Wave propagation in a non-homogeneous anisotropic elastic medium occupying a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with boundary $\Gamma$, is described by the linear wave equation:

$$\rho \frac{\partial^2 v}{\partial t^2} - \nabla \cdot \tau = f, \quad \text{in } \Omega \times (0, T),$$  \hspace{1cm} (1)

$$\tau = C \epsilon,$$  \hspace{1cm} (2)

$$v = v_0, \quad \frac{\partial v}{\partial t} = 0,$$  \hspace{1cm} (3)

where $v(x, t) \subset \mathbb{R}^d$, is the displacement, $\tau$ is the stress tensor, $\rho(x)$ is the density of the material depending on $x \in \Omega$, $t$ is the time variable, $T$ is a final time, and $f(x, t) \subset \mathbb{R}^d$, is a given source function.
Further, $\epsilon$ is the strain tensor with components

$$
\epsilon_{ij} = \epsilon_{ij}(v) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)
$$

(4)

coupled to $\tau$ by Hooke’s law

$$
\tau_{ij} = \sum_{k=1}^{d} \sum_{l=1}^{d} C_{ijkl} \epsilon_{kl},
$$

(5)

where $C$ is a cyclic symmetric tensor, satisfying

$$
C_{ijkl} = C_{klij} = C_{jkli}.
$$

(6)

If the constants $C_{ijkl}(x)$ do not depend on $x$, the material of the body is said to be homogeneous. If the constants $C_{ijkl}(x)$ do not depend on the choice of the coordinate system, the material of the body is said to be isotropic at the point $x$. Otherwise, the material is anisotropic at the point $x$. 
In the isotropic case $C$ can be written as

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left( \delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk} \right),$$  \hspace{1cm} (7)

where $\delta_{ij}$ is Kronecker symbol, in which case (5) takes the form of Hooke’s law

$$\tau_{ij} = \lambda \delta_{ij} \sum_{k=1}^{d} \epsilon_{kk} + 2\mu \epsilon_{ij},$$  \hspace{1cm} (8)

where $\lambda$ and $\mu$ are the Lamé’s coefficients, depending on $x$, given by

$$\mu = \frac{E}{2(1 + \nu)}, \quad \lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)},$$  \hspace{1cm} (9)

where $E$ is the modulus of elasticity (Young modulus) and $\nu$ is the Poisson’s ratio of the elastic material. We have that

$$\lambda > 0, \mu > 0 \iff E > 0, 0 < \nu < 1/2.$$  \hspace{1cm} (10)
\[ \rho \frac{\partial^2 v_1}{\partial t^2} - \frac{\partial}{\partial x_1} ( (\lambda + 2\mu) \frac{\partial v_1}{\partial x_1} + \lambda \frac{\partial v_2}{\partial x_2} + \lambda \frac{\partial v_3}{\partial x_3} ) \\
- \frac{\partial}{\partial x_2} ( \mu ( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} ) ) \\
- \frac{\partial}{\partial x_3} ( \mu ( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} ) ) = f_1, \]

\[ \rho \frac{\partial^2 v_2}{\partial t^2} - \frac{\partial}{\partial x_2} ( (\lambda + 2\mu) \frac{\partial v_2}{\partial x_2} + \lambda \frac{\partial v_1}{\partial x_1} + \lambda \frac{\partial v_3}{\partial x_3} ) \\
- \frac{\partial}{\partial x_1} ( \mu ( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} ) ) \\
- \frac{\partial}{\partial x_3} ( \mu ( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} ) ) = f_2, \]

\[ \rho \frac{\partial^2 v_3}{\partial t^2} - \frac{\partial}{\partial x_3} ( (\lambda + 2\mu) \frac{\partial v_3}{\partial x_3} + \lambda \frac{\partial v_2}{\partial x_2} + \lambda \frac{\partial v_1}{\partial x_1} ) \\
- \frac{\partial}{\partial x_2} ( \mu ( \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} ) ) \\
- \frac{\partial}{\partial x_1} ( \mu ( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} ) ) = f_3, \]
or in more compact form

\[
\rho \frac{\partial^2 v}{\partial t^2} - \nabla \cdot (\mu \nabla v) - \nabla ((\lambda + \mu) \nabla \cdot v) = f. \tag{11}
\]
The inverse scattering problem

We consider the elastics system in a non-homogeneous isotropic medium in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with boundary $\Gamma$:

$$
\rho \frac{\partial^2 v}{\partial t^2} - \nabla \cdot (\mu \nabla v) - \nabla((\lambda + \mu) \nabla \cdot v) = f \quad \text{in} \quad \Omega \times (0, T),
$$

$$
v(x, 0) = v_0(t), \quad \frac{\partial v(x, t)}{\partial t} = 0, v|_{\Gamma} = 0.
$$

where $v = (v_1, v_2, v_3)^T$, $\nabla v = (\nabla v_1, \nabla v_2, \nabla v_3)^T$, and $\nabla \cdot v$ is the divergence of the vector field $v$. 
The inverse problem for (12 - 13) can be formulated as follows: find a controls $\rho, \mu, \lambda$ which belongs to the set of admissible controls

$$U = \{ \rho, \mu, \lambda \in L^2(\Omega); 0 < \rho\min < \rho(x) < \rho\max, \\
0 < \lambda\min < \lambda(x) < \lambda\max, \\
0 < \mu\min < \mu(x) < \mu\max \}$$

and minimizes the quantity

$$E(v, \rho, \lambda, \mu) = \frac{1}{2} \int_0^T \int_{\Omega} (v - \tilde{v})^2 \delta_{obs} \ dx dt + \frac{1}{2} \gamma_1 \int_{\Omega} \rho^2 \ dx \\
+ \frac{1}{2} \gamma_2 \int_{\Omega} \mu^2 \ dx + \frac{1}{2} \gamma_3 \int_{\Omega} \lambda^2 \ dx,$$

(15)

where $\tilde{v}$ is observed data at a finite set of observation points $x_{obs}$, $v$ satisfies (12) and thus depends on $\rho, \mu, \lambda$, $\delta_{obs} = \sum \delta(x_{obs})$ is a sum of delta-functions corresponding to the observation points, and $\gamma_i, \ i = 1, 2, 3$ are a regularization parameters.
To approach this minimization problem, we introduce the Lagrangian

\[ L(\alpha, v, \rho, \mu, \lambda) = E(v, \rho, \lambda, \mu) + \]

\[ \int_0^T \int_\Omega \left( -\rho \frac{\partial \alpha}{\partial t} \frac{\partial v}{\partial t} + \mu \nabla \alpha \nabla v + (\lambda + \mu) \nabla \cdot v \nabla \cdot \alpha - f \alpha \right) \, dx \, dt, \]

and search for a stationary point with respect to \((\alpha, v, \rho, \mu, \lambda)\) satisfying for all \((\bar{\alpha}, \bar{v}, \bar{\rho}, \bar{\lambda}, \bar{\mu})\)

\[ L'(\alpha, v, \rho, \mu, \lambda)(\bar{\alpha}, \bar{v}, \bar{\rho}, \bar{\mu}, \bar{\lambda}) = 0, \quad (16) \]

where \(L'\) is the gradient of \(L\) and we assume that \(\alpha(\cdot, T) = \bar{\alpha}(\cdot, T) = 0, \alpha = 0\) and \(v(\cdot, 0) = \bar{v}(\cdot, 0) = 0.\)
\[ 0 = \frac{\partial L}{\partial \alpha}(\alpha, v, \rho, \mu, \lambda)(\bar{\alpha}) = \int_0^T \int_\Omega \left( -\rho \frac{\partial \bar{\alpha}}{\partial t} \frac{\partial v}{\partial t} + \mu \nabla \bar{\alpha} \nabla v \right) dt + (\lambda + \mu) \nabla \cdot v \nabla \cdot \bar{\alpha} - f \bar{\alpha} \] dx dt, \tag{17} \]

\[ 0 = \frac{\partial L}{\partial v}(\alpha, v, \rho, \mu, \lambda)(\bar{v}) = \int_0^T \int_\Omega (v - \bar{v}) \bar{v} \delta_{obs} \] dx dt \tag{18} \]

\[ + \int_0^T \int_\Omega -\rho \frac{\partial \alpha}{\partial t} \frac{\partial \bar{v}}{\partial t} + \mu \nabla \alpha \nabla \bar{v} + (\lambda + \mu) \nabla \cdot \bar{v} \] \nabla \cdot \alpha \] dx dt,
\[ 0 = \frac{\partial L}{\partial \rho}(\alpha, v, \rho, \mu, \lambda)(\bar{\rho}) = \int_0^T \int_\Omega \frac{\partial \alpha(x, t)}{\partial t} \frac{\partial v(x, t)}{\partial t} \bar{\rho} \, dx \, dt \quad (19) \]

\[ + \quad \gamma_1 \int_\Omega \rho \bar{\rho} \, dx, \quad x \in \Omega. \]

\[ 0 = \frac{\partial L}{\partial \mu}(\alpha, v, \rho, \mu, \lambda)(\bar{\mu}) = \int_0^T \int_\Omega (\nabla \alpha \nabla v + \nabla \cdot v \nabla \cdot \alpha) \bar{\mu} \, dx \, dt \]

\[ + \quad \gamma_2 \int_\Omega \mu \bar{\mu} \, dx, \quad x \in \Omega. \]

\[ 0 = \frac{\partial L}{\partial \lambda}(\alpha, v, \rho, \mu, \lambda)(\bar{\lambda}) = \int_0^T \int_\Omega \nabla \cdot v \nabla \cdot \alpha \bar{\lambda} \, dx \, dt \quad (20) \]

\[ + \quad \gamma_3 \int_\Omega \lambda \bar{\lambda} \, dx, \quad x \in \Omega. \]
The equation (17) is a weak form of the state equation (12 - 13), the equation (18) is a weak form of the adjoint state equation

\[
\rho \frac{\partial^2 \alpha}{\partial t^2} - \nabla \cdot (\mu \nabla \alpha) - \nabla ((\lambda + \mu) \nabla \cdot \alpha) = -(v - \tilde{v}) \delta_{obs}, \ x \in \Omega, \ 0 < t < T,
\]

\[
\alpha(T) = \frac{\partial \alpha(T)}{\partial t} = 0,
\]

\[
\alpha = 0 \ on \ \Gamma \times (0, T),
\]

and (19) - (20) expresses stationarity with respect to \( \rho(x), \mu(x), \lambda(x) \).
To solve the minimization problem we shall use a discrete form of the following steepest descent or gradient method starting from an initial guess $\rho^0, \mu^0, \lambda^0$ and computing a sequence $\rho^n, \mu^n, \lambda^n$ in the following steps:

1. Compute the solution $v^n = (v^n_1, v^n_2, v^n_3)$ of the forward problem (12 - 13) with $\rho = \rho^n, \mu = \mu^n, \lambda = \lambda^n$.

2. Compute the solution $\alpha^n = (\alpha^n_1, \alpha^n_2, \alpha^n_3)$ of the adjoint problem (22).

3. Update the $\rho, \mu, \lambda$ according to

$$
\begin{align*}
\rho^{n+1}(x) & = \rho^n(x) - \beta^n \left( \int_0^T \frac{\partial \alpha^n(x, t)}{\partial t} \frac{\partial v^n(x, t)}{\partial t} dt + \gamma_1 \rho^n(x) \right), \\
\mu^{n+1}(x) & = \mu^n(x) - \beta^n \left( \int_0^T \nabla \alpha^n \nabla v^n + \nabla \cdot v^n \nabla \cdot \alpha^n + \gamma_2 \mu^n(x) \right), \\
\lambda^{n+1}(x) & = \lambda^n(x) - \beta^n \left( \int_0^T \nabla \cdot v^n \nabla \cdot \alpha^n + \gamma_3 \lambda^n(x) \right),
\end{align*}
$$
We now formulate a finite element method for (16) based on using continuous piecewise linear functions in space and time. We discretize $\Omega \times (0, T)$ in the usual way denoting by $K_h = \{ K \}$ a partition of the domain $\Omega$ into elements $K$ (triangles in $\mathbb{R}^2$ and tetrahedra in $\mathbb{R}^3$ with $h = h(x)$ being a mesh function representing the local diameter of the elements), and we let $J_k = \{ J \}$ be a partition of the time interval $I = (0, T)$ into time intervals $J = (t_{k-1}, t_k]$ of uniform length $\tau = t_k - t_{k-1}$. In fully discrete form the resulting method corresponds to a centered finite difference approximation for the second order time derivative and a usual finite element approximation of the Laplacian.

To formulate the finite element method for (16) we introduce the finite
element spaces $V_h$, $W^\nu_h$ and $W^\alpha_h$ defined by:

\[
V_h := \{ v \in L_2(\Omega) : v \in P_0(K), \forall K \in K_h \},
\]
\[
W^\nu := \{ v \in [H^1(\Omega \times I)]^3 : v(\cdot, 0) = 0, v|\Gamma = 0 \},
\]
\[
W^\alpha := \{ \alpha \in [H^1(\Omega \times I)]^3 : \alpha(\cdot, T) = \alpha|\Gamma = 0 \},
\]
\[
W^\nu_h := \{ v \in W^\nu : v|_{K \times J} \in [P_1(K) \times P_1(J)]^3, \forall K \in K_h, \forall J \in J_k \},
\]
\[
W^\alpha_h := \{ v \in W^\alpha : v|_{K \times J} \in [P_1(K) \times P_1(J)]^3, \forall K \in K_h, \forall J \in J_k \},
\]

where $P_1(K)$ and $P_1(J)$ are the set of linear functions on K and J, respectively.
The finite element method now reads: Find
\[ \rho_h \in V_h, \mu_h \in V_h, \lambda_h \in V_h, \alpha_h \in W_h^\alpha, v_h \in W_h^v, \] such that
\[ L'(\alpha_h, v_h, \rho_h, \mu_h, \lambda_h)(\bar{\alpha}, \bar{v}, \bar{\rho}, \bar{\mu}, \bar{\lambda}) = 0 \]
\[ \forall \bar{\rho} \in V_h, \bar{\mu} \in V_h, \bar{\lambda} \in V_h, \bar{\alpha} \in W_h^\alpha, \bar{v} \in W_h^v. \]
Expanding $v, \alpha$ in terms of the standard continuous piecewise linear functions $\varphi_i(x)$ in space and $\psi_i(t)$ in time and substituting this into (17 - 18), the following system of linear equations is obtained:

$$M(v^{k+1} - 2v^k + v^{k-1}) = \frac{\tau^2}{\rho} F^k - \frac{\tau^2}{\rho} \mu K(\frac{1}{6}v^{k-1} + \frac{2}{3}v^k + \frac{1}{6}v^{k+1})$$

$$- \frac{\tau^2}{\rho} (\lambda + \mu) Dv^k,$$

(22)

$$M(\alpha^{k+1} - 2\alpha^k + \alpha^{k-1}) = -\frac{\tau^2}{\rho} S^k - \frac{\tau^2}{\rho} \mu K(\frac{1}{6}\alpha^{k-1} + \frac{2}{3}\alpha^k + \frac{1}{6}\alpha^{k+1})$$

$$- \frac{\tau^2}{\rho} (\lambda + \mu) D\alpha^k,$$

(23)
with initial conditions:

\[ v(0) = 0, \quad \dot{v}(0) \approx 0, \]
\[ \alpha(T) = 0, \quad \dot{\alpha}(T) \approx 0. \]  

Here, \( M \) is the mass matrix in space, \( K \) is the stiffness matrix, \( D \) is the divergence matrice, \( k = 1, 2, 3 \ldots \) denotes the time level, \( F^k, S^k \) are the load vectors, \( \mathbf{v} \) is the unknown discrete field values of \( v \), \( \mathbf{\alpha} \) is the unknown discrete field values of \( \alpha \) and \( \tau \) is the time step.
The explicit formulas for the entries in system (22 - 23) at the element level can be given as:

\[
M_{i,j}^e = (\varphi_i, \varphi_j)_e, \quad (26)
\]
\[
K_{i,j}^e = (\nabla \varphi_i, \nabla \varphi_j)_e, \quad (27)
\]
\[
D_{i,j}^e = (\nabla \cdot \varphi_i, \nabla \cdot \varphi_j)_e, \quad (28)
\]
\[
F_{j,m}^e = (f, \varphi_j \psi_m)_{e \times J}, \quad (29)
\]
\[
S_{j,m}^e = (v - \tilde{v}, \varphi_j \psi_m)_{e \times J}, \quad (30)
\]

where \((.,.)_e\) denotes the \(L_2(e)\) scalar product. The matrix \(M_e\) is the contribution from element \(e\) to the global assembled matrix in space \(M\), \(K_e\) is the contribution from element \(e\) to the global assembled matrix \(K\), \(D_e\) is the contribution from element \(e\) to the global assembled matrix \(D\), \(F^e\) and \(S^e\) are the contributions from element \(e\) to the assembled source vectors \(F\) and vector of the right hand side of (22), correspondingly.
To obtain an explicit scheme we approximate $M$ with the lumped mass matrix $M^L$, the diagonal approximation obtained by taking the row sum of $M$. By multiplying (22) - (23) with $(M^L)^{-1}$ and replacing the terms $\frac{1}{6}v^{k-1} + \frac{2}{3}v^k + \frac{1}{6}v^{k+1}$ and $\frac{1}{6}\alpha^{k-1} + \frac{2}{3}\alpha^k + \frac{1}{6}\alpha^{k+1}$ by $v^k$ and $\alpha^k$, respectively, we obtain an efficient explicit formulation:

\[
\begin{align*}
v^{k+1} &= \frac{\tau^2}{\rho}(M^L)^{-1}F^k + 2v^k - \frac{\tau^2}{\rho}(M^L)^{-1}Kv^k \\
&\quad - \frac{\tau^2}{\rho}(\lambda + \mu)(M^L)^{-1}Dv^k - v^{k-1},
\end{align*}
\]

\[
\begin{align*}
\alpha^{k-1} &= -\frac{\tau^2}{\rho}(M^L)^{-1}S^k + 2\alpha^k - \frac{\tau^2}{\rho}(M^L)^{-1}K\alpha^k \\
&\quad - \frac{\tau^2}{\rho}(\lambda + \mu)(M^L)^{-1}D\alpha^k - \alpha^{k+1}.
\end{align*}
\]
The discrete version of gradients takes the form:

\[
0 = \frac{\partial L}{\partial \rho}(\alpha, v, \rho(x), \mu(x), \lambda(x))(\bar{\rho}) = \int_0^T \int_\Omega \frac{\partial \alpha_h}{\partial t} \frac{\partial v_h}{\partial t} \bar{\rho} \, dx \, dt + \gamma_1 \int_\Omega \rho_h \bar{\rho} \, dx, \forall \bar{\rho} \in V_h. \tag{33}
\]

\[
0 = \frac{\partial L}{\partial \mu}(\alpha, v, \rho(x), \mu(x), \lambda(x))(\bar{\mu}) = \int_0^T \int_\Omega (\nabla \alpha_h \nabla v_h + \nabla \cdot v_h \nabla \cdot \alpha_h)\bar{\mu} \, dx \, dt
+ \gamma_2 \int_\Omega \mu_h \bar{\mu} \, dx, \, x \in \Omega. \tag{34}
\]

\[
0 = \frac{\partial L}{\partial \lambda}(\alpha, v, \rho(x), \mu(x), \lambda(x))(\bar{\lambda}) = \int_0^T \int_\Omega \nabla \cdot v_h \nabla \cdot \alpha_h \bar{\lambda} \, dx \, dt + \gamma_3 \int_\Omega \lambda_h \bar{\lambda} \, dx, \, x \in \Omega. \tag{35}
\]
An a posteriori error estimate for the Lagrangian

We start by writing an equation for the error $e$ in the Lagrangian as

$$e = L(\alpha, v, \rho, \mu, \lambda) - L(\alpha_h, v_h, \rho_h, \mu_h, \lambda_h)$$

$$= \frac{1}{2} L'(\alpha_h, v_h, \rho_h, \mu_h, \lambda_h)((\alpha, v, \rho, \mu, \lambda) - (\alpha_h, v_h, \rho_h, \mu_h, \lambda_h)) + R$$

$$= \frac{1}{2} L'(\alpha_h, v_h, \rho_h, \mu_h, \lambda_h)(\alpha - \alpha_h, v - v_h, \rho - \rho_h, \mu - \mu_h, \lambda - \lambda_h) + R,$$

where $R$ denotes (a small) second order term. Using the Galerkin orthogonality and the splitting

$$\alpha - \alpha_h = (\alpha - \alpha^I_h) + (\alpha^I_h - \alpha_h), \quad v - v_h =$$

$$(v - v^I_h) + (v^I_h - v_h), \quad \rho - \rho_h = (\rho - \rho^I_h) + (\rho^I_h - \rho_h), \quad \mu - \mu_h =$$

$$(\mu - \mu^I_h) + (\mu^I_h - \mu_h), \quad \lambda - \lambda_h = (\lambda - \lambda^I_h) + (\lambda^I_h - \lambda_h),$$

where

$$(\alpha^I_h, v^I_h, \rho^I_h, \mu^I_h, \lambda^I_h)$$

denotes an interpolant of

$$(\alpha, v, \rho, \mu, \lambda) \in W^{\alpha}_h \times W^v_h \times V_h \times V_h \times V_h,$$

and neglecting the term $R,$
we get:

\begin{equation}
\begin{aligned}
e & \approx \frac{1}{2} L'(\alpha_h, \nu_h, \rho_h, \mu_h, \lambda_h)(\alpha - \alpha_h, \nu - \nu_h, \rho - \rho_h, \mu - \mu_h, \lambda - \lambda_h) \\
& = \frac{1}{2} (I_1 + I_2 + I_3 + I_4 + I_5),
\end{aligned}
\end{equation}
where

\[ I_1 = \int_0^T \int_\Omega -\rho_h \frac{\partial (\alpha - \alpha_h^I)}{\partial t} \frac{\partial v_h}{\partial t} + \mu_h \nabla (\alpha - \alpha_h^I) \nabla v_h + (\lambda_h + \mu_h) (\nabla \cdot v_h \nabla \cdot (\alpha - \alpha_h^I)) - f(\alpha - \alpha_h^I) \, dx \, dt, \]

\[ I_2 = \int_0^T \int_\Omega (v_h - \tilde{v})(v - v_h^I) \delta_{obs} \, dx \, dt \]

\[ + \int_0^T \int_\Omega -\rho_h \frac{\partial \alpha_h}{\partial t} \frac{\partial (v - v_h^I)}{\partial t} + \mu_h \nabla \alpha_h \nabla (v - v_h^I) + (\lambda_h + \mu_h) \nabla \cdot (v - v_h^I) \nabla \cdot \alpha_h \, dx \, dt, \]

\[ I_3 = -\int_0^T \int_\Omega \frac{\partial \alpha_h(x, t)}{\partial t} \frac{\partial v_h(x, t)}{\partial t} (\rho - \rho_h^I) \, dx \, dt + \int_\Omega \rho_h (\rho - \rho_h^I) \, dx, \]

\[ I_4 = \int_0^T \int_\Omega (\nabla \alpha_h \nabla v_h + \nabla \cdot v_h \nabla \cdot \alpha_h) (\mu - \mu_h^I) \, dx \, dt + \int_\Omega \mu_h (\mu - \mu_h^I) \, dx, \]

\[ I_5 = \int_0^T \int_\Omega (\nabla \cdot v_h \nabla \cdot \alpha_h)(\lambda - \lambda_h^I) \, dx \, dt + \int_\Omega \lambda_h (\lambda - \lambda_h^I) \, dx. \]
Defining the residuals

\[ R_{v_1} = |f|, \quad R_{v_2} = \frac{\mu_h}{2} \max_{S \subset \partial K} h^{-1}_k |[\partial_s v_h]|, \quad R_{v_3} = \frac{\rho_{h}}{2} \tau^{-1} |[\partial v_{ht}]|, \]

\[ R_{\alpha_1} = |v_h - \tilde{v}|, \quad R_{\alpha_2} = \frac{\mu_h}{2} \max_{S \subset \partial K} h^{-1}_k |[\partial_s \alpha_h]|, \quad R_{\alpha_3} = \frac{\rho_{h}}{2} \tau^{-1} |[\partial \alpha_{ht}]|, \]

\[ R_{\rho_1} = \left| \frac{\partial \alpha_h}{\partial t} \frac{\partial v_h}{\partial t} \right|, \quad R_{\rho_2} = |\rho_{h}|, \]

\[ R_{\mu_1} = |\nabla \alpha_h \nabla v_h + \nabla \cdot \alpha_h \nabla \cdot v_h|, \quad R_{\mu_2} = |\mu_{h}|, \]

\[ R_{\lambda_1} = |\nabla \cdot \alpha_h \nabla \cdot v_h|, \quad R_{\lambda_2} = |\lambda_{h}|, \]

\[(37)\]
and interpolation errors in the form

\[
\sigma_\alpha = C_T \left| \left[ \frac{\partial \alpha_h}{\partial t} \right] \right| + C_h \left| \left[ \frac{\partial \alpha_h}{\partial n} \right] \right|, \quad (38)
\]

\[
\sigma_v = C_T \left| \left[ \frac{\partial v_h}{\partial t} \right] \right| + C_h \left| \left[ \frac{\partial v_h}{\partial n} \right] \right|, \quad (39)
\]

\[
\sigma_\rho = C |[\rho_h]|, \quad \sigma_\mu = C |[\mu_h]|, \quad \sigma_\lambda = C |[\lambda_h]| \quad (40)
\]
we obtain the following a posteriori estimate

\[ |e| \leq \frac{1}{2} \left( \int_0^T \int_\Omega R_{v_1} \sigma_\alpha \ dx dt + \int_0^T \int_\Omega R_{v_2} \sigma_\alpha \ dx dt + \int_0^T \int_\Omega R_{v_3} \sigma_\alpha \ dx dt \right. \\
+ \left. \int_0^T \int_\Omega R_{\alpha_1} \sigma_v \ dx dt + \int_0^T \int_\Omega R_{\alpha_2} \sigma_v \ dx dt + \int_0^T \int_\Omega R_{\alpha_3} \sigma_v \ dx dt \right. \\
+ \left. \int_0^T \int_\Omega R_{\rho_1} \sigma_\rho \ dx dt + \int_\Omega R_{\rho_2} \sigma_\rho \ dx + \int_0^T \int_\Omega R_{\mu_1} \sigma_\mu \ dx dt \right. \\
+ \left. \int_\Omega R_{\mu_2} \sigma_\mu \ dx + \int_0^T \int_\Omega R_{\lambda_1} \sigma_\lambda \ dx dt + \int_\Omega R_{\lambda_2} \sigma_\lambda \ dx \right). \]
Adaptive algorithm

In the computations below we use the following variant of the gradient method with adaptive mesh selection:

1. Choose an initial mesh $K_h$ and an initial time partition $J_k$ of the time interval $(0, T)$.

2. Compute the solution $v^n = (v^n_1, v^n_2, v^n_3)$ on $K_h$ and $J_k$ of the forward problem (12 - 13) with $\rho = \rho^n, \mu = \mu^n, \lambda = \lambda^n$.

3. Compute the solution $\alpha^n = (\alpha^n_1, \alpha^n_2, \alpha^n_3)$ of the adjoint problem

$$\rho \frac{\partial^2 \alpha}{\partial t^2} - \mu \Delta \alpha - (\lambda + \mu) \nabla (\nabla \cdot \alpha) = -(v - \tilde{v}) \delta_{obs}, \quad x \in \Omega, \ 0 < t < T$$

on $K_h$ and $J_k$. 
4. Update the $\rho, \mu, \lambda$ according to

\[ \rho^{n+1}(x) = \rho^n(x) - \beta^n \left( \int_0^T \frac{\partial \alpha^n(x,t)}{\partial t} \frac{\partial v^n(x,t)}{\partial t} dt + \gamma_1 \rho^n(x) \right), \]

\[ \mu^{n+1}(x) = \mu^n(x) - \beta^n \left( \int_0^T \nabla \alpha^n \cdot \nabla v^n + \nabla \cdot v^n \nabla \cdot \alpha^n + \gamma_2 \mu^n(x) \right), \]

\[ \lambda^{n+1}(x) = \lambda^n(x) - \beta^n \left( \int_0^T \nabla \cdot v^n \nabla \cdot \alpha^n + \gamma_3 \lambda^n(x) \right). \]

Make steps 1 – 4 as long the gradient quickly decreases.

5. Refine all elements, where

\[ (R_{\rho_1} + R_{\rho_2}) \sigma_\rho + (R_{\mu_1} + R_{\mu_2}) \sigma_\mu + (R_{\lambda_1} + R_{\lambda_2}) \sigma_\lambda > tol \]

and construct a new mesh $K_h$ and a new time partition $J_k$. Here $tol$ is a tolerance chosen by the user. Return to 1.