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Mean square convergence of a semidiscrete scheme for SPDEs of Zakai type driven by square integrable martingales

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Abstract

In this short note, a direct proof of L^2 convergence of an Euler–Maruyama approximation of a Zakai equation driven by a square integrable martingale is shown. The order of convergence is as known for real-valued stochastic differential equations and for less general driving noises $O(\sqrt{\Delta t})$ for a time discretization step size Δt.

Keywords: Euler–Maruyama scheme, stochastic partial differential equations, Zakai equation, numerical scheme, Lévy process, mean square convergence
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1. Introduction

A large amount of literature on the numerical study of real-valued stochastic differential equations (SDEs) exists (cf. [1], [2]), while Hilbert space valued stochastic differential equations (or SPDEs) have just been treated in recent years. SDEs appear in various models in financial mathematics. However, in the last years an increasing number of problems have surfaced for which infinite dimensional noise seems to be more appropriate, and which are then modeled by SPDEs such as interest rate modeling ([3], [4]) and energy markets ([5]). To calculate prices one needs a numerical approximation of infinite dimensional Hilbert space valued stochastic differential equations.

For a numerical treatment of SPDEs, which will be seen in the more general framework of Hilbert space valued SDEs, approximation has to be done in space and time. There are various approaches possible. In this paper we study a semidiscrete Euler–Maruyama scheme which approximates the solution of a stochastic partial differential equation of the form

$$du_t = (A + B)u_t \, dt + G(u_t) \, dM_t, \qquad u_0 = v, \tag{1.1}$$

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in time. The generalization of this approach to fully discrete schemes is currently under investigation. Here M is a — not necessarily continuous — square integrable martingale with values in a separable Hilbert space U. Probably the most popular examples of such stochastic processes are Wiener and Lévy processes. The operators A and B act on a separable Hilbert space H, where A is generator of a C_0 semigroup of contractions which is a necessary condition below, and the operator G is a mapping from H into the linear operators from U to H.

In the literature Euler–Maruyama schemes are approximations of the stochastic integral of an SDE which are derived from the Itô–Taylor expansion [1]. For the simplest example of a real-valued SDE

$$dX_t = X_t\, dB_t, \qquad X_0 = x$$

with Brownian motion B, it is given by

$$X^{j+1} = (1 + \Delta B_j)X^j$$

and converges of order $\mathrm{O}(\sqrt{\Delta t})$ where Δt denotes the time discretization step size. The elements ΔB_j are the normal distributed increments of the Brownian motion. For an SPDE as introduced here but driven by a Q–Wiener process a similar time discretization was derived in [6] and [7]. The scheme introduced in these papers converges in L^2 and almost surely of order $\mathrm{O}(\sqrt{\Delta t})$ and has the iterative form

$$X^{j+1} = S_{\Delta t}X^j + S_{\Delta t}BX^j\,\Delta t + S_{\Delta t}G\big(X^j\big)\,(W(t_{j+1}) - W(t_j)),$$

where S denotes the semigroup generated by A. There, higher order schemes are also developed. In this paper we will use a similar scheme but the driving noise is a square integrable martingale. Furthermore, in [7] and [8] the authors proved almost sure convergence by Chebyshev's inequality and the Borel–Cantelli lemma, giving estimates in L^p for $p > 2$. These estimates implied L^2 convergence immediately but the methods did not work for L^2 estimates. Here, we present an approach to prove L^2 convergence directly for a much simpler scheme. It is subject of future work to generalize this idea to fully discrete approximations and higher order schemes. The reason why we look at L^2 convergence instead of L^p for $p > 2$ is that for non-continuous martingales Burkholder–Davis–Gundy type inequalities that preserve the known orders of convergence for continuous martingales do not exist and therefore worse orders of L^2 convergence will be achieved. This can be seen by looking at the moments of a Poisson process for $p \geq 2$. For small times t they all behave like t independently of the moment. So far, to the knowledge of the author, there do not exist results for almost sure convergence for Zakai's equation with that type of noise.

The type of equation studied in this paper appears naturally in the study of filtering problems with Zakai's equation (cf. [9]). Fully discrete approximations of its solution were already studied in [6], while a semidiscrete time approximation with higher order of convergence was presented in [7] and a Galerkin–Milstein approximation was done in [8]. Zakai's classical nonlinear filtering problem transformed to an SPDE and extended to square integrable martingales is given by

$$du_t(x) = L^* u_t(x)\, dt + G(u_t(x))\, dM_t(x) \tag{1.2}$$

on a bounded domain $D \subset \mathbb{R}^d$ with zero Dirichlet boundary conditions on ∂D and initial condition $u_0(x) = v(x)$. The operator L^* is a second order elliptic differential operator of the form

$$L^* u = \frac{1}{2}\sum_{i,j=1}^{d} \partial_i\partial_j a_{ij}u - \sum_{i=1}^{d}\partial_i f_i u$$

for $u \in C_c^2(D)$ and it can be split into the operators A and B in Equation (1.1). This will be done explicitly in Section 2. Originally the operator G denotes a pointwise multiplication with a suitable function $g \in H$. This is included in the more general assumptions on G in Equation (1.1) which will be introduced in detail in the next section.

The work is organized as follows: Section 2 sets up the framework including the properties and the regularity of the SPDEs to be approximated. In Section 3 the time discretization scheme is introduced and the main result that the approximation converges of order $O(\sqrt{\Delta t})$ in L^2 is shown. Finally, the last section presents future work that should be done using the presented estimation methods.

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2. Framework

Let H denote the Hilbert space $L^2(D)$, where $D \subset \mathbb{R}^d$ is a bounded domain with piecewise smooth boundary ∂D. We are interested in developing a numerical algorithm to approximate the solution of equation

$$du_t = (A + B)u_t\,dt + G(u_t)\,dM_t \tag{2.1}$$

on the finite time interval $[0, T]$ with initial condition $u_0 = v$ and zero Dirichlet boundary conditions on ∂D. M is a square integrable martingale — not necessarily continuous — on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with values in a separable Hilbert space $(U, (\cdot, \cdot)_U)$. The space of all square integrable martingales on U with respect to $(\mathcal{F}_t)$ is denoted by $\mathcal{M}^2(U)$. We restrict ourselves to the following class of square integrable martingales

$$\mathcal{C} := \{M \in \mathcal{M}^2(U) : \exists\, Q \in L_1^+ \text{ such that } \forall t \geq s \geq 0,\ \langle\!\langle M, M\rangle\!\rangle_t - \langle\!\langle M, M\rangle\!\rangle_s \leq (t - s)Q\},$$

where L_1^+ denotes the space of all nuclear, symmetric, positive-definite operators. The operator angle bracket process $\langle\!\langle M, M\rangle\!\rangle_t$ is defined as

$$\langle\!\langle M, M\rangle\!\rangle_t = \int_0^t Q_s\, d\langle M, M\rangle_s,$$

where $\langle M, M\rangle_t$ denotes the unique angle bracket process from the Doob–Meyer decomposition. The process $(Q_s, s \geq 0)$ is called the martingale covariance.

Since $Q \in L_1^+(U)$, there exists an orthonormal basis $(e_n, n \in \mathbb{N})$ of U consisting of eigenvectors of Q. Therefore we have the representation $Qe_n = \gamma_n e_n$, where $\gamma_n \geq 0$ is the eigenvalue corresponding to e_n. The square root of Q is defined as

$$Q^{1/2}x := \sum_n (x, e_n)_U\, \gamma_n^{1/2}\, e_n, \qquad x \in U$$

and $Q^{-1/2}$ is the pseudo inverse of $Q^{1/2}$. Let us denote by $(\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$ the Hilbert space defined by $\mathcal{H} = Q^{1/2}(U)$ endowed with the inner product $(x, y)_{\mathcal{H}} = (Q^{-1/2}x, Q^{-1/2}y)_U$ for $x, y \in \mathcal{H}$.

In what follows we introduce a generalization of the Itô "isometry" for square integrable martingales of class $\mathcal{C}$, where $L_{HS}(\mathcal{H}, H)$ refers to the space of all Hilbert–Schmidt operators from $\mathcal{H}$ to H and $\|\cdot\|_{L_{HS}(\mathcal{H},H)}$ denotes the corresponding norm. Let the space of integrands be given by $\mathbb{L}^2_{\mathcal{H},T}(H) := L^2(\Omega \times [0, T], \mathcal{P}_{[0,T]}, \mathbb{P} \otimes d\lambda; L_{HS}(\mathcal{H}, H))$, where $\mathcal{P}_{[0,T]}$ denotes the σ–field of predictable sets in $\Omega \times [0, T]$ and $d\lambda$ is the Lebesgue measure, then by Equation (1.6) in [10], we have, as a generalization of Proposition 8.16 in [11], for every $X \in \mathbb{L}^2_{\mathcal{H},T}(H)$ an Itô type inequality

$$\mathbb{E}\big(\sup_{0\le t\le T} \| \int_0^t X_s \, dM_s\|_H^2\big) \le C\,\mathbb{E}\Big(\int_0^T \|X_s\|^2_{L_{HS}(\mathcal{H},H)}\, ds\Big). \tag{2.2}$$

For a full introduction to Hilbert space valued stochastic differential equations we refer the reader to [11, 12, 13].

The operators A and B in Equation (2.1) are derived from L^* in Equation (1.2). We assume that the functions a_{ij}, for $i, j = 1, \ldots, d$, are twice continuously differentiable on D with continuous extension to the closure $\bar{D}$. The operator A is the unique self-adjoint extension of the differential operator

$$\sum_{i,j=1}^{d} \partial_i(a_{ij}\,\partial_j u), \qquad u \in C_c^2(D),$$

where ∂_i denotes the derivative in the ith coordinate direction of $\mathbb{R}^d$ and $C_c^2(D)$ is the space of all twice continuously differentiable functions on D with compact support, to the second order Sobolev space H_0^2 with elements satisfying zero Dirichlet boundary conditions. B is a first order differential operator given by

$$Bu := \sum_{i=1}^{d} \partial_i(b_i u), \qquad u \in C_c^1(D),$$

for f continuously differentiable on D with continuous extension to $\bar{D}$, with elements b_i that are defined as

$$b_i := \frac{1}{2}\sum_{j=1}^{d} \partial_j\, a_{ij} - f_i.$$

With the following assumptions the right hand side of Equation (2.1) is well defined.

Assumptions 2.1. The coefficients of A and B and the initial condition u_0 satisfy the following conditions:

(a) a_{ij} and f_i, $i, j = 1, \ldots, d$ belong to the space of all twice continuously differentiable functions with bounded derivatives on D denoted by $C_b^2(D)$ with continuous extension to $\bar{D}$,
(b) $\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge \delta\|\xi\|^2_{\mathbb{R}^d}$ for all $x \in D$ and $\xi \in \mathbb{R}^d$,
(c) u_0 is $\mathcal{F}_0$-measurable and $\mathbb{E}(\|u_0\|^2_{H^1}) < +\infty$.

Assumption 2.1(b) implies that the operator A is dissipative, see e.g. [14]. Then by the Lumer–Phillips theorem, e.g. [15], A generates a strongly continuous contraction semigroup on H which we denote by $S = (S_t, t \ge 0)$. By Corollary 2 in [16], S is analytic in the right half-plane. Therefore fractional powers of A are well defined, cf. [15], and we denote for simplicity reasons $A_{-\alpha} = (-A)^{-\alpha}$ and $A_\alpha = A_{-\alpha}^{-1}$ for $\alpha > 0$. With this notion we make the following assumptions:

Assumptions 2.1 (cont'd)**.** The operator G satisfies for a constant $C \in \mathbb{R}_+$ the following conditions:

(d) $\|G(\phi)\|_{L_{HS}(\mathcal{H},H)} \le C(1 + \|\phi\|_H)$ for $\phi \in H$,
(e) $\|G(\phi) - G(\psi)\|_{L_{HS}(\mathcal{H},H)} \le C\|\phi - \psi\|_H$ for $\phi, \psi \in H$,
(f) $\|A_{1/2}G(\phi)\|_{L_{HS}(\mathcal{H},H)} \le C\,(1 + \|\phi\|_{H^1})$ for $\phi \in H^1$.

We shall make use of the following lemma — whose statement is also known as *Kato's conjecture* — and which was proved in [17].

Lemma 2.2. *The domain of $A_{1/2}$ satisfies that $\mathcal{D}(A_{1/2}) = H_0^1$, where H_0^m denotes the Sobolev space of order m with zero Dirichlet boundary conditions, and the norm $\|A_{1/2}\cdot\|_H$ is equivalent to $\|\cdot\|_{H^1}$, i.e. there exists $C > 0$ such that*

$$\|A_{1/2}\,\phi\|_H \leq C\,\|\phi\|_{H^1} \;\; and \;\; \|\phi\|_{H^1} \leq C\,\|A_{1/2}\,\phi\|_H$$

for all $\phi \in H^1$.

Definition 2.3. Let u_0 be a $\mathcal{F}_0$-measurable square integrable random variable with values in H. A predictable process $u : \mathbb{R}_+ \times \Omega \to H$ is called a *mild solution* to (2.1), if

$$\sup_{t\in[0,T]} \mathbb{E}(\|u_t\|_H^2) < +\infty$$

for all $T \in (0, +\infty)$, and if for all $t > 0$

$$u_t = S_t u_0 + \int_0^t S_{t-s} B u_s\, ds + \int_0^t S_{t-s} G(u_s)\, dM_s. \tag{2.3}$$

It follows from Assumptions 2.1 that these integrals are well defined and Equation (2.1) has a unique mild solution by results in Chapter 9 of [11]. Furthermore we have similarly to [18], [19], [20], [10] that Equation (2.2) implies for all $X \in \mathbb{L}^2_{\mathcal{H},T}(H)$

$$\mathbb{E}\big(\sup_{0\leq t\leq T} \|\int_0^t S_{t-s} X_s\, dM_s\|_H^2\big) \leq C\,\mathbb{E}\Big(\int_0^T \|X_s\|^2_{L_{HS}(\mathcal{H},H)}\, ds\Big). \tag{2.4}$$

To simplify the notation we introduce the following norm for a mapping Φ from $[0,T]\times\Omega$ into H with finite p-th moment for fixed $p \geq 1$

$$\|\Phi\|_{p,H,T} := \Big(\mathbb{E}\big(\sup_{0\leq t\leq T} \|\Phi(t)\|_H^p\big)\Big)^{1/p}.$$

The next Lemma provides some insight on the regularity of the mild solution.

Lemma 2.4. *Under Assumptions 2.1 the mild solution satisfies $\|u\|_{2,H^1,T} < +\infty$.*

Proof. From here on C denotes a constant that may vary from line to line.

$$\begin{aligned}
\|u\|^2_{u,H^1,T} &= \|S_t u_0 + \int_0^t S_{t-s} B u_s\, ds + \int_0^t S_{t-s} G(u_s)\, dM_s\|^2_{2,H^1,T} \\
&\leq C\Big(\mathbb{E}(\|u_0\|^2_{H^1}) + \|\int_0^t S_{t-s} B u_s\, ds\|^2_{2,H^1,T} + \|A_{1/2}\int_0^t S_{t-s} G(u_s)\, dM_s\|^2_{2,H,T}\Big) \\
&\leq C\,\Big(\mathbb{E}(\|u_0\|^2_{H^1}) + \mathbb{E}\big(\sup_{0\leq t\leq T}\big(\int_0^t \|S_{t-s} B u_s\|_{H^1}\, ds\big)^2\big) \\
&\qquad\qquad + \mathbb{E}\big(\int_0^T \|A_{1/2} G(u_s)\|^2_{L_{HS}(\mathcal{H},H)}\, ds\big)\Big) \\
&\leq C\,\Big(\mathbb{E}(\|u_0\|^2_{H^1}) + \mathbb{E}\big(\sup_{0\leq t\leq T}\big(\int_0^t (t-s)^{-1/2}\|u_s\|_{H^1}\, ds\big)^2\big) + \mathbb{E}\big(\int_0^t (1+\|u_s\|_{H^1})^2\, ds\big)\Big) \\
&\leq C\,\Big(\mathbb{E}(\|u_0\|^2_{H^1}) + T + \int_0^T (1 + (T-s)^{-1/2})\|u\|^2_{2,H^1,s}\, ds\Big) \\
&\leq C\,(1 + \mathbb{E}(\|u_0\|^2_{H^1})) < +\infty,
\end{aligned}$$

where we used the property of the contraction semigroup and Equation (2.4) in the second step, Lemma 2.2, Corollary 1.4 in [21] and the definition of the Bochner integral in the third one, and Hölder's and Gronwall's inequality in the fourth. The last steps are shown in more detail in the proof of the main result. □

The regularity of the solution is given in the following lemma, where we set, for an H-valued stochastic process on $[0,T]$ with finite p-th moment

$$\|\Phi\|_{p,H,r,R} = \Big(\mathbb{E}\big(\sup_{r\le t\le R} \|\Phi(t)\|_H^p\big)\Big)^{1/p}.$$

Lemma 2.5. *If u is the mild solution of Equation* (2.3)*, then for* $0 \le r \le R \le T$

$$\|u - u_r\|^2_{2,H,r,R} \le C(1 + \|u\|^2_{2,H^1,T})\,(R-r).$$

Proof. Let

$$\phi(t) = \int_0^t S_{t-s} B u_s \, ds$$

and

$$\psi(t) = \int_0^t S_{t-s} G(u_s)\, dM_s,$$

then we estimate employing Assumptions 2.1, Theorem 2.6.13 in [21], Equation (2.4), and Lemma 2.2

$$\begin{aligned}
\|u - u_r\|^2_{2,H,r,R} &\le 4\Big(\|(S - S(r))u_0\|^2_{2,H,r,R} + \|(S(\cdot - r) - \mathbb{1})(\phi(r) + \psi(r))\|^2_{2,H,r,R} \\
&\qquad + \|\phi - \phi(r)\|^2_{2,H,r,R} + \|\psi - \psi(r)\|^2_{2,H,r,R}\Big) \\
&\le C\left(\mathbb{E}(\|A_{1/2}u_0\|^2_H) + 1 + \|u\|^2_{2,H,T} + \|u\|^2_{2,H^1,T}\right)(R-r) \\
&\le C(1 + \|u\|^2_{2,H^1,T})\,(R-r).
\end{aligned}$$

□

3. Discretization and main result

In this section we give a time discretization of Euler–Maruyama type that approximates Equation (2.1). This semidiscrete scheme gives the main idea how to approach L^2 convergence if L^p convergence for $p > 2$ is not available or not optimal due to the properties of the not necessarily continuous square integrable martingale as driving noise. Generalizations to higher order schemes as in [7] and fully discrete schemes as in [8] will be possible with this approach and are subject to future work.

We shall always consider a finite time horizon: $t \in [0,T]$ with $T < +\infty$. Let $\mathcal{T} = (\mathcal{T}_m, m \in \mathbb{N})$ be a sequence of partitions $\mathcal{T}_m$, $m \in \mathbb{N}$, of the interval $[0,T]$ whose mesh Δ_m tends to zero as m tends to $+\infty$. We write $\mathcal{T}_m$ as $\{t_0^m, t_1^m, \dots, t_{n_m}^m\}$ with $n_m \in \mathbb{N}$, $0 = t_0^m < t_1^m < \cdots < t_{n_m}^m = T$, and

$$\Delta_m = \max_i (t_{i+1}^m - t_i^m),$$

the maximum being taken over $i \in \{0, \dots, n_m - 1\}$. For $m \in \mathbb{N}$, we define the map $\pi_m : [0,T] \to \{t_i^m,\ i = 0, \dots, n_m\}$ by $\pi_m(s) = t_i^m$ if $t_i^m \le s < t_{i+1}^m$.

Then we define a time discretization of Equation (2.1) by

$$u_t^m = S_t u_0 + \int_0^t S_{t-\pi_m(s)} B u^m_{\pi_m(s)}\, ds + \int_0^t S_{t-\pi_m(s)} G(u^m_{\pi_m(s)})\, dM_s. \tag{3.1}$$

The corresponding recursive scheme is given by

$$u^{i+1} = S_{\Delta^i} u^i + \Delta^i S_{\Delta^i} B u^i + S_{\Delta^i} G(u^i)(M_{t^m_{i+1}} - M_{t^m_i})$$

with $u^i = u^m_{t^m_i}$ and $\Delta^i = t^m_{i+1} - t^m_i$.

This Euler–Maruyama type scheme converges in L^2 of order $\mathrm{O}(\sqrt{\Delta_m})$ as for similar schemes for SDEs (cf. [1]) and for less general driving noises (cf. [6], [7], [8]):

Theorem 3.1. *Under Assumptions 2.1, there exists a constant $C \in \mathbb{R}_+$ depending on the properties of the solution of the SPDE* (2.1) *and T, such that for all $m \in \mathbb{N}$*

$$\|u - u^m\|^2_{2,H,T} \leq C\, \Delta_m.$$

Proof. To prove this theorem we split the mild form of the SPDE (2.3) and the discretization scheme given by (3.1) as follows:

$$u_t - u_t^m = \xi^m(t) + \eta^m(t)$$

with

$$\xi^m(t) = \int_0^t S_{t-s} B u_s\, ds - \int_0^t S_{t-\pi_m(s)} B u^m_{\pi_m(s)}\, ds = \xi_1^m(t) + \xi_2^m(t) + \xi_3^m(t),$$
$$\eta^m(t) = \int_0^t S_{t-s} G(u_s)\, dM_s - \int_0^t S_{t-\pi_m(s)} G(u^m_{\pi_m(s)})\, dM_s = \eta_1^m(t) + \eta_2^m(t) + \eta_3^m(t),$$

where

$$\xi_1^m(t) = \int_0^t (S_{t-s} - S_{t-\pi_m(s)}) B u_s\, ds,$$
$$\xi_2^m(t) = \int_0^t S_{t-\pi_m(s)} B(u_s - u_{\pi_m(s)})\, ds,$$
$$\xi_3^m(t) = \int_0^t S_{t-\pi_m(s)} B(u_{\pi_m(s)} - u^m_{\pi_m(s)})\, ds$$

and

$$\eta_1^m(t) = \int_0^t (S_{t-s} - S_{t-\pi_m(s)}) G(u_s)\, dM_s,$$
$$\eta_2^m(t) = \int_0^t S_{t-\pi_m(s)} (G(u_s) - G(u_{\pi_m(s)}))\, dM_s,$$
$$\eta_3^m(t) = \int_0^t S_{t-\pi_m(s)} (G(u_{\pi_m(s)}) - G(u^m_{\pi_m(s)}))\, dM_s.$$

Next we give estimates on the six different expressions separately. We start with ξ_1^m and apply first the properties of the Bochner integral:

$$\|\xi_1^m\|^2_{2,H,T} \leq \mathbb{E}\Big(\sup_{0\leq t\leq T}\Big(\int_0^t \|(\mathbb{1} - S_{s-\pi_m(s)}) S_{t-s} B u_s\|_H\, ds\Big)^2\Big).$$

The properties of the semigroup in Theorem 2.6.13 in [21] lead to

$$\begin{aligned}\|\xi_1^m\|_{2,H,T}^2 &\le C\,\mathbb{E}\big(\sup_{0\le t\le T}\big(\int_0^t (s-\pi_m(s))^{1/2}\|A_{1/2}S_{t-s}Bu_s\|_H\,ds\big)^2\big)\\ &\le C\Delta_m\,\mathbb{E}\big(\sup_{0\le t\le T}\big(\int_0^t (t-s)^{-1/2}\|Bu_s\|_H\,ds\big)^2\big),\end{aligned}$$

where C denotes a constant varying from line to line and independent of m. Finally, we take the supremum of $\|Bu_s\|_H$ and get

$$\|\xi_1^m\|_{2,H,T}^2 \le C\Delta_m\,\big(\int_0^T (T-s)^{-1/2}\,ds\big)^2\,\|Bu\|_{2,H,T}^2\big) \le C\,4T\,\|u\|_{2,H^1,T}^2\,\Delta_m.$$

For the second term we use again Theorem 2.6.13 in [21] and split the singularity in zero. Hölder's inequality is applied and leads to

$$\begin{aligned}\|\xi_2^m\|_{2,H,T}^2 &\le C\,\mathbb{E}\big(\sup_{0\le t\le T}\big(\int_0^t (t-\pi_m(s))^{2\cdot(-1/4)}\|u_s-u_{\pi_m(s)}\|_H\,ds\big)^2\big)\\ &\le C\,\int_0^T (T-\pi_m(s))^{-1/2}\,ds\,\mathbb{E}\big(\sup_{0\le t\le T}\int_0^t (t-\pi_m(s))^{-1/2}\|u_s-u_{\pi_m(s)}\|_H^2\,ds\big)\\ &\le C\,2\sqrt{T}\,\sum_{i=1}^{n_m}\|u-u_{t_{i-1}^m}\|_{2,H,t_{i-1}^m,t_i^m}^2\int_{t_{i-1}^m}^{t_i^m}(T-\pi_m(s))^{-1/2}\,ds.\end{aligned}$$

The properties of the solution from Lemma 2.4 imply

$$\|\xi_2^m\|_{2,H,T}^2 \le C\,2\sqrt{T}\Delta_m\,\|u\|_{2,H^1,T}^2\int_0^T (T-\pi_m(s))^{-1/2}\,ds \le C\,\|u\|_{2,H^1,T}^2\,\Delta_m.$$

The expression ξ_3^m is first estimated in the same way as ξ_2^m, which leads to

$$\begin{aligned}\|\xi_3^m\|_{2,H,T}^2 &\le C\,2\sqrt{T}\,ds\,\mathbb{E}\big(\sup_{0\le t\le T}\int_0^t (t-\pi_m(s))^{-1/2}\|u_{\pi_m(s)}-u_{\pi_m(s)}^m\|_H^2\,ds\big)\\ &\le C\,2\sqrt{T}\,\mathbb{E}\big(\int_0^T (T-\pi_m(s))^{-1/2}\sup_{0\le t\le s}\|u_t-u_t^m\|_H^2\,ds\big)\\ &= C\,2\sqrt{T}\,\int_0^T (T-\pi_m(s))^{-1/2}\|u-u^m\|_{2,H,s}^2\,ds.\end{aligned}$$

Next, we give the estimates on the expressions with respect to the stochastic integrals. The estimates on the semigroup are the same as in the previous calculations. For the first term we have

$$\begin{aligned}\|\eta_1^m\|_{2,H,T}^2 &= \mathbb{E}\big(\sup_{0\le t\le T}\|\int_0^t S_{t-s}(\mathbb{1}-S_{s-\pi_m(s)})G(u_s)\,dM_s\|_H^2\big)\\ &\le C\,\mathbb{E}\big(\int_0^T \|(\mathbb{1}-S_{s-\pi_m(s)})G(u_s)\|_{L_{HS}(\mathcal{H},H)}^2\,ds\big),\end{aligned}$$

where we applied Equation (2.4). Next, Theorem 2.6.13 in [21] leads similarly to the estimates for ξ_1^m to

$$\|\eta_1^m\|_{2,H,T}^2 \le C\,\Delta_m\,\mathbb{E}\big(\int_0^T \|A_{1/2}G(u_s)\|_{L_{HS}(\mathcal{H},H)}^2\,ds\big).$$

Finally, Assumption 2.1(f) and the regularity of the solution imply

$$\|\eta_1^m\|_{2,H,T}^2 \le C\,\Delta_m\,\mathbb{E}\Big(\int_0^T (1+\|u_s\|_{H^1})^2\,ds\Big) \le C\,2T\big(1+\|u\|_{2,H^1,T}^2\big)\,\Delta_m.$$

Applying again Equation (2.4) and Assumption 2.1(e), we get for the second stochastic integral expression

$$\|\eta_2^m\|_{2,H,T}^2 \le C\,\mathbb{E}\Big(\int_0^T \|G(u_s)-G(u_{\pi_m(s)})\|_{L_{HS}(\mathcal{H},H)}^2\,ds\Big) \le C\,\mathbb{E}\Big(\int_0^T \|u_s-u_{\pi_m(s)}\|_H^2\,ds\Big).$$

The regularity of the solution from Lemma 2.5 leads to

$$\|\eta_2^m\|_{2,H,T}^2 \le C\,T\Big(1+\|u\|_{2,H^1,T}^2\Big).$$

The last expression is estimated similarly to η_2^m, which gives

$$\|\eta_3^m\|_{2,H,T}^2 \le C\,\mathbb{E}\Big(\int_0^T \|u_{\pi_m(s)}-u_{\pi_m(s)}^m\|_H^2\,ds\Big).$$

Fubini's theorem and the properties of the supremum finally imply

$$\|\eta_3^m\|_{2,H,T}^2 \le C\int_0^T \|u-u^m\|_{2,H,s}^2\,ds.$$

So overall we have

$$\begin{aligned}\|u-u^m\|_{2,H,T}^2 &\le 6\sum_{i=1}^3\big(\|\xi_i^m\|_{2,H,T}^2+\|\eta_i^m\|_{2,H,T}^2\big)\\ &\le C_1\big(1+\|u\|_{2,H^1,T}^2\big)\,\Delta_m + C_2\int_0^T \big(1+(T-s)^{-1/2}\big)\|u-u^m\|_{2,H,s}^2\,ds\end{aligned}$$

and Gronwall's inequality yields

$$\|u-u^m\|_{2,H,T}^2 \le C_1\big(1+\|u\|_{2,H^1,T}^2\big)\,\Delta_m\exp\Big(C_2\int_0^T \big(1+(T-s)^{-1/2}\big)\,ds\Big) \le C\Delta_m$$

due to the properties of the solution from Lemma 2.4. This proves the theorem. □

4. Future work

In this short note, the problems that arise when looking at the approximation of SPDEs driven by non-continuous square integrable martingales are presented. As the absence of similar Burkholder–Davis–Gundy type inequalities for non-continuous driving noises as for continuous ones causes problems especially in connection with time approximations, we show here a way how to prove mean square convergence directly for a simple time discretization scheme. Previously, this was done by proving L^p convergence for $p > 2$ and applying Hölder's inequality. This approach leads without Burkholder–Davis–Gundy type inequalities that have the same convergence properties as for continuous martingales to worse upper bounds for the order of convergence than with direct estimates. The approach presented here has to be generalized to fully discrete schemes and higher order approximations like Milstein schemes in future work, which shall be straight forward similarly to [7] and [8].

[1] P. E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, Applications of Mathematics. 23. Berlin: Springer, 1992.

[2] D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, SIAM Rev. 43 (3) (2001) 525–546.

[3] R. A. Carmona, M. R. Tehranchi, Interest Rate Models: An Infinite Dimensional Stochastic Analysis Perspective, Springer Finance. Berlin: Springer, 2006.

[4] I. Ekeland, E. Taflin, A theory of bond portfolios, Ann. Appl. Probab. 15 (2) (2005) 1260–1305.

[5] F. E. Benth, J. Šaltyte Benth, S. Koekebakker, Stochastic modelling of electricity and related markets, World Scientific, Advanced Series on Statistical Science and Applied Probability - Vol. 11, 2008.

[6] P.-L. Chow, J.-L. Jiang, J.-L. Menaldi, Pathwise convergence of approximate solutions to Zakai's equation in a bounded domain, Da Prato, G. (ed.) et al., Stochastic partial differential equations and applications. Proceedings of the third meeting on stochastic partial differential equations and applications held at Villa Madruzzo, Trento, Italy, January 1990. Harlow: Longman Scientific & Technical. Pitman Res. Notes Math. Ser. 268, 111–123 (1992).

[7] A. Lang, P.-L. Chow, J. Potthoff, Almost sure convergence of a semidiscrete Milstein scheme for SPDEs of Zakai type, to appear in Stochastics (April 2009).

[8] A. Barth, A. Lang, Almost sure convergence of a Galerkin–Milstein approximation for stochastic partial differential equations, submitted (June 2009).

[9] M. Zakai, On the optimal filtering of diffusion processes, Z. Wahrsch. verw. Geb. 11 (1969) 230–243.

[10] E. Hausenblas, J. Seidler, Stochastic convolutions driven by martingales: Maximal inequalities and exponential integrability, Stochastic Anal. Appl. 26 (1) (2008) 98–119.

[11] S. Peszat, J. Zabczyk, Stochastic Partial Differential Equations with Lévy Noise. An Evolution Equation Approach, Encyclopedia of Mathematics and Its Applications 113. Cambridge: Cambridge University Press, 2007.

[12] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Encyclopedia of Mathematics and Its Applications. 44. Cambridge: Cambridge University Press, 1992.

[13] P.-L. Chow, Stochastic Partial Differential Equations, Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series. Boca Raton, FL: Chapman & Hall/CRC, 2007.

[14] T. Kato, Perturbation Theory for Linear Operators. Corr. printing of the 2nd ed., Grundlehren der mathematischen Wissenschaften, 132. Berlin – Heidelberg – New York: Springer, 1980.

[15] K.-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics. 194. Berlin: Springer, 2000.

[16] S. Kantorovitz, On liu's analyticity criterion for semigroups, Semigroup Forum 53 (2) (1996) 262–265.

[17] P. Auscher, P. Tchamitchian, Square roots of elliptic second order divergence operators on strongly Lipschitz domains: L^2 theory, J. Anal. Math. 90 (2003) 1–12.

[18] E. Hausenblas, J. Seidler, A note on maximal inequality for stochastic convolutions, Czech. Math. J. 51 (4) (2001) 785–790.

[19] C. Marinelli, Local well-posedness of Musiela's SPDE with Lévy noise, arXiv:0704.2380, to appear in Math. Finance (April 2007).

[20] L. Tubaro, An estimate of Burkholder type for stochastic processes defined by the stochastic integral, Stochastic Anal. Appl. 2 (1984) 187–192.

[21] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, 44. New York – Berlin – Heidelberg – Tokyo: Springer, 1983.