CHALMERS | UNIVERSITY OF GOTHENBURG

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Simulation of Random Fields – LP2 2015/16

Project 2

1. The goal of this task is to do error analysis for the random field generation via FFT in Project 1, Task 2. Therefore let us consider centered isotropic Gaussian random fields on the square $[0,1)^2$ with the class of covariance functions

$$C(x,y) = \int_{\mathbb{R}^2} e^{-2\pi i (p, x-y)} \gamma(p) \, \mathrm{d}p, \qquad x, y \in \mathbb{R}^2,$$

where

$$\gamma(p) = (1 + (p_1^{2k} + p_2^{2k})^l)^{-n}$$

for $k, l, n \in \mathbb{N}$.

a) Choose your favorite algorithm from Project 1, Task 2 and γ to generate sufficiently many samples on grids with different numbers of points. (The sequence of uniform subdivisions might be a suitable one.) Compute for one point, e.g., the middle one, the statistical covariance on all grids. Plot your obtained two-dimensional functions.

b) Compute (theoretically oder numerically) the "exact" covariance of your random field. Take the supremum over the pointwise errors for all grids and plot them in a loglog plot with respect to the total number of grid points N. Derive the order of convergence by fitting a line $C \cdot N^{-\alpha}$ to your obtained errors in the plot.

c) What do you observe is a "sufficiently large" number of samples? Why do you think that it is sufficient?

2. Isotropic Gaussian random fields on the sphere are implemented in this task. In the lecture we have seen that this class of random fields admits a Karhunen–Loève expansion with respect to the spherical harmonic functions, i.e.,

$$T(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m},$$

where the properties of the random coefficients were specified in Theorem 3.0.4. We approximated the random field in the lecture by the finite series truncation at $\varkappa \in \mathbb{N}$

$$T^{\varkappa}(x) = \sum_{\ell=0}^{\varkappa} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}.$$

Please turn!

Denote by $(A_\ell, \ell \in \mathbb{N}_0)$ the angular power spectrum. Instead of the expansion with respect to the spherical harmonic functions you might also want to consider using the expansion from Lemma 3.1.1 for your implementation.

a) Write an algorithm that approximates T for variable $\varkappa$ and $A_\ell \simeq \ell^{-\alpha}$. Plot samples and check for large α and $\varkappa$, if your results look how you expect them to look like. If they do not do so, what might be the reason for that? (Hint: Consider the sizes of the numbers that you are adding.)

b) For two different fixed α and varying $\varkappa$, simulate for your favorite $x \in \mathbb{S}^2$

$$\mathbb{E}(|T(x) - T^\varkappa(x)|^2)^{1/2}$$

by a Monte Carlo method. How do you generate samples of $T(x)$? Plot the obtained convergence and check if it confirms the theory from the lecture.

c) Generate samples of $T^\varkappa$ and transform them to lognormal random fields, i.e., compute $\exp(T^\varkappa)$. Plot the random fields by generating a deformed sphere that has radius $T^\varkappa(x)$ at the original point $x \in \mathbb{S}^2$. Generate several plots for different α and $\varkappa$. What do you observe? (Hint: In Matlab "surf" and the given example might help you‘ to generate the pictures.)

- **Deadline:** Monday, December 21, 2015, 10:00
- **Webpage:**
`http://www.math.chalmers.se/~langa/teaching/SimRF15`
- **Requirement:** To pass the course, both projects as well as the final exam have to be passed.
- **Formalities:** You are allowed to work in groups up to two. You are asked to email the code and your solutions to annika.lang 'at' chalmers.se.