(t) = (0, e^t)$. Recall that the intersection of the upper half-plane with the Euclidean circles orthogonal to the real line are lines in this model. It is easy to see that for $z = (\rho \cos \theta, \rho \sin \theta)$, we have $\gamma(t_z) = (0, \rho)$. Moreover, the distance from z to γ is

$$\Big|\int_\theta^{\pi/2} \frac{\rho\, d\psi}{\rho \sin\psi}\Big| = \big|\log\tan(\theta/2)\big|.$$

Thus, if we choose $\theta \in (0, \pi/2)$ such that $\tan(\theta/2) = e^{-R}$, then N_0 consists of the set $\big\{(\rho\cos\psi, \rho\sin\psi) : \rho \in [1, e^r], \psi \in (\theta, \pi - \theta)\big\}$. Thus,

$$\operatorname{area}(N_0) = \int_\theta^{\pi-\theta}\int_1^{e^r} \frac{\rho\, d\rho\, d\psi}{\rho^2 \sin^2\psi} = 2\, r\cot\theta = r(\cot\tfrac{\theta}{2} - \tan\tfrac{\theta}{2}) = 2\, r \sinh R\,.$$

The result follows. □

From Lemma 4.2 we see that for $\mathcal{W}$ we have $\alpha = 1$ when $\lambda = 1/(2\sinh(R))$, which means $\lambda_{\mathrm{gv}}(R) = 1/(2\sinh(R))$.

Remark 4.3. Let $\lambda_{\mathrm{c}}(R)$ be the infimum of the set of intensities $\lambda \geq 0$ such that $\mathcal{B}$ contains unbounded components a.s. Proposition 4.7 in [17] says that for R large, $\lambda_{\mathrm{c}}(R) \leq Ke^{-2R}$ for some constant K which means $\lambda_{\mathrm{gv}}(R) > \lambda_{\mathrm{c}}(R)$ for R large. Theorem 4.1 in [17], therefore implies that for R large, there are intensities for which there are lines in $\mathcal{W}$, but also infinitely many unbounded components in both $\mathcal{W}$ and $\mathcal{B}$. On the other hand, by Lemma 4.6 in [17] we have $\lambda_{\mathrm{c}}(R) \geq 1/(2\pi(\cosh(2R) - 1))$. Therefore, for R small enough, we have $\lambda_{\mathrm{c}}(R) > \lambda_{\mathrm{gv}}(R)$. So for R small, there are no intensities for which lines in $\mathcal{W}$ coexist with unbounded components in $\mathcal{B}$. Moreover, Theorem 4.2 in [17] says that when there are no unbounded components in $\mathcal{B}$, there is a unique unbounded component in $\mathcal{W}$. Therefore, for R small, there are intensities for which there is a unique unbounded component in $\mathcal{W}$, but still no lines in $\mathcal{W}$.

Lemma 4.4. *In the setting of line percolation in $\mathcal{B}$, α is the unique solution of the equation*

$$1 = \int_0^{2R} e^{\alpha t}\, H'_{R,\lambda}(t)\, dt\,, \tag{4.1}$$

where

$$H_{R,\lambda}(t) := -\exp\left(-4\,\lambda\int_0^{t/2} \sinh\left(\cosh^{-1}\left(\frac{\cosh R}{\cosh s}\right)\right) ds\right).$$

Proof. Consider a line $\gamma : \mathbb{R} \to \mathbb{H}^2$, parameterized by arclength. Recall that X is the underlying Poisson process. We now derive an integral equation satisfied by

$$f(r) = \mathbf{P}\big[\gamma[0, r] \subset \mathcal{B}\big].$$

For a point x in the R-neighborhood of γ, let $u_+(x) := \sup\{s : \gamma(s) \in B(x, R)\}$ and $u_-(x) := \inf\{s : \gamma(s) \in B(x, R)\}$. Let $X_0 := \{x \in X : u_-(x) < 0 < u_+(x)\}$. This is the set of $x \in X$ such that $\gamma(0) \in B(x, R)$. Also set

$$S := \begin{cases} \inf\{u_+(x) : x \in X_0\} & X_0 \neq \emptyset, \\ -\infty & X_0 = \emptyset\,. \end{cases}$$

Assume that $r \geq 2R$. A.s., if $S = -\infty$, then $\gamma[0,r]$ is not contained in $\mathcal{B}$. On the other hand, if we condition on $S = s$, where $s \in (0, 2R)$ is fixed, then $\gamma[0,s) \subset \mathcal{B}$ and the conditional distribution of $\gamma[s,r] \cap \mathcal{B}$ is the same as the unconditional distribution. (Of course, $S = s$ has probability zero, and so this conditioning should be understood as a limit.) Therefore, we get

$$\mathbf{P}\big[\gamma[0,r] \subset \mathcal{B} \bigm| S\big] = f(r - S), \tag{4.2}$$

where, of course, $f(\infty) = 0$.

Let $G(t) := \mathbf{P}\big[S \in (0,t)\big]$. Shortly, we will show that $G(t) = H_{R,\lambda}(t) + 1$. But presently, we just assume that $G'(t)$ is continuous and derive (4.1) with G in place of H. Since the probability density for S in $(0, 2R)$ is given by $G'(t)$, we get from (4.2)

$$f(r) = \int_0^{2R} f(r - s)\, G'(s)\, ds\,. \tag{4.3}$$

Suppose that $\beta > 0$ satisfies

$$1 = \int_0^{2R} e^{\beta s}\, G'(s)\, ds\,. \tag{4.4}$$

Since $\int_0^{2R} G'(s)\, ds = \mathbf{P}\big[S > 0\big] < 1$, continuity implies that there is some such β. Suppose that there is some $r > 0$ such that $f(r) \leq e^{-\beta r} f(2R)$, then let r_0 be the infimum of all such r. Clearly, $r_0 \geq 2R$. By the definition of r_0 and (4.3), we get

$$f(r_0) > \int_0^{2R} e^{-\beta(r_0 - s)}\, f(2R)\, G'(s)\, ds \overset{(4.4)}{=} e^{-\beta r_0}\, f(2R)\,.$$

Since $f(r)$ is continuous on $(0,\infty)$, this contradicts the definition of r_0. A similar contradiction is obtained if one assumes that there is some $r > 0$ satisfying $f(r) \geq e^{-\beta(r-2R)}$. Hence $e^{-\beta r} f(2R) \leq f(r) \leq e^{-\beta(r-2R)}$, which gives $\alpha = \beta$.

It remains to prove that $G(t) = H_{R,\lambda}(t) + 1$. Let $Q_t := B\big(\gamma(0),R\big) \setminus B\big(\gamma(t),R\big)$. Observe that

$$G(t) = \mathbf{P}\big[X \cap Q_t \neq \emptyset\big] = 1 - \mathbf{P}\big[X \cap Q_t = \emptyset\big] = 1 - e^{-\lambda\ \mathrm{area}(Q_t)}. \tag{4.5}$$

Hence, we want to calculate $\mathrm{area}(Q_t)$. For $z \in \mathbb{H}^2$ let $u(z)$ denote the $t \in \mathbb{R}$ that minimizes $d\big(z, \gamma(t)\big)$, and let $\phi(t,y)$ denote the point in $\mathbb{H}^2$ satisfying $u(z) = t$ which is at distance y to the left of γ if $y \geq 0$, or $-y$ to the right of γ otherwise. Observe that $\big\{z \in B\big(\gamma(0),R\big) : u(z) < -t/2\big\}$ is isometric to (see Figure 4.1)

$$\big\{z \in B\big(\gamma(t),R\big) : u(z) < t/2\big\} = \big\{z \in B\big(\gamma(0),R\big) : u(z) < t/2\big\} \setminus \overline{Q_t}\,.$$

Therefore,

$$\mathrm{area}(Q_t) = \mathrm{area}\big\{z \in B\big(\gamma(0),R\big) : u(z) \in [-t/2, t/2]\big\}. \tag{4.6}$$

By the hyperbolic Pythagorian theorem, we have

$$\cosh d\big(\gamma(0), \phi(s,y)\big) = \cosh s\, \cosh y\,.$$

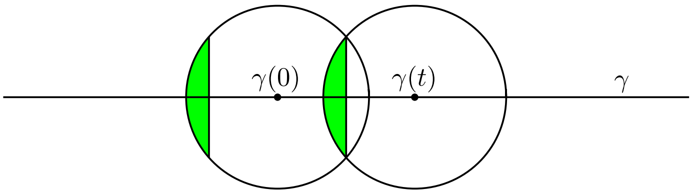

Figure 4.1: Calculating the area of Q_t. The set Q_t is the left ball minus the right ball. The area is calculated by first exchanging the left cap by its "shift".

Hence, the set on the right hand side of (4.6) is

$$\big\{\phi(s,y) : s \in [-t/2, t/2], \cosh y \le \cosh R / \cosh s\big\}. \tag{4.7}$$

At the end of the proof of Lemma 4.2, we saw that the area of a set of the form $\big\{\phi(s,y) : s \in [0,r], |y| \le R\big\}$ is $2\,r\,\sinh R$. Hence, the area of (4.7) (and also the area of Q_t) is given by

$$\int_{-t/2}^{t/2} 2\,\sinh\big(\cosh^{-1}(\cosh R/\cosh s)\big)\,ds\,.$$

The result follows by (4.4) and (4.5), since $\alpha = \beta$. $\square$

5 No planes in higher dimensions

It is natural to ask for high dimensional variants. Fix some $d \in \mathbb{N}$, $d > 2$. Let $\lambda, R > 0$. Let $\mathcal{B} := \bigcup_{x \in X} B(x,R)$, where X is a Poisson point process of intensity λ in $\mathbb{H}^d$. Let $\mathcal{W}$ be the closure of $\mathbb{H}^d \setminus \mathcal{B}$. A 2-dimensional plane $L \subset \mathbb{H}^d$ is a set which is isometric to $\mathbb{H}^2$. The Grassmannian of planes in $\mathbb{H}^d$ is the space of all planes in $\mathbb{H}^d$. To the Grassmannian it is possible to assign an invariant measure (a volume measure), which will be denoted by Φ. We assume that Φ is normalized so that the set of planes that intersect $B(o,2)$ has Φ-measure 1. If L is a plane that contains o, then the probability that $L \cap B(o,r)$ is contained in $\mathcal{B}$ (or $\mathcal{W}$) decays much faster than the probability that a line segment of length r is contained in $\mathcal{B}$ (or $\mathcal{W}$). Therefore, it is reasonable to guess that every plane intersects both $\mathcal{B}$ and $\mathcal{W}$.

Proposition 5.1. *For every $d \in \mathbb{N} \cap [3,\infty)$, $\lambda, R > 0$, a.s. there are no 2-dimensional planes in $\mathbb{H}^d$ that are contained in $\mathcal{B}$. Similarly, there are no 2-dimensional planes in $\mathbb{H}^d$ that are contained in $\mathcal{W}$.*

Proof. Let $\mathcal{Z}$ be $\mathcal{B}$. Fix some $o \in \mathbb{H}^d$, and let $r > 0$ be large. Let Y_r be the set of planes L intersecting the ball $B(o,2)$ such that $L \cap B(o,r)$ is contained in the 1-neighborhood of $\mathcal{Z}$. Let Z_r be the set of planes L intersecting $B(o,1)$ such that $L \cap B(o,r) \subset \mathcal{Z}$. If $L \in Z_r$, then Y_r contains the set of planes L' such that the Hausdorff distance between $L \cap B(o,r)$ and $L' \cap B(o,r)$ is less than 1. Consequently,

$$\mathbf{E}[\Phi(Y_r)|Z_r \neq \emptyset] \geq \exp\bigl(-O(r)\bigr). \tag{5.1}$$

Now fix a plane L that intersects $B(o,2)$. If the points x, y fulfill $d(x,y) \geq 2R+4$, the event that x belongs to the 1-neighborhood of $\mathcal{Z}$ is independent of the corresponding event for y. Since there are order e^r points in $L \cap B(o,r)$ such that the distance between any two is larger than $2R+4$, we get that

$$\mathbf{P}\bigl[L \in Y_r\bigr] \leq \exp(-c\, e^r)$$

for some constant $c = c(d,R,\lambda) > 0$. This means that

$$\mathbf{E}[\Phi(Y_r)] \leq \exp(-c\, e^r). \tag{5.2}$$

From (5.1) and (5.2) we see that $\mathbf{P}[Z_r \neq \emptyset] \to 0$ as $r \to \infty$. Since $\mathbb{H}^d$ can be covered by a countable collection of balls of radius 1, it follows that a.s. there are no planes contained in $\mathcal{Z}$. The case $\mathcal{Z} = \mathcal{W}$ is proved in the same way. $\square$

6 Connectivity of lines

In this section, we consider a somewhat different model using a Poisson process on the Grassmannian $\mathbb{G}$ of lines in $\mathbb{H}^2$. For this purpose, we first recall the form of an isometry-invariant measure on $\mathbb{G}$. Consider the upper half-plane model for $\mathbb{H}^2$. Let $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} = \partial\mathbb{H}^2$ denote the boundary at infinity of $\mathbb{H}^2$. To each unoriented line $L \subset \mathbb{H}^2$ we may associate the pair of points of L on the boundary at infinity $\hat{\mathbb{R}}$. This defines a bijection between $\mathbb{G}$ and

$$\mathbb{M} := \bigl\{\{x,y\} : x, y \in \hat{\mathbb{R}}, x \neq y\bigr\}.$$

(Though we will not use this fact, $\mathbb{M}$ is an open Möbius band, or a punctured projective plane.) In the following, we often identify $\mathbb{M}$ and $\mathbb{G}$ via this bijection, and will not always be careful to distinguish between them.

The set $\mathbb{M}$ inherits a locally Euclidean metric coming from the 2 to 1 projection from $\hat{\mathbb{R}} \times \hat{\mathbb{R}} \setminus$ diagonal. Let Φ be the measure on $\mathbb{M}$ whose density at a point $\{x,y\} \in \mathbb{M}$ such that $x, y \neq \infty$ with respect to the Euclidean area measure is

$$d\Phi = \frac{dx\, dy}{(x-y)^2},$$

and $\Phi\bigl(\bigl\{\{x,\infty\} : x \in \mathbb{R}\bigr\}\bigr) = 0$. An isometry $\psi : \mathbb{H}^2 \to \mathbb{H}^2$ induces a map $\mathbb{G} \mapsto \mathbb{G}$. In the upper half plane coordinate system, each such ψ is a transformation of the form $z \mapsto (a\,z+b)/(c\,z+d)$, with $a,b,c,d \in \mathbb{R}$ and $a\,d - b\,c \neq 0$. Moreover, ψ extends to a self-homeomorphism of $\hat{\mathbb{R}}$, and therefore there is an induced map from $\mathbb{M}$ to $\mathbb{M}$. It is easy to verify using the integration change of variables formula that ψ preserves the measure Φ. Hence, Φ is an isometry-invariant measure on $\mathbb{G}$.

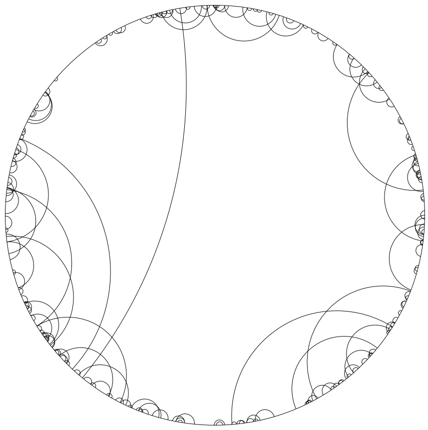

Figure 6.1: A realization of a Poisson process on the Grassmannian of lines in $\mathbb{H}^2$ in the Poincaré disk model.

Let Y be a Poisson point process of intensity λ on $\mathbb{G}$ with respect to Φ. With a slight abuse of notation, we will also write Y for the union of all lines in Y, when viewed as a subset of $\mathbb{H}^2$. Let $\mathcal{Z}$ be the complement of Y. Observe that a.s. Y is connected if and only if $\mathcal{Z}$ contains no lines.

Proposition 6.1. *If $\lambda \geq 1$ then $\mathcal{Z}$ contains no lines a.s. If $\lambda < 1$ then $\mathcal{Z}$ contains lines a.s.*

One motivation for this model comes from long range percolation on $\mathbb{Z}$. Fix some $c < 1$. For each pair $x, y \in \mathbb{Z}$, let there be an edge between x and y with probability c (independently for different pairs) if there is a line in Y with one endpoint in $[x, x+1]$ and the other in $[y, y+1]$. Then a calculation shows that if $\lambda = 1$ (which is the critical value), the probability that there is an edge between x and y is asymptotic to $c/|x-y|^2$ as $|x-y| \to \infty$, that is, we have recovered the standard long range percolation model on $\mathbb{Z}$ with critical exponent 2 (see [1]). The critical case of long range percolation is not well understood and it might be of interest to further study the connection between it and the line process.

Observe that $\mathcal{Z}$ is not a well-behaved percolation (in the sense of our definition 3.1), since there is no independence at any distance, and moreover, $\mathcal{Z}$ is open. Therefore, several statements in Section 3 cannot be used directly to prove Proposition 6.1. Nevertheless, it is possible to adapt the proofs without much difficulty.

Proof. First, we calculate $f(r)$. Let $\gamma(t) = (0, e^t) \in \mathbb{H}^2$, where we think of $\mathbb{H}^2$ in the upper half plane model. Let A_r be the set of lines that intersect $\gamma[0, r]$. Then it is easy to see that under the identification $\mathbb{G} = \mathbb{M}$,

$$A_r = \big\{\{x, y\} \,:\, 1 \leq -xy \leq e^{2r}\big\}.$$

An easy calculation shows that $\Phi(A_r) = r$. Therefore, $f(r) = e^{-\lambda\Phi(A_r)} = e^{-\lambda r}$.

It will now be convenient to use the Poincaré disk model. An unoriented line in $\mathbb{H}^2$ in the Poincaré disk model corresponds to an unordered pair of distinct points on the unit circle, $x, y \in \partial\mathbb{H}^2$. Thus, the measure we have on the Grassmannian induces a measure on $(\partial\mathbb{H}^2)^2$. By using an isometry between the hyperbolic plane in the upper half plane model and the hyperbolic plane in the Poincaré disk model it is easy to verify that the density of this measure is again given locally by $|x-y|^{-2}$ times the product of the length measure on the circle with itself.

Suppose that $0 < \lambda < 1$. For $\theta \in [0, 2\pi)$ and $r > 0$ let $L_r(\theta)$ denote the geodesic ray of hyperbolic length r started from 0 whose continuation meets $\partial\mathbb{H}^2$ at $e^{i\theta}$. Let K_r be the set of $\theta \in [0, 2\pi)$ such that $L_r(\theta) \subset \mathcal{Z}$. Then $\mathbf{P}\big[\theta \in K_r\big] = e^{-\lambda r}$. To apply the second moment method, we need to estimate $\mathbf{P}\big[\theta, \theta' \in K_r\big]$ from above for $\theta, \theta' \in [0, 2\pi)$. Suppose first that $\theta' = 0$ and $\theta \in [0, \pi]$. Let $L(\theta)$ be the hyperbolic line $L_\infty(\theta) \cup L_\infty(\theta + \pi)$, which contains $L_r(\theta)$. The set of pairs $\{x, y\} \in (\partial\mathbb{H}^2)^2$ such that the line connecting them intersects both $L(\theta)$ and $L(\theta')$ is precisely that set of pairs that are separated by these two lines. The measure of this set is

$$\int_0^\theta \int_\pi^{\pi+\theta} + \int_\theta^\pi \int_{\pi+\theta}^{2\pi} \big|e^{i\alpha} - e^{i\beta}\big|^{-2}\, d\alpha\, d\beta = -2\log\frac{\sin\theta}{2}\,.$$

The measure of the set of lines that intersect both $L_r(\theta)$ and $L_r(0)$ is bounded by the measure of the set of lines that intersect both $L_\infty(0)$ and $L_\infty(\theta)$, which is bounded by half the measure calculated above. The measure of the set of lines that intersect $L_r(\theta) \cup L_r(0)$ is the sum of the measures of the lines intersecting each of these segments minus the measure of the set of lines intersecting both. Thus, it is at least

$$2\,r + \log\frac{\sin\theta}{2}\,.$$

This gives

$$\mathbf{P}\big[\theta, \theta' \in K_r\big] \le \Big(\frac{2}{\sin|\theta - \theta'|}\Big)^{\lambda} e^{-2\lambda r}\,,$$

and by symmetry this will also hold if we drop the assumptions that $\theta' = 0$ and $\theta \in [0, \pi]$. Since $\sin^{-\lambda}\theta$ is integrable when $\lambda < 1$, this facilitates the second moment argument, which shows that $\inf_{r>0} \mathbf{P}\big[K_r \ne \emptyset\big] > 0$. Let $K := \bigcap_{r>0} \overline{K}_r$. Then $\mathbf{P}\big[K \ne \emptyset\big] = \inf_{r>0} \mathbf{P}\big[K_r \ne \emptyset\big] > 0$, because $K_r \supset K_{r'}$ when $r' > r$. Now note that a.s. $\partial K_r \cap \overline{K}_{r'} = \emptyset$ when $r' > r$. (The set ∂K_r consists of points in the intersection of Y with the circle of hyperbolic radius r about 0.) Hence $K \subset K_r$ holds a.s. for each $r > 0$. Thus, with positive probability there will be some ray $L_\infty(\theta)$ that is contained in $\mathcal{Z}$. Clearly, this implies that with positive probability there are at least three rays corresponding to angles $\theta \in (0, \pi)$. Since the interior of the convex hull of the union of such rays is in $\mathcal{Z}$, it follows that $\mathcal{Z}$ contains lines with positive probability. Since the Poisson line process Y is ergodic (which is easy to verify), any event which is determined by Y and is invariant under $\mathrm{Isom}(\mathbb{H}^2)$ has probability 0 or 1. Consequently there are lines in $\mathcal{Z}$ a.s. when $\lambda < 1$.

We now consider the case $\lambda \geq 1$ and show that in this case there are a.s. no lines contained in $\mathcal{Z}$. In that case, we can follow the proof of Lemma 3.5 with only minor modifications. For $x, y \in \mathbb{H}^2$ and $s > 0$ let $\bar{A}(x, y, s)$ be the event that there is some line in $\mathcal{Z}$ which intersects both $B(x, s)$ and $B(y, s)$. For $x, y \in \mathbb{H}^2$ with $d(x, y) \geq 4$ it is not hard to show that the set of lines that separate the hyperbolic 1-ball around x from the hyperbolic 1-ball around y has Φ-measure $d(x, y) - O(1)$. Each such line will obviously intersect any line meeting both these balls. Thus, the probability that there is a line in $\mathcal{Z}$ meeting both these balls is $e^{-d(x,y)+O(1)}$. This means that for $d(x, y) \geq 4$ we have

$$\mathbf{P}[\bar{A}(z, z', 1)] = e^{-d(x,y)+O(1)}. \tag{6.1}$$

Equation (6.1) is the analog of (3.14).

The next detail requiring modification is that in the proof of (3.16) independence at a distance was mentioned. Let r be large and let $x \in \partial B(o, r)$. Let $\mathcal{L}(o, x)$ be the set of lines that pass through both $B(o, 1)$ and $B(x, 1)$. Let $r' > r + 4$. It is easy to see that set of lines that are disjoint from $B(o, r)$ and intersect every line in $\mathcal{L}(o, x)$ somewhere in $B(o, r') \setminus B(o, r + 1)$ has Φ-measure bounded away from 0. The lines in Y that intersect $B(o, r + 1)$ are independent of the set of lines in Y that do not intersect $B(o, r + 1)$. Consequently, if $\mathcal{Z}$ contains geodesic line segments that intersect both $B(o, 1)$ and $B(x, 1)$, there is probability bounded away from 0 that none of these line segments can be extended to a line segment which also reaches $\partial B(o, r')$ without hitting some line in Y. From this, we can deduce that the analog of (3.16) holds in our setting. With the analogs of (3.14) and (3.16) established, the argument is then completed as in Lemma 3.5. $\square$

7 Further Problems

We first consider a generalization of Theorem 1.1.

Conjecture 7.1. *Let $B \subset \mathbb{H}^2$ be some fixed open ball of radius 1. There is a constant $\delta > 0$ such that if $\mathcal{Z} \subset \mathbb{H}^2$ is any open random set with isometry-invariant law and $\mathbf{E}\big[\mathrm{length}(B \setminus \mathcal{Z})\big] < \delta$, then with positive probability $\mathcal{Z}$ contains a hyperbolic line.*

It is easy to verify that the conjecture implies the theorem. Indeed, if $\mathbf{P}\big[B \subset \mathcal{Z}\big]$ is close to 1, then one can show that there is a union of unit circles whose law is isometry invariant, where the interiors cover the complement of $\mathcal{Z}$, and where the expected length of the intersection of the circles with B is small.

Next, we consider quantitative aspects of Theorem 1.1.

Conjecture 7.2. *Fix some $o \in \mathbb{H}^2$. For every $r > 0$ let p_r be the least $p \in [0, 1]$ such that for every random closed $\mathcal{Z} \subset \mathbb{H}^2$ with an isometry-invariant law and*

$\mathbf{P}\big[B(o,r) \subset \mathcal{Z}\big] > p$ *there is positive probability that* $\mathcal{Z}$ *contains a hyperbolic line. Theorem 1.1 implies that* $p_r < 1$ *for every* $r > 0$*. We conjecture that* $\limsup_{r \searrow 0}(1 - p_r)/r < \infty$.

It is easy to see that $\liminf_{r \searrow 0}(1 - p_r)/r > 0$; for example, take a Poisson point process $X \subset \mathbb{H}^2$ with intensity λ sufficiently large and let $\mathcal{Z}$ be the complement of the ϵ-neighborhood of $\bigcup_{x \in X} \partial B(x, 1)$, where $0 < \epsilon < r$.

Problem 7.3. *Does the limit* $\lim_{r \searrow 0}(1 - p_r)/r$ *exist? If it does, what is its value?*

The behavior of p_r as $r \to \infty$ seems to be an easier problem, though potentially of some interest as well.

We now move on to problems related to Theorem 1.2 and its proof.

Question 7.4. For either $\mathcal{W}$ or $\mathcal{B}$, is there some pair (λ, R) for which there is with positive probability a percolating ray such that every other percolating ray with the same endpoint at infinity is contained in it? (Note, such a ray must be exceptional among the percolating rays.)

Question 7.5. Is it true that when $\mathcal{B}$ (or $\mathcal{W}$) has a unique infinite connected component, the union of the lines in $\mathcal{B}$ (or $\mathcal{W}$) is connected as well? We believe that there is some pair (λ, R) such that $\mathcal{B}$ contains a unique infinite component but no lines (we know this for $\mathcal{W}$, see Remark 4.3).

Question 7.6. For which homogenous spaces $\mathcal{W}$ or $\mathcal{B}$ a.s. contain infinite geodesics for some parameters (λ, R)?

Note that since $\mathbb{H}^2 \times \mathbb{R}$ contains $\mathbb{H}^2$, it follows that for every R there is some λ such that $\mathcal{W}$ on $\mathbb{H}^2 \times \mathbb{R}$ contains lines within an $\mathbb{H}^2$ slice, and the same holds for $\mathcal{B}$.

Question 7.7. Let V be the orbit of a point $x \in \mathbb{H}^2$ under a group of isometries Γ. Suppose that V is discrete and $\mathbb{H}^2/\Gamma$ is compact. (E.g., V is a co-compact lattice in $\mathbb{H}^2$.) Let $\mathcal{W}_V(R) := \mathbb{H}^2 \setminus \bigcup_{v \in V} B(v, R)$, and let R_c^V denote the supremum of the set of R such that $\mathcal{W}_V(R)$ contains uncountably many lines. Does $\mathcal{W}_V(R_c^V)$ contain uncountably many lines?

It might be interesting to determine the value of R_c^V for some lattices V.

Problem 7.8. *It is not difficult to adapt our proof to show that in* $\mathbb{H}^d$*,* $d \geq 2$*, for every* $R > 0$ *when* λ *is critical for the existence of lines in* $\mathcal{W}$*, there are a.s. no lines inside* $\mathcal{W}$*. This should also be true for* $\mathcal{B}$*, but we presently do not know a proof. It seems that what is missing is an analog of Lemma 3.3.*

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References

[1] I. Benjamini and N. Berger, *The diameter of long-range percolation clusters on finite cycles*, Rand. Struct. Alg. **19**:2 (2001), 102-111.

[2] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm, *Group-invariant percolation on graphs*, Geom. Funct. Anal. **9** (1999), 29-66.

[3] I. Benjamini and O. Schramm, *Percolation in the hyperbolic plane*, J. Amer. Math. Soc. **14** (http://uppsalainitiativet.blogspot.com/2001), 487-507.

[4] I. Benjamini and O. Schramm, *Percolation beyond $\mathbb{Z}^d$, many questions and a few answers*, Electronic Commun. Probab. **1** (1996), 71-82.

[5] J.W. Cannon, W.J. Floyd, R. Kenyon and W.R. Parry. Hyperbolic geometry. In *Flavors of geometry*, pp. 59-115, Cambridge University Press, 1997.

[6] O. Häggström, *Infinite clusters in dependent automorphism invariant percolation on trees*, Ann. Probab. **25** (1997), 1423-1436.

[7] O. Häggström and J. Jonasson, *Uniqueness and non-uniqueness in percolation theory*, Probability Surveys **3** (2006), 289-344.

[8] J. Jonasson, *Dynamical circle covering with homogeneous Poisson updating,* (2007), Statist. Probab. Letters **78** (2008), 2400-2403.

[9] J. Jonasson and J. Steif, *Dynamical models for circle covering*, Ann. Probab. **36** (2008), 739-764.

[10] J.-P. Kahane, *Recouvrements altoires et thorie du potentiel*, Colloq. Math. **60/61** (1990), 387-411.

[11] J.-P. Kahane, *Produits de poids alatoires indpendants et application*, in *Fractal Geometry and Analysis*, ed. J. Belair and S. Dubuc, Kluwer Acad. Publ., Dordrecht (1991), pp. 277-324.

[12] S. Lalley, *Percolation Clusters in Hyperbolic Tessellations*, Geom. Funct. Anal. **11** (2001), 971-1030.

[13] R. Lyons, *Diffusion and Random Shadows in Negatively-Curved Manifolds*, J. Funct. Anal. **138** (1996), 426-448.

[14] R. Lyons with Y. Peres, *Probability on trees and networks*, (2007), preprint. http://mypage.iu.edu/ rdlyons/prbtree/prbtree.html

[15] R. Meester and R. Roy, *Continuum Percolation*, Cambridge University Press, New York, 1996.

[16] L.A. Shepp, *Covering the circle with random arcs*, Israel J. Math. 11 (1972), 328-345.

[17] J. Tykesson, *The number of unbounded components in the Poisson Boolean model of continuum percolation in hyperbolic space*, Electron J. Prob. 12 (2007), 1379-1401.