A posteriori error estimate in Lagrangian and parameter identification for time-dependent inverse scattering problems

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A posteriori error estimation for the Lagrangian

We obtain an a posteriori error estimate for error in the Lagrangian by noting that

$$
\begin{aligned}
L(u) - L(u_h) &= \int_0^1 \frac{d}{d\epsilon} L(\epsilon u + (1-\epsilon)u_h) d\epsilon \\
&= \int_0^1 L'(\epsilon u + (1-\epsilon)u_h; u - u_h) d\epsilon \\
&= L'(u_h; u - u_h) + R,
\end{aligned}
$$

where R is a second order remainder term. Using now the Galerkin orthogonality

$$L'(u_h; \bar{u}) = 0 \ \ \forall \bar{u} \in U_h$$

with the splitting

$$u - u_h = (u - u_h^I) + (u_h^I - u_h),$$

where $u_h^I \in U_h$ denotes an interpolant of u, and neglecting the term R, we get the following error representation:

$$L(u) - L(u_h) \approx L'(u_h; u - u_h^I), \tag{1}$$

involving the residual $L'(u_h; \cdot)$ with $u - u_h^I$ appearing as a weight.

Acoustic wave propagation

The scalar wave equation modeling acoustic wave propagation in a bounded domain $\Omega \subset \mathbf{R}^d,\ d = 2, 3$, with boundary Γ, takes the following form:

$$
\begin{aligned}
\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \triangle p = f, \qquad \text{in } \Omega \times (0, T), \\
p(\cdot, 0) = 0, \ \ \frac{\partial p}{\partial t}(\cdot, 0) = 0, \ \text{ in } \ \Omega, \\
p\big|_{\Gamma} = 0, \ \text{ on } \ \Gamma \times (0, T),
\end{aligned}
\tag{2}
$$

where $p(x, t) \in \mathbf{R}$ is the pressure satisfying homogeneous boundary and initial conditions, $c(x)$ is the wave speed depending on $x \in \Omega$, t is the time variable and T is a final time, and $f(x, t)$ is a given source function.

Inverse acoustic scattering

Our goal is to find the function $c(x)$ which minimizes the quantity

$$E(p,c) = \frac{1}{2}\int_0^T \int_\Omega (p-\tilde{p})^2 \delta_{obs} dx dt + \frac{1}{2}\gamma \int_\Omega |\nabla c|^2 \ dx, \qquad (3)$$

where $\tilde{p}$ is observed data at x_{obs}, p satisfies (2) and thus depends on c, $\delta_{obs} = \sum \delta(x_{obs})$ is a sum of multiples of delta-functions $\delta(x_{obs})$ corresponding to the observation points, and γ is a regularization parameter (small).

To approach this minimization problem, we introduce the Lagrangian

$$L(u) = E(p, c) - ((\frac{1}{c^2} Dp, D\varphi)) + ((\nabla p, \nabla \varphi)) - ((f, \varphi)),$$

where $u = (p, \varphi, c)$, and search for a stationary point with respect to u satisfying $\forall \bar{u}$

$$L'(u; \bar{u}) = 0, \tag{4}$$

where $L'(u; \cdot)$ is the Jacobian of L at u, and we assume that $\varphi(\cdot, T) = \bar{\varphi}(\cdot, T) = 0$ and $p(\cdot, 0) = \bar{p}(\cdot, 0) = 0$, together with homogeneous Dirichlet boundary conditions.

The equation (4) expresses that in $\Omega \times (0,T)$

$$
\begin{aligned}
\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \triangle p &= f, && (5)\\
\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \triangle \varphi &= -(p-\tilde{p})\delta_{obs}, && (6)\\
-\gamma \triangle c - \frac{2}{c^3}\int_0^T \frac{\partial p}{\partial t}\frac{\partial \varphi}{\partial t}\ dt &= 0, && (7)
\end{aligned}
$$

together with homogeneous boundary and initial conditions.

Finite element discretization.

To formulate the finite element method for (4) we introduce the finite element spaces V_h, W_h^p and W_h^φ defined by :

$$\begin{aligned}
V_h &:= \{v \in L_2(\Omega) : v \in P_0(K), \forall K \in K_h\},\\
W^p &:= \{p \in H^1(\Omega \times J) : p(\cdot, 0) = 0, p|_\Gamma = 0\},\\
W^\varphi &:= \{\varphi \in H^1(\Omega \times J) : \varphi(\cdot, T) = 0, \varphi|_\Gamma = 0\},\\
W_h^p &:= \{v \in W^p : v|_{K\times J} \in P_1(K) \times P_1(J), \forall K \in K_h, \forall J \in J_k\},\\
W_h^\varphi &:= \{v \in W^\varphi : v|_{K\times J} \in P_1(K) \times P_1(J), \forall K \in K_h, \forall J \in J_k\},
\end{aligned}$$

where $P_1(K)$ and $P_1(J)$ are the set of linear functions on K and J, respectively.

The finite element method now reads: Find
$c_h \in V_h, \varphi_h \in W_h^{\varphi}, p_h \in W_h^p$, such that

$$L'(\varphi_h, p_h, c_h)(\bar{\varphi}, \bar{p}, \bar{c}) = 0 \ \ \forall \bar{c} \in V_h, \bar{\varphi} \in W_h^{\lambda}, \bar{p} \in W_h^p. \tag{8}$$

An a posteriori error estimate for the Lagrangian

Using the Galerkin orthogonality (4) and the splitting $\varphi - \varphi_h = (\varphi - \varphi_h^I) + (\varphi_h^I - \varphi_h),\ \ p - p_h = (p - p_h^I) + (p_h^I - p_h),\ \ c - c_h = (c - c_h^I) + (c_h^I - c_h)$, where $(\varphi_h^I, p_h^I, c_h^I)$ denotes an interpolant of $(\varphi, p, c) \in W_h^{\varphi} \times W_h^{p} \times V_h$, and neglecting the term R, we get:

$$e \approx L'(\varphi_h, p_h, c_h)(\varphi - \varphi_h^I, p - p_h^I, c - c_h^I) = (I_1 + I_2 + I_3), \qquad (9)$$

where

$$
\begin{aligned}
I_1 &= \int_0^T \int_\Omega (-\frac{1}{c_h^2} \frac{\partial(\varphi - \varphi_h^I)}{\partial t} \frac{\partial p_h}{\partial t} + \nabla(\varphi - \varphi_h^I) \nabla p_h \\
&\quad - f(\varphi - \varphi_h^I)) \; dxdt, \\
I_2 &= \int_0^T \int_\Omega (p_h - \tilde{p})(p - p_h^I) \, \delta_{obs} \; dxdt \\
&+ \int_0^T \int_\Omega \left(-\frac{1}{c_h^2} \frac{\partial \varphi_h}{\partial t} \frac{\partial (p - p_h^I)}{\partial t} + \nabla \varphi_h \nabla (p - p_h^I) \right) \; dxdt, \\
I_3 &= \frac{2}{c_h^3} \int_0^T \int_\Omega \frac{\partial \varphi_h(x,t)}{\partial t} \frac{\partial p_h(x,t)}{\partial t} (c - c_h^I) \; dxdt - \gamma \int_\Omega \triangle c_h (c - c_h^I) \; dx.
\end{aligned}
$$

To estimate (**??**) we integrate by parts in the first and second terms to get:

$$
\begin{aligned}
|I_1| &= \Big| \int_0^T \int_\Omega \left(\frac{1}{c_h^2} \frac{\partial^2 p_h}{\partial t^2} (\varphi - \varphi_h^I) - \triangle p_h (\varphi - \varphi_h^I) - f(\varphi - \varphi_h^I) \right) \; dxdt \\
&+ \sum_K \int_0^T \int_{\partial K} \frac{\partial p_h}{\partial n_K} (\varphi - \varphi_h^I) \, dsdt \qquad (10) \\
&- \sum_k \int_\Omega \frac{1}{c_h^2} \Big[\frac{\partial p_h}{\partial t}(t_k) \Big] (\varphi - \varphi_h^I)(t_k) \, dx \Big|,
\end{aligned}
$$

where the terms $\frac{\partial p_h}{\partial n_K}$ and $\left[\frac{\partial p_h}{\partial t}\right]$ appear during the integration by parts and denote the derivative of p_h in the outward normal direction n_K of the boundary ∂K of element K, and the jump of the derivative of p_h in time, respectively.

In the second term of the (10) we sum over the element boundaries, and each internal side $S \in S_h$ occurs twice. Denoting by $\partial_s p_h$ the derivative of a function p_h in one of the normal directions of each side S, we can write

$$\sum_K \int_{\partial K} \frac{\partial p_h}{\partial n_K}(\varphi - \varphi_h^I)\ ds = \sum_S \int_S \left[\partial_s p_h\right](\varphi - \varphi_h^I)\ ds, \tag{11}$$

where $\left[\partial_s p_h\right]$ is jump in the derivative $\partial_s p_h$ computed from the two triangles sharing S.

We distribute each jump equally to the two sharing triangles and return to a sum over elements edges ∂K :

$$\sum_S \int_S [\partial_s p_h] \cdot (\varphi - \varphi_h^I) \ ds = \sum_K \frac{1}{2} h_K^{-1} \int_{\partial K} \big[\partial_s p_h\big](\varphi - \varphi_h^I)\, h_K \ ds. \tag{12}$$

We formally set $dx = h_K ds$ and replace the integrals over the element boundaries ∂K by integrals over the elements K, to get:

$$\left| \sum_K \frac{1}{2} h_K^{-1} \int_{\partial K} \big[\partial_s p_h\big](\varphi - \varphi_h^I)\, h_K \ ds \right| \leq C \max_{S \subset \partial K} h_K^{-1} \int_\Omega \big|\big[\partial_s p_h\big]\big| \big|(\varphi - \varphi_h^I)\big| \ dx, \tag{13}$$

where $\big[\partial_s p_h\big]\big|_K = \max_{S \subset \partial K} \big[\partial_s p_h\big]\big|_S$.

In a similar way we can estimate the third term in (10):

$$\begin{aligned} &\left|\sum_k \int_\Omega \frac{1}{c_h^2}\left[\frac{\partial p_h}{\partial t}(t_k)\right](\varphi-\varphi_h^I)(t_k)\,dx\right| \le \\ &\sum_k \int_\Omega \frac{1}{c_h^2}\tau^{-1}\cdot\left|\left[\frac{\partial p_h}{\partial t}(t_k)\right]\right|\cdot\left|(\varphi-\varphi_h^I)(t_k)\right|\ \tau dx \\ \le\ \ & C\sum_k \int_{J_k}\int_\Omega \frac{1}{c_h^2}\tau^{-1}\cdot\left|[\partial p_{h\,t_k}]\right|\cdot\left|(\varphi-\varphi_h^I)\right|\ dxdt \\ =\ \ & C\int_0^T\int_\Omega \frac{1}{c_h^2}\tau^{-1}\cdot\left|[\partial p_{h\,t}]\right|\cdot\left|(\varphi-\varphi_h^I)\right|\ dxdt, \end{aligned}$$

where

$$[\partial p_{h\,t_k}] = \max_k\left(\left[\frac{\partial p_h}{\partial t}(t_k)\right],\ \left[\frac{\partial p_h}{\partial t}(t_{k+1})\right]\right), \tag{14}$$

$$[\partial p_{h\,t}] = [\partial p_{h\,t_k}]\ \text{ on }\ J_k. \tag{15}$$

Substituting both above expressions for the second and third terms in (10), we get:

$$
\begin{aligned}
\left|I_1\right| \quad &\leq \quad \left|\int_0^T \int_\Omega \big(\frac{1}{c_h^2}\frac{\partial^2 p_h}{\partial t^2} - \triangle p_h - f\big)(\varphi - \varphi_h^I) \ \ dxdt\right| \qquad (16)\\
&+ \quad C\int_0^T \int_\Omega \max_{S\subset\partial K} h_k^{-1}\cdot\big|\big[\partial_s p_h\big]\big|\cdot\big|(\varphi-\varphi_h^I)\big| \ \ dxdt\\
&+ \quad \frac{C}{c_h^2}\int_0^T \int_\Omega \tau^{-1}\cdot\big|\big[\partial p_{h\,t}\big]\big|\cdot\big|\big(\varphi-\varphi_h^I)\big| \ \ dxdt\\
&\leq \quad C\ \int_0^T \int_\Omega \left|\frac{1}{c_h^2}\frac{\partial^2 p_h}{\partial t^2} - \triangle p_h - f\right|\cdot\left(\tau^2\left|\frac{\partial^2\varphi}{\partial t^2}\right| + h^2\big|D_x^2\varphi\big|\right) \ \ dxdt\\
&+ \quad C\int_0^T \int_\Omega \max_{S\subset\partial K} h_k^{-1}\cdot\big|\big[\partial_s p_h\big]\big|\cdot\left(\tau^2\left|\frac{\partial^2\varphi}{\partial t^2}\right| + h^2\big|D_x^2\varphi\big|\right) \ \ dxdt\\
&+ \quad \frac{C}{c_h^2}\int_0^T \int_\Omega \tau^{-1}\cdot\big|\big[\partial p_{h\,t}\big]\big|\cdot\left(\tau^2\left|\frac{\partial^2\varphi}{\partial t^2}\right| + h^2\big|D_x^2\varphi\big|\right) \ \ dxdt,
\end{aligned}
$$

where we used standard interpolation estimates for $\varphi - \varphi_h^I$, and C denotes interpolation constants. Next, the terms $\frac{\partial^2 p_h}{\partial t^2}$ and $\triangle p_h$ disappears in the first integral in (17) (p_h is continuous piecewise linear function).

We estimate $\frac{\partial^2 \varphi}{\partial t^2} \approx \frac{\left[\frac{\partial \varphi_h}{\partial t}\right]}{\tau}$ and $D_x^2 \varphi \approx \frac{\left[\frac{\partial \varphi_h}{\partial n}\right]}{h}$ to get:

$$
\begin{aligned}
\left|I_1\right| \leq \quad & C \int_0^T \int_\Omega \left|f\right| \cdot \left(\tau^2 \left| \frac{\left[\frac{\partial \varphi_h}{\partial t}\right]}{\tau} \right| + h^2 \left| \frac{\left[\frac{\partial \varphi_h}{\partial n}\right]}{h} \right| \right) \, dxdt \\
+ \quad & C \int_0^T \int_\Omega \max_{S \subset \partial K} h_k^{-1} \left| \left[\partial_s p_h\right] \right| \cdot \left(\tau^2 \left| \frac{\left[\frac{\partial \varphi_h}{\partial t}\right]}{\tau} \right| + h^2 \left| \frac{\left[\frac{\partial \varphi_h}{\partial n}\right]}{h} \right| \right) \, dxdt \\
+ \quad & \frac{C}{c_h^2} \int_0^T \int_\Omega \tau^{-1} \left| \left[\partial p_{h\,t}\right] \right| \cdot \left(\tau^2 \left| \frac{\left[\frac{\partial \varphi_h}{\partial t}\right]}{\tau} \right| + h^2 \left| \frac{\left[\frac{\partial \varphi_h}{\partial n}\right]}{h} \right| \right) \, dxdt.
\end{aligned} \tag{17}
$$

We estimate I_2 similarly:

$$
\begin{aligned}
\left|I_2\right| &\leq \int_0^T \int_\Omega \left| \left(\frac{1}{c_h^2} \frac{\partial^2 \varphi_h}{\partial t^2} (p - p_h^I) - \triangle \varphi_h (p - p_h^I) - (p_h - \tilde{p})(p - p_h^I) \right) \right| \, dxdt \\
&+ C \int_0^T \int_\Omega \max_{S \subset \partial K} \, h_k^{-1} \cdot \left| \left[\partial_s \varphi_h \right] \right| \cdot \left| (p - p_h^I) \right| \, dxdt \\
&+ \frac{C}{c_h^2} \int_0^T \int_\Omega \tau^{-1} \cdot \left| \left[\partial \varphi_{h\,t} \right] \right| \cdot \left| (p - p_h^I) \right| \, dxdt \\
&\leq C \int_0^T \int_\Omega \left| \left(\frac{1}{c_h^2} \frac{\partial^2 \varphi_h}{\partial t^2} - \triangle \varphi_h - (p_h - \tilde{p}) \right) \right| \cdot \left| (p - p_h^I) \right| \, dxdt \\
&+ C \int_0^T \int_\Omega \max_{S \subset \partial K} h_k^{-1} \cdot \left| \left[\partial_s \varphi_h \right] \right| \cdot \left| (p - p_h^I) \right| \, dxdt \\
&+ \frac{C}{c_h^2} \int_0^T \int_\Omega \tau^{-1} \cdot \left| \left[\partial \varphi_{h\,t} \right] \right| \cdot \left| (p - p_h^I) \right| \, dxdt
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & C\int_0^T\int_\Omega \left|\left(\frac{1}{c_h^2}\frac{\partial^2\varphi_h}{\partial t^2}-\triangle\varphi_h-(p_h-\tilde{p})\right)\right|\left(\tau^2\left|\frac{\partial^2 p}{\partial t^2}\right|+h^2\left|D_x^2 p\right|\right)\ dxdt\\
+\ & C\int_0^T\int_\Omega \max_{S\subset\partial K}\ h_k^{-1}\cdot\left|\left[\partial_s\varphi_h\right]\right|\left(\tau^2\left|\frac{\partial^2 p}{\partial t^2}\right|+h^2\left|D_x^2 p\right|\right)\ dxdt\\
+\ & \frac{C}{c_h^2}\int_0^T\int_\Omega \tau^{-1}\cdot\left|\left[\partial\varphi_{h\,t}\right]\right|\cdot\left(\tau^2\left|\frac{\partial^2 p}{\partial t^2}\right|+h^2\left|D_x^2 p\right|\right)\ dxdt\\
\leq\ & C\int_0^T\int_\Omega \left|(p_h-\tilde{p})\right|\cdot\left(\tau^2\left|\frac{\left[\frac{\partial p_h}{\partial t}\right]}{\tau}\right|+h^2\left|\frac{\left[\frac{\partial p_h}{\partial n}\right]}{h}\right|\right)\ dxdt\\
+\ & C\int_0^T\int_\Omega \max_{S\subset\partial K}\ h_k^{-1}\left|\left[\partial_s\varphi_h\right]\right|\cdot\left(\tau^2\left|\frac{\left[\frac{\partial p_h}{\partial t}\right]}{\tau}\right|+h^2\left|\frac{\left[\frac{\partial p_h}{\partial n}\right]}{h}\right|\right)\ dxdt\\
+\ & \frac{C}{c_h^2}\int_0^T\int_\Omega \tau^{-1}\cdot\left|\left[\partial\varphi_{h\,t}\right]\right|\cdot\left(\tau^2\left|\frac{\left[\frac{\partial p_h}{\partial t}\right]}{\tau}\right|+h^2\left|\frac{\left[\frac{\partial p_h}{\partial n}\right]}{h}\right|\right)\ dxdt.
\end{aligned}
$$

To estimate I_3 we use a standard approximation estimate of the form $c - c_h^I \approx hD_x c$ to get:

$$
\begin{aligned}
\left|I_3\right| &\leq \frac{2}{c_h^3} \int_0^T \int_\Omega \left| \frac{\partial \varphi_h(x,t)}{\partial t} \cdot \frac{\partial p_h(x,t)}{\partial t} \right| \cdot h \cdot \left|D_x c\right| \; dxdt - \gamma \int_\Omega \left|\triangle c_h\right| \cdot h \cdot \left|D_x c\right| \text{(1} \\
&\leq C \, \frac{2}{c_h^3} \int_0^T \int_\Omega \left| \frac{\partial \varphi_h(x,t)}{\partial t} \cdot \frac{\partial p_h(x,t)}{\partial t} \right| \cdot h \cdot \left| \frac{[c_h]}{h} \right| \; dxdt - \gamma \int_\Omega \left|\triangle c_h h \cdot \right| \frac{[c_h]}{h} \left| \; d \right. \\
&\leq C \, \frac{2}{c_h^3} \int_0^T \int_\Omega \left| \frac{\partial \varphi_h(x,t)}{\partial t} \cdot \frac{\partial p_h(x,t)}{\partial t} \right| \cdot \left|[c_h]\right| \; dxdt - \gamma \int_\Omega \left|\triangle c_h\right|[c_h]\left| \; dx. \right.
\end{aligned}
$$

Defining the residuals

$$
\begin{aligned}
R_{p_1} &= \big|f\big|, \;\; R_{p_2} = \frac{1}{2} \max_{S\subset\partial K} h_k^{-1}\big|\big[\partial_s p_h\big]\big|, \;\; R_{p_3} = \frac{1}{2\, c_h^2}\tau^{-1}\big|\big[\partial p_{h\,t}\big]\big|, \\
R_{\varphi_1} &= \big|p_h - \tilde{p}\big|, R_{\varphi_2} = \frac{1}{2} \max_{S\subset\partial K} h_k^{-1}\big|\big[\partial_s \varphi_h\big]\big|, \;\; R_{\varphi_3} = \frac{1}{2\, c_h^2}\tau^{-1}\big|\big[\partial \varphi_{h\,t}\big]\big|, \\
R_{c_1} &= \frac{2}{{c_h}^3}\left|\frac{\partial \varphi_h}{\partial t}\right| \cdot \left|\frac{\partial p_h}{\partial t}\right|, \;\; R_{c_2} = \big|\triangle c_h\big|,
\end{aligned}
$$

and interpolation errors in the form

$$
\sigma_\varphi = C\tau\left|\left[\frac{\partial \varphi_h}{\partial t}\right]\right| + Ch\left|\left[\frac{\partial \varphi_h}{\partial n}\right]\right|, \tag{19}
$$

$$
\sigma_p = C\tau\left|\left[\frac{\partial p_h}{\partial t}\right]\right| + Ch\left|\left[\frac{\partial p_h}{\partial n}\right]\right|, \tag{20}
$$

$$
\sigma_c = C\big|[c_h]\big|, \tag{21}
$$

we obtain the following a posteriori estimate

$$
\begin{aligned}
|e| \quad &\leq \quad \int_0^T \int_\Omega R_{p_1} \sigma_\varphi \;\; dxdt + \int_0^T \int_\Omega R_{p_2} \sigma_\varphi \;\; dxdt + \int_0^T \int_\Omega R_{p_3} \sigma_\varphi \;\; dxdt + \\
&+ \quad \int_0^T \int_\Omega R_{\varphi_1} \sigma_p \;\; dxdt + \int_0^T \int_\Omega R_{\varphi_2} \sigma_p \;\; dxdt + \int_0^T \int_\Omega R_{\varphi_3} \sigma_p \;\; dxdt \\
&+ \quad \int_0^T \int_\Omega R_{c_1} \sigma_c \;\; dxdt - \int_\Omega R_{c_2} \sigma_c \;\; dx
\end{aligned}
$$

An a posteriori error estimate for parameter identification

Now we present a more general a posteriori error estimate, which may be used to estimate the error in the parameter identification, our prime quantity of interest. This estimate involves the solution $\tilde{u}$ of the problem:

$$-L''(u_h; \bar{u}, \tilde{u}) = (\psi, \bar{u}) \;\; \forall \bar{u}, \tag{22}$$

where ψ acts as given data, and $L''(u; \cdot, \cdot)$ is the Hessian of the Lagrangian at u, which expresses the sensitivity of the Jacobian $L'(u; \cdot)$ with respect to changes in u. Assuming this problem can be solved, we obtain choosing here $\bar{u} = u - u_h$ and using the fact that $L''(u; \bar{u}, \tilde{u})$ is symmetric in $\bar{u}$ and $\tilde{u}$, the following error representation:

$$\begin{aligned}
((\psi, u - u_h)) &= -L''(u_h; u - u_h, \tilde{u}) \\
&= -L'(u; \tilde{u}) + L'(u_h; \tilde{u}) + R \\
&= L'(u_h; \tilde{u}) + R = L'(u_h; \tilde{u} - \tilde{u}^I) + R,
\end{aligned}$$

where $\tilde{u}^I$ is an interpolant of $\tilde{u}$ and again R is a second order remainder. Neglecting R we obtain the following analog of (1)

$$((\psi, u - u_h)) \approx L'(u_h; \tilde{u} - \tilde{u}^I),$$

with $\tilde{u}$ replacing u in the second argument. With proper choice of ψ and estimating $\tilde{u} - \tilde{u}^I$ as above by solving approximately for $\tilde{u}$, we may this way obtain, for example, an a posteriori error estimate for a mean value of the error in the parameter identification. The concrete form of this estimate is the same as that given above for the Lagrangian with only u replaced by $\tilde{u}$ in the weights. We will consider now scalar wave equation in the form

$$\begin{aligned} \alpha \frac{\partial^2 p}{\partial t^2} - \triangle p = f, \qquad \text{in } \ \Omega \times (0,T), \\ p(\cdot, 0) = 0, \ \ \frac{\partial p}{\partial t}(\cdot, 0) = 0, \ \ \text{in } \ \Omega, \\ p\big|_{\Gamma} = 0, \ \ \text{on } \ \Gamma \times (0,T), \end{aligned} \tag{23}$$

where we define $\alpha = \frac{1}{c^2}$.

The Hessian for the acoustic wave equation

In the acoustic case the second derivative L'' takes the form

$$
\begin{aligned}
L''(u;\bar{u},\tilde{u}) = &-((\alpha D\tilde{p}, D\bar{\varphi})) + ((\nabla\tilde{p}, \nabla\bar{\varphi})) \\
&+ ((\bar{p},\tilde{p}))_{\delta_{obs}} - ((\bar{\alpha}D\tilde{p}, D\varphi)) \\
&- ((\alpha D\bar{p}, D\tilde{\varphi})) + ((\nabla\bar{p}, \nabla\tilde{\varphi})) - ((\bar{\alpha}Dp, D\tilde{\varphi})) \\
&- ((\tilde{\alpha}Dp, D\bar{\varphi})) - ((\tilde{\alpha}D\bar{p}, D\varphi)) + \gamma(\nabla\bar{\alpha}, \nabla\tilde{\alpha}),
\end{aligned}
$$

and the Hessian problem takes the following strong form:

$$
\begin{aligned}
\alpha D^2\tilde{\varphi} - \Delta\tilde{\varphi} + \tilde{p}_{\delta_{obs}} + D^2\varphi\tilde{\alpha} &= \psi_1, \\
\alpha D^2\tilde{p} - \Delta\tilde{p} + D^2 p\tilde{\alpha} &= \psi_2, \\
\int_0^T D^2\varphi\tilde{p}\,dt + \int_0^T \tilde{\varphi}D^2 p\,dt - \gamma\triangle\tilde{\alpha} &= \psi_3,
\end{aligned}
\tag{24}
$$

together with initial and boundary conditions. The stability properties of this linear system determines the sensitivity in the parameter identification to perturbations. Thus, we may say that the secret of parameter identification is reflected by the stability (or solvability) properties of the linear system (24).

With correct data, the dual solution φ will be small and thus we may expect to be able to neglect the terms with $D^2\varphi$ as coefficient in (24). In this case one can prove uniqueness of the solution with $\gamma = 0$. We may further expect the stability properties of this system to improve (the sensitivity to decrease), with increasing number of observation points and correct observations.

Algorithm

To get error estimator, we solve iteratively system (24). The iterative algorithm is:

0. Find (p, α, φ), using quasi-Newton method, see [**?**], where p is solution of the state problem, φ is solution of the adjoint problem and α are discrete values of the identification parameter found from the optimality condition.

1. Choose guess for $\psi = (\psi_1, \psi_2, \psi_3)$, for example, $\psi = (0, 0, 1)$.

2. From third equation of system (24) eliminate $\tilde{\alpha}$ using equation

$$\tilde{\alpha}^{new} = \tilde{\alpha}^{old} + \rho(\psi_3 - \int_0^T \tilde{\varphi} D^2 p \, dt - \gamma \tilde{\alpha}^{old}) \tag{25}$$

with already computed (p, α, φ).

3. From second equation eliminate $\tilde{p}$ solving scalar wave equation

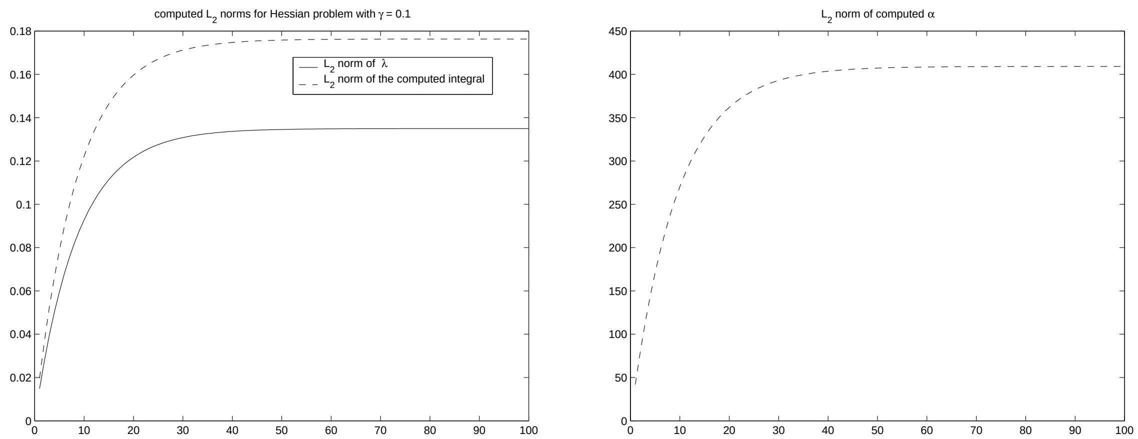
$$\alpha D^2 \tilde{p} - \Delta \tilde{p} = \psi_2 - D^2 p \tilde{\alpha} \tag{26}$$

4. From first equation eliminate $\tilde{\varphi}$ solving scalar wave equation

$$\alpha D^2 \tilde{\varphi} - \Delta \tilde{\varphi} = \psi_1 - \tilde{p}_{\delta_{obs}} \tag{27}$$

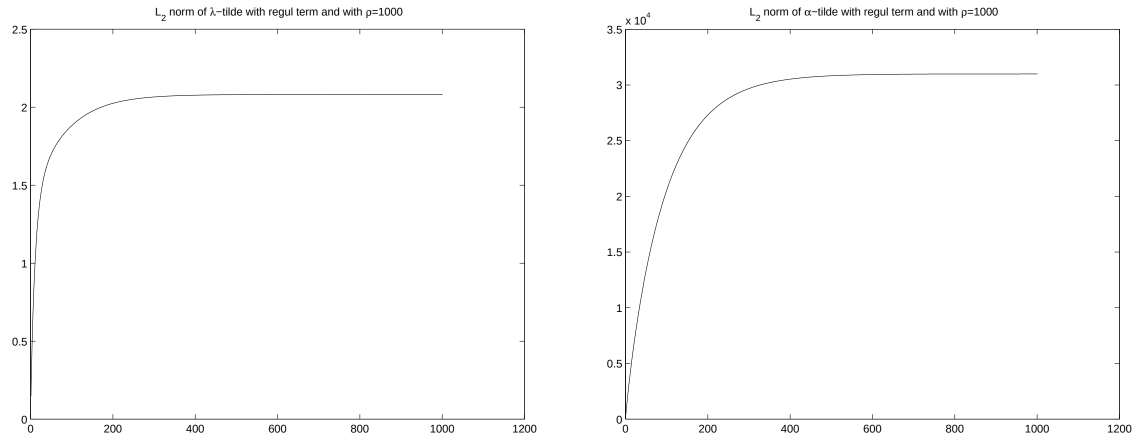
5. Repeat steps $2 - 4$ until desired convergence is achieved.

1 Numerical examples



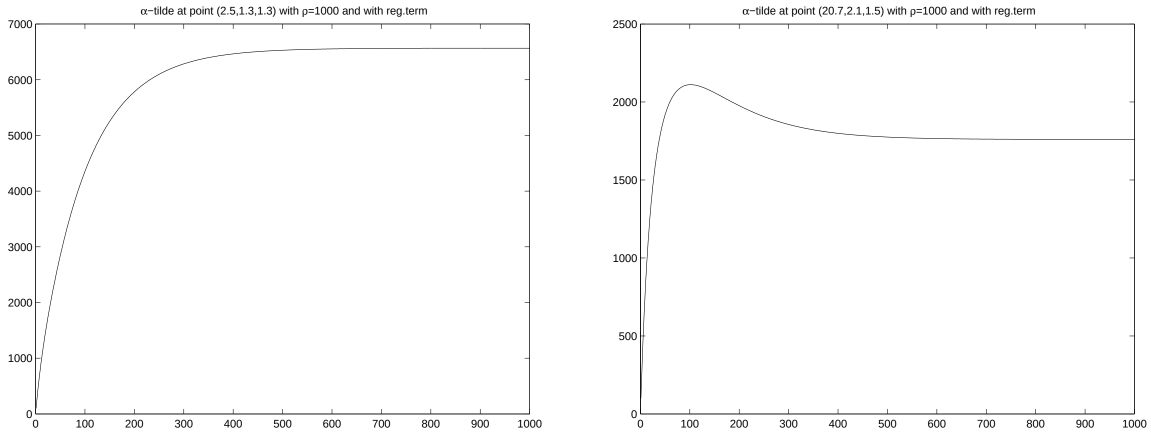
a) b)

Figure 1: Computed L_2 norms of $\tilde{\varphi}$ and $\int_0^T D^2 p\tilde{\varphi}$ are given in a), and L_2 norms of $\tilde{\alpha}$ are presented in b).



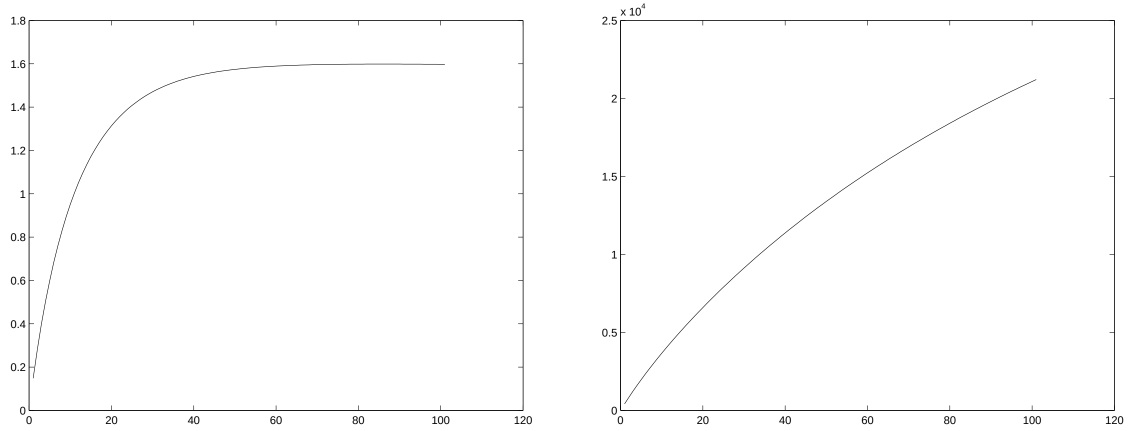
a) b)

Figure 2: Computed L_2 norms of $\tilde{\varphi}$ are given in a), and L_2 norms of $\tilde{\alpha}$ are presented in b).



a) b)

Figure 3: Values of $\tilde{\alpha}$ in one point (2.5,1.3,1.3) are given in a), and in point (0.7,2.1,1.5) are presented in b).



a) b)

Figure 4: Computed L_2 norms with $\gamma = 0$ of $\tilde{\varphi}$ are given in a), and of $\tilde{\alpha}$ are presented in b).

We now present numerical tests to prove convergence of the presented above algorithm. We simulate inverse acoustic scattering problem in three dimensions to identify parameter α in (23).

The computational domain is $\Omega = [0, 5.0] \times [0, 2.5] \times [0, 2.5]$, which is split into a finite element domain $\Omega_{FEM} = [0.3, 4.7] \times [0.3, 2.3] \times [0.3, 2.3]$ with a nonstructured mesh, and a surrounding domain Ω_{FDM} with a structured mesh. The space mesh in Ω_{FEM} consists of tetrahedra and in Ω_{FDM} of hexahedra with mesh size $h = 0.2$. We apply the hybrid finite element/difference method presented in [**?**] with finite elements in Ω_{FEM} and finite differences in Ω_{FDM} with absorbing boundary conditions on the boundary of Ω.

We present example of the reconstruction of a single cube with spherical pulses, generated at different points in Ω_{FDM}, which are given by the source function

$$f_1(x, x_0) = \begin{cases} 10^3 \sin^2 \pi t & \text{if } 0 \leq t \leq 0.1 \text{ and } |x - x_0| < r, \\ 0 & \text{otherwise;} \end{cases} \tag{28}$$

The experiments are performed with 6 spherical pulses, initialized in Ω_{FDM} at the points with coordinates $(0.45, 2.2, 1.25)$, $(1.25, 2.2, 1.25)$, $(2.05, 2.2, 1.25)$, $(2.95, 2.2, 1.25)$, $(3.75, 2.2, 1.25)$ and $(4.55, 2.2, 1.25)$, using absorbing boundary conditions on the outer boundary of Ω_{FDM}. In Fig. 5 - 6 we present the computed exact solution of the problem (23) inside Ω_{FEM} and Ω_{FDM}. The observation points are placed at the surface of the Ω_{FDM} such that they are located at the opposite side to the initialized pulses. We use a total of 22 observation points for this experiment.

To get data at the observation points we solve the acoustic wave equation with 6 pulses, initialized as described above, with the exact value of the parameter $\alpha = 2$ inside a single cube, and $\alpha = 1$ in the rest of the domain. We perform tests with $T = 3.0$ and 300 time steps.

The optimization algorithm is started with quess value of the parameter $\alpha = 1.0$ at all points of the computational domain. The computations was performed on five times adaptively refined meshes. In Table 1 we shown computed L_2 norms of $p - p_{obs}$ on adaptively refined meshes. The computational tests show, that the best results of the identification of the parameter α are obtained on 5 times adaptively refined mesh.

opt.it.	2783 nodes	2847 nodes	3183 nodes	3771 nodes	4283 nodes	6613 nodes
1	0.00694825	0.00692864	0.00693746	0.007015554	0.00708052	0.00719459
2	0.00693482	0.00688032	0.00681395	0.006836000	0.00687631	0.00687631
3	0.00692904	0.00685980	0.00673734	0.006691750	0.00667551	0.00667551
4	0.00692904	0.00685377	0.00670842	0.00665982	0.00663715	0.00663764
5		0.00685107	0.00670227	0.00665705	0.00663510	0.00661256

Table 1: L_2 norm of computed $p - p_{obs}$ for number of stored corrections $m = 5$ on adaptively refined meshes.

1.1 Example 1

To prove the convergence of the Hessian problem (24) first we take values of (p, α, φ) from the solution of the system (4) on a 4 times adaptively refined mesh. Then we perform all the steps of the algorithm (23) until L_2 norms of $\tilde{\varphi}, \tilde{\alpha}$ are stabilized.

Computed L_2 norms of $\tilde{\alpha}, \tilde{\varphi}$ and $\int_0^T D^2 p\tilde{\varphi}$ with $\rho = 10$ and with parameter in regularization term $\gamma = 0.1$ are presented in Fig. 10, and for $\rho = 100$ and $\gamma = 0.01$ – in Fig. 2. Computed L_2 norms without regularization parameter ($\gamma = 0$) and with $\rho = 100$ are presented in Fig. 4.

We tested the same algorithm and with different values of parameters ρ and γ. In Fig. 7 we show computed L_2 norms of $\tilde{\alpha}, \tilde{\varphi}$ and $\int_0^T D^2 p\tilde{\varphi}$ with $\rho = 10$ and $\gamma = 0.01$, and also values of $\tilde{\alpha}$ in one point.

Better convergence of the algorithm can be obtained by adding a new regularization term $\rho\gamma\triangle\tilde{\alpha}$. Then the equation for computation $\tilde{\alpha}$ takes the following form:

$$\tilde{\alpha}^{new}(1+\rho\gamma) = \tilde{\alpha}^{old} + \rho(\psi_3 - \int_0^T \tilde{\varphi} D^2 p \, dt) + \rho\gamma\triangle\tilde{\alpha}^{old}. \qquad (29)$$

In Fig. 8 we show computed L_2 norms of $\tilde{\alpha}, \tilde{\varphi}$ with adding new regularization term and without it. In Fig. 9 are presented computational results for regularization parameters $\rho = 1.0, \gamma = 0.01$.

1.2 Example 2

In the second example the data (p, α, φ) for Hessian problem are obtained from the solution of the identification problem on the same meshes as for example 1. The solution at the observation points obtained on 5 times refined mesh.

opt.it.	2783 nodes	2847 nodes	3183 nodes	3771 nodes	4283 nodes	6613 nodes
1	0.00694825	0.00692864	0.00693746	0.007015554	0.00708052	0.00719459
2	0.00693361	0.00687162	0.00678214	0.00677909	0.00681245	0.00692611
3	0.0068053	0.00664284	0.00643874	0.00636594	0.00642106	0.00646023
4	0.00676193	0.00662587		0.0063419	0.00634826	0.00645059
5					0.00630912	0.00643267
						0.00622704
						0.00619646
						0.00618079

Table 2: L_2 norm of computed $p - p_{obs}$ for number of stored corrections $m = 5$ on adaptively refined meshes.

In this example the new values of the parameter α in reconstruction algorithm are computed as

$$\alpha^{(n+1)} = (1+\gamma)\alpha^n + \beta\left(\int_0^T \int_\Omega \frac{\partial \varphi^n(x,t)}{\partial t} \cdot \frac{\partial p^n(x,t)}{\partial t} \ dxdt - \gamma \int_\Omega \triangle \alpha^n \ dx\right), \tag{30}$$

where β is step length.

In Table 2 we show computed L_2 norms of $p - p_{obs}$ on adaptively refined meshes with regularization parameter $\gamma = 0.0001$.

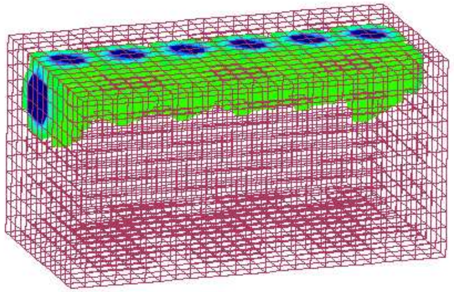
We tested again convergence of $\tilde{\alpha}$ in algorithm for Hessian problem with different values of parameters ρ and γ, when $\tilde{\alpha}$ is computed using (29). In Fig. 11-e),f) are presented L_2 norms of $\tilde{\alpha}$ and $\tilde{\varphi}$ computed with different regularization parameters.

After computing $\tilde{\alpha}$ we can define error in the computed parameter α using last two terms in a posteriori error for Lagrangian (**??**)

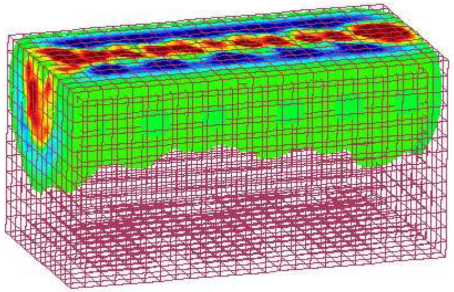
$$E_\alpha = \int_0^T \int_\Omega \left| \frac{\partial \varphi_h(x,t)}{\partial t} \cdot \frac{\partial p_h(x,t)}{\partial t} \right| \cdot \left| [\tilde{\alpha}_h] \right| \, dxdt - \gamma \int_\Omega |\triangle \alpha_h| | \cdot \left| [\tilde{\alpha}_h] \right| \, dx, \tag{31}$$

where $[\tilde{\alpha}_h]$ is jump of $\tilde{\alpha}_h$ across inter-element boundaries and is computed from two elements sharing common side.

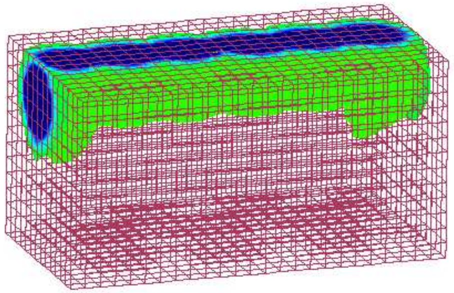
In this example the computed error in the parameter is $E_\alpha = 0.0324972$, when the exact value of the reconstructed parameter $\alpha \approx 1.97$, see Fig. 11-a),b).



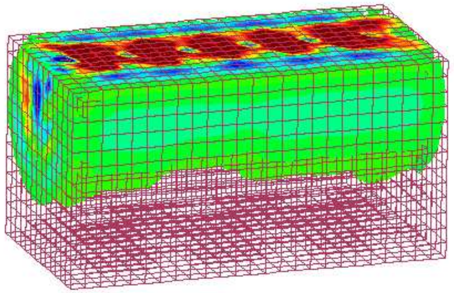
t=0.3



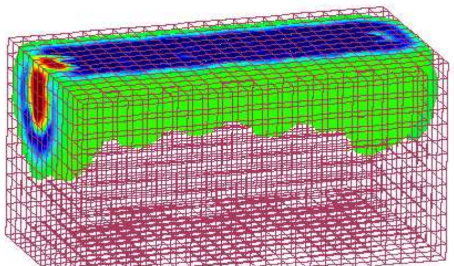
t=0.9

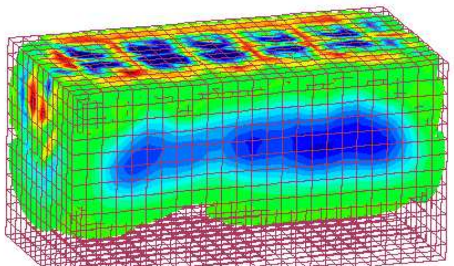


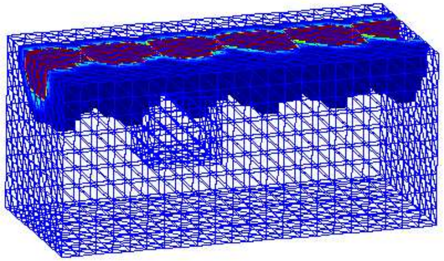
t=0.5

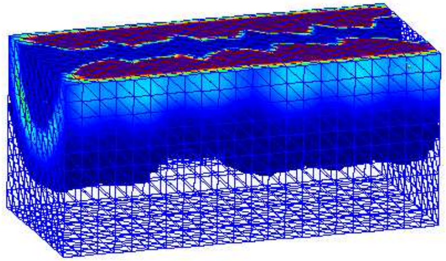


t=1.1



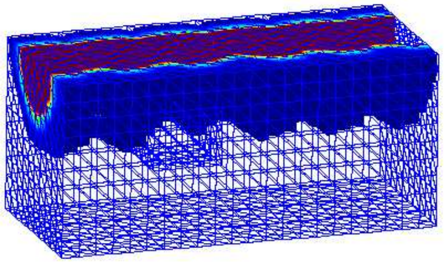


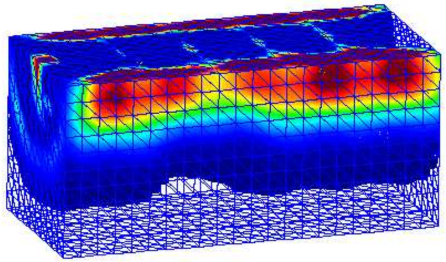




t=0.3

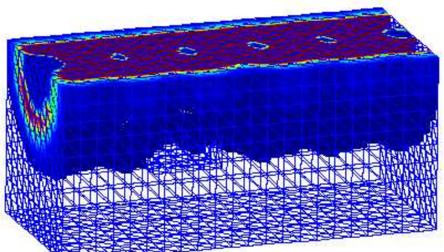
t=0.9

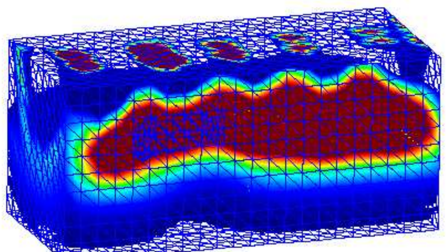


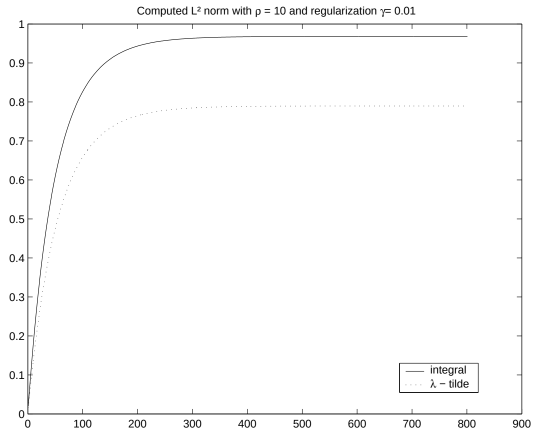


t=0.5

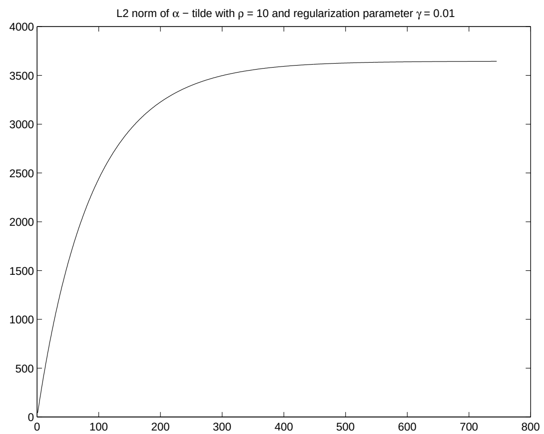
t=1.1



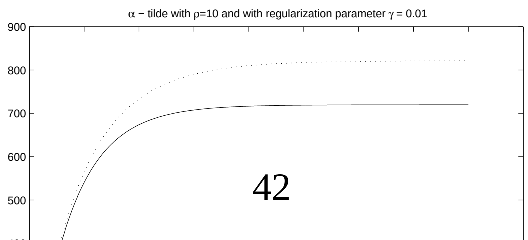


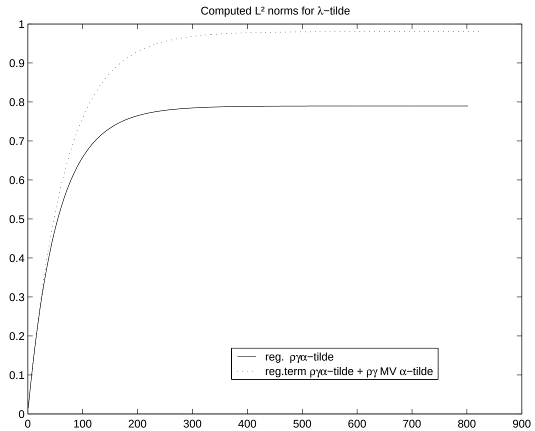

Computed L² norm with ρ = 10 and regularization γ= 0.01
integral
λ − tilde

a)

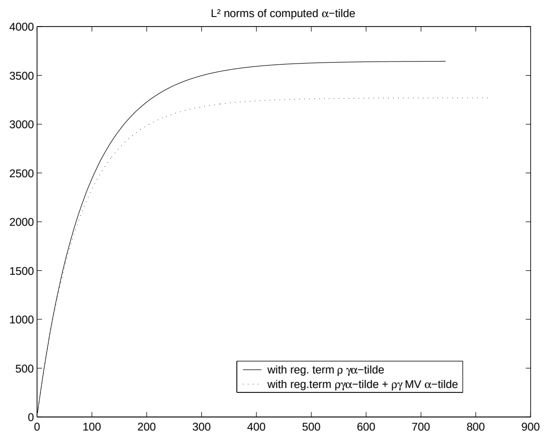

L2 norm of α − tilde with ρ = 10 and regularization parameter γ = 0.01

b)

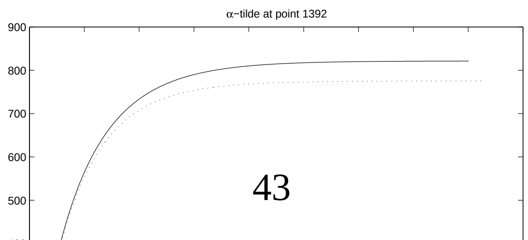

α − tilde with ρ=10 and with regularization parameter γ = 0.01

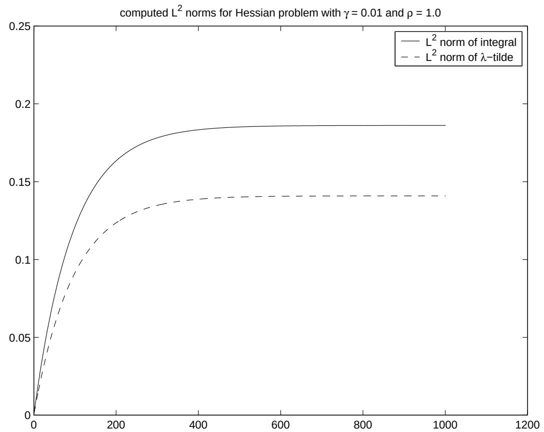

Computed L² norms for λ−tilde
reg. ργα−tilde
reg.term ργα−tilde + ργ MV α−tilde

a)

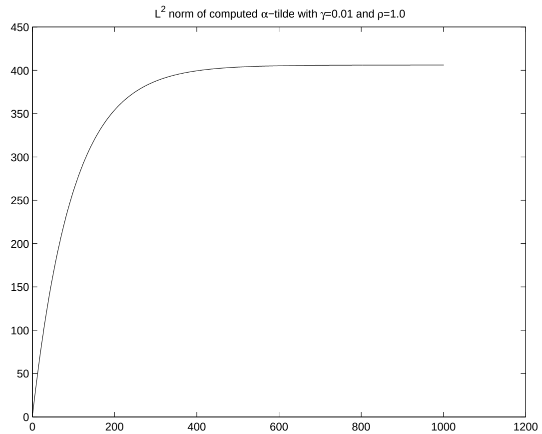

L² norms of computed α−tilde
with reg. term ρ γα−tilde
with reg.term ργα−tilde + ργ MV α−tilde

b)

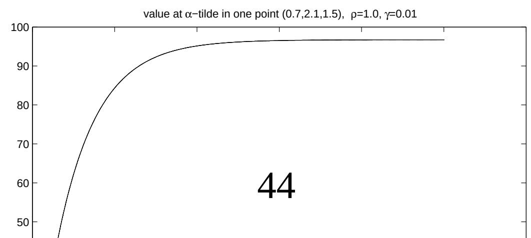

α−tilde at point 1392

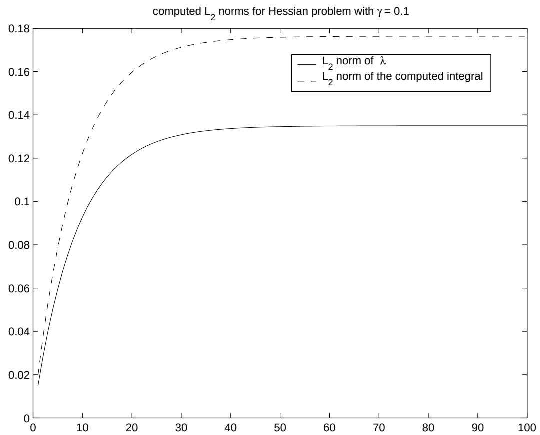

computed L2 norms for Hessian problem with γ = 0.01 and ρ = 1.0
L2 norm of integral
L2 norm of λ−tilde
0.25
0.2
0.15
0.1
0.05
0
0
200
400
600
800
1000
1200

a)

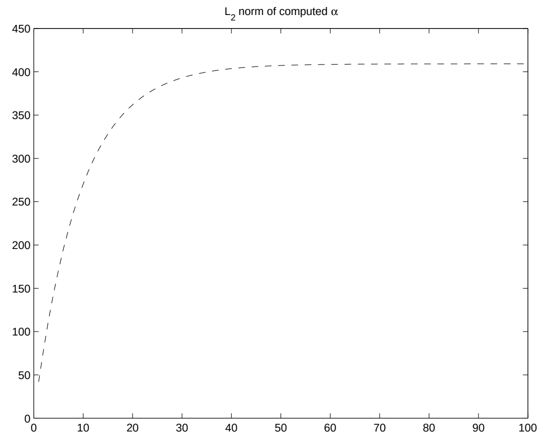

L2 norm of computed α−tilde with γ=0.01 and ρ=1.0
450
400
350
300
250
200
150
100
50
0
0
200
400
600
800
1000
1200

b)

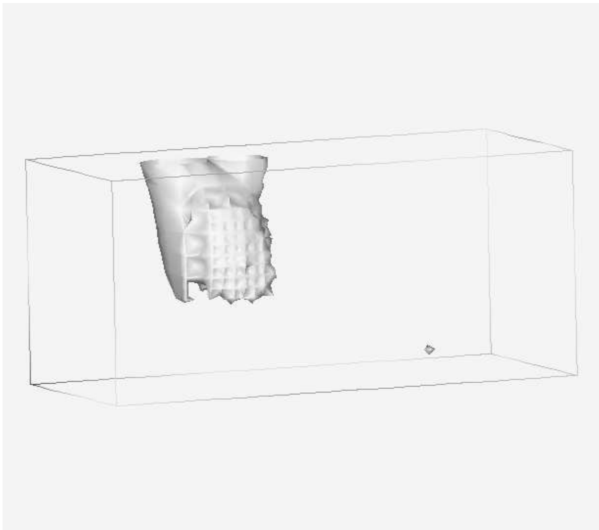

value at α−tilde in one point (0.7,2.1,1.5), ρ=1.0, γ=0.01
100
90
80
70
60
50



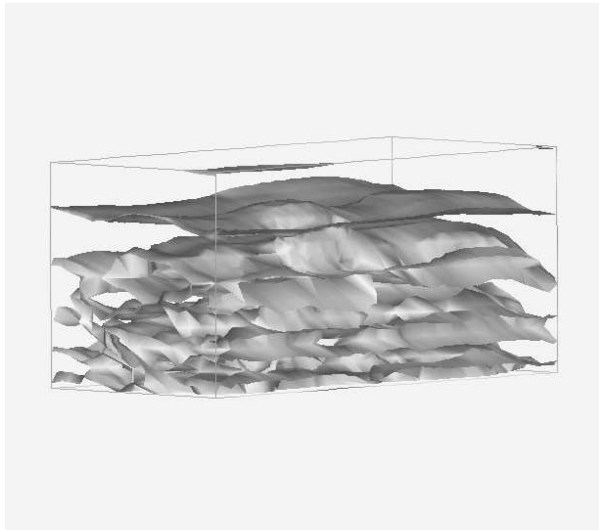
a)



b)

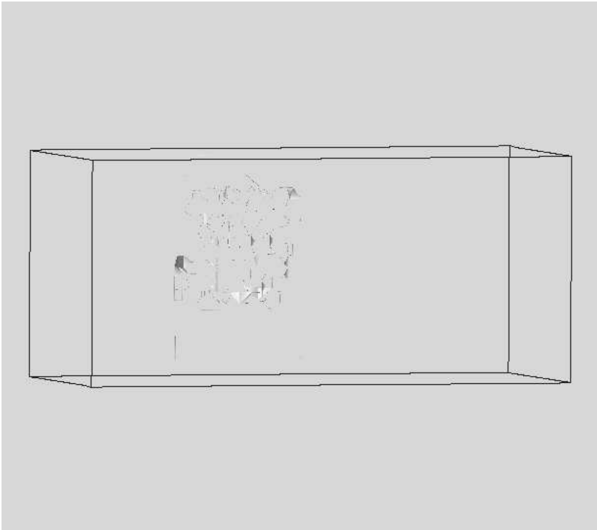


c) $\tilde{\alpha}$



d) $\tilde{\lambda}$

Figure 10: a)-b) Computed L_2 norms of $\tilde{\varphi}$ and $\int_0^T D^2 p\tilde{\varphi}$ in Example 1 are



a)

b)

c)

d)