The Finite Element Method for the Wave Equation
We consider the scalar wave equation modelling acoustic wave propagation in a bounded domain $\Omega^3$, with boundary $\Gamma$:

$$\frac{1}{c(x)^2} \frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \quad \text{in} \; \Omega \times (0, T),$$

$$u(\cdot, 0) = 0, \quad \frac{\partial u}{\partial t}(\cdot, 0) = f, \quad \text{in} \; \Omega,$$

$$\partial_n u \big|_{\Gamma_1} = 0, \quad \text{on} \; \Gamma \times (0, T),$$

where $u(x, t)$ is the pressure and $c(x)$ is the wave speed depending on $x = (x_1, x_2, x_3) \in \Omega$, $t$ is the time variable, $T$ is a final time, and $f$ is a load vector.
Finite Element Discretization

We now formulate a finite element method for (1) based on using continuous piecewise linear functions in space and time. We discretise $\Omega \times (0, T)$ in the usual way denoting by $K_h = \{K\}$ a partition of the domain $\Omega$ into tetrahedra $K$ ($h = h(x)$ being a mesh function representing the local diameter of the elements), and we let $J_k = \{J\}$ be a partition of the time interval $(0, T)$ into time intervals $J = (t_{k-1}, t_k]$ of uniform length $\tau = t_k - t_{k-1}, k = 1, ..., N$.

To formulate the finite element method for (1) we introduce the finite element space $W^u_h$ defined by:

$$W^u := \{u \in H^1(\Omega \times J) : u(\cdot, 0) = 0, \partial_n u|_{\Gamma} = 0\},$$

$$W^*_u := \{u \in W^u : u|_{K \times J} \in P_1(K) \times P_1(J), \forall K \in K_h, \forall J \in J_k\},$$

(2)
where $P_1(K)$ and $P_1(J)$ are the set of piecewise linear functions on $K$ and $J$, respectively.
To obtain discrete scheme we apply CG(1) in space and time and seek a discrete solution in the space $W_{h}^u$ for $u$ spanned by the functions

$$u(x, t) = \sum_{l=0}^{N} \sum_{i=1}^{M} u_i^l \varphi_i(x) \psi_l(t),$$

where $\varphi_i(x)$ and $\psi_l(t)$ are standard continuous piecewise linear functions in space and time, respectively.
Substituting this into (1), we obtain the following system of linear equations:

\[M(u^{k+1} - 2u^k + u^{k-1}) = \tau^2 F^k - \tau^2 A\left(\frac{1}{6} u^{k-1} + \frac{2}{3} u^k + \frac{1}{6} u^{k+1}\right), \quad k = 1, \ldots, N-1,\]  

with initial conditions:

\[u(0) = \frac{\partial u}{\partial t}\bigg|_{t=0} = 0.\]  

Here, \(M\) is the mass matrix in space, \(A\) is the stiffness matrix, \(F^k\) is the load vector at time level \(t_k\), \(k = 1, 2, 3 \ldots\), \(u\) is the unknown discrete field values of \(u\), and \(\tau\) is the time step.
The explicit formulas for the entries in system (3) at each element $e$ can be given as:

$$
M_{i,j}^e = \left( \frac{1}{c^2} \varphi_i, \varphi_j \right)_e, \\
K_{i,j}^e = (\nabla \varphi_i, \nabla \varphi_j)_e, \\
F_j^e = (f, \varphi_j)_e,
$$

(5)

where $(.,.)_e$ denotes the $L_2(e)$ scalar product. The matrix $M_e$ is the contribution from element $e$ to the global assembled matrix in space $M$, $K_e$ is the contribution from element $e$ to the global assembled matrix $K$. 


$W^u_h$ is a finite dimensional vector space, $\dim W^u_h = M \times N$, with a basis consisting of the standard continuous piecewise linear functions $\{\varphi_i\}^M_{i=1}$ in space and hat functions $\{\psi_l\}^N_{l=1}$ in time, where:

$$\psi_l(t) = \begin{cases} 
0, & \text{if } t \notin [t_{l-1}, t_{l+1}], \\
\frac{t-t_{l-1}}{t_i-t_{l-1}}, & \text{if } t \in [t_{l-1}, t_l], \\
\frac{t_{l+1}-t}{t_{l+1}-t_l}, & \text{if } t \in [t_l, t_{l+1}]; 
\end{cases}$$

(6)
The discrete system of equations

Using basis of functions \( \{ \varphi_i \}_{i=1}^M \) in space and \( \{ \psi_l \}_{l=1}^N \) in time we have:

\[
u_h = \sum_{l=0}^N \sum_{i=1}^M u_{hi} \varphi_i(x) \psi_l(t) .\]

Substituting into (1) we get:

\[
- \sum_{l=0}^N \sum_{i=1}^M u_{hi} \int_\Omega \frac{1}{c^2} \varphi_i(x) \int_{t_{l-1}}^{t_{l+1}} \frac{\partial \psi_l(t)}{\partial t} \frac{\partial v(x,t)}{\partial t} \, dx \, dt
\]

\[
+ \sum_{l=0}^N \sum_{i=1}^M u_{hi} \int_\Omega \int_{t_{l-1}}^{t_{l+1}} \nabla \varphi_i(x) \nabla v(x,t) \psi_l(t) \, dx \, dt
\]

\[
= \int_\Omega \int_0^T f(x)v(x,t) \, dx \, dt \quad \forall v \in W_h^u .
\]
We take \( v(x, t) = \varphi_j(x)\psi_m(t) \) and get:

\[
- \sum_{l=0}^{N} \sum_{i=1}^{M} u_{h_i}^l \int_\Omega \frac{1}{c^2} \varphi_i(x) \varphi_j(x) \int_{t_{l-1}}^{t_{l+1}} \frac{\partial \psi_l(t)}{\partial t} \frac{\partial \psi_m(t)}{\partial t} \, dx \, dt \\
+ \sum_{l=0}^{N} \sum_{i=1}^{M} u_{h_i}^l \int_\Omega \nabla \varphi_i(x) \nabla \varphi_j(x) \int_{t_{l-1}}^{t_{l+1}} \psi_l(t)\psi_m(t) \, dx \, dt
\]

\( = \int_\Omega \int_0^T f(x)v(x, t) \, dx \, dt \quad \forall v \in W_h^u. \)
Let $U = u_{n_i}$ denote the vector of unknown coefficients, $M = (m_{ij})$, $A = (a_{ij})$ are mass and stiffness matrices $M \times M$ in space, correspondingly, with coefficients

\[ m_{ij} = \int_{\Omega} \varphi_i(x) \varphi_j(x) \, dx, \]

\[ a_{ij} = \int_{\Omega} \nabla \varphi_i(x) \nabla \varphi_j(x) \, dx, \]

\[ P = (p_{lm}), K = (k_{lm}) \] are stiffness and mass matrices $N \times N$ in time with coefficients

\[ k_{lm} = \int_{t_{l-1}}^{t_{l+1}} \frac{\partial \psi_l(t)}{\partial t} \frac{\partial \psi_m(t)}{\partial t} \, dt, \]

\[ k_{lm} = \int_{t_{l-1}}^{t_{l+1}} \psi_l(t) \psi_m(t) \, dt, \]
and the load vector $b = (b_{jm})$ with coefficients

$$b_{jm} = \int_{t_{l-1}}^{t_{l+1}} \int_{\Omega} f(x) \varphi_j(x) \psi_m(t) \, dx \, dt. \quad (11)$$
First, we compute $K = (k_{lm})$ and $P = (p_{lm})$. Note, that
$k_{lm} = 0, p_{lm} = 0$ unless $l = m - 1, l = m, l = m + 1$. Using the
definition of test functions (6), we compute first diagonal elements $k_{ll}$:

$$
k_{ll} = \int_{t_{l-1}}^{t_{l+1}} \psi'_l(t)\psi'_l(t) \, dt
$$

$$
= \int_{t_{l-1}}^{t_l} \left( \frac{1}{\tau} \right)^2 \, dt + \int_{t_l}^{t_{l+1}} \left( \frac{-1}{\tau} \right)^2 \, dt = \frac{2}{\tau},
$$

(12)

$$
p_{ll} = \int_{t_{l-1}}^{t_{l+1}} \psi_l(t)\psi_l(t) \, dt
$$

$$
= \int_{t_{l-1}}^{t_l} \left( \frac{t - t_{l-1}}{\tau} \right)^2 \, dt + \int_{t_l}^{t_{l+1}} \left( \frac{t_{l+1} - t}{\tau} \right)^2 \, dt = \frac{2}{3}\tau,
$$
Similarly,

\[ k_{l,l+1} = \int_{t_l}^{t_{l+1}} \psi'_l(t) \psi'_l(t+1) \, dt \]
\[ = \int_{t_l}^{t_{l+1}} -\frac{1}{\tau} \frac{1}{\tau} \, dt \]
\[ = -\frac{1}{\tau}, \]  

(13)

\[ k_{l-1,l} = \int_{t_{l-1}}^{t_l} \psi'_l(t) \psi'_l(t) \, dt \]
\[ = \int_{t_{l-1}}^{t_l} -\frac{1}{\tau} \frac{1}{\tau} \, dt \]
\[ = -\frac{1}{\tau}. \]

Verification: write (6) for \( \psi_{l+1} \) and \( \psi_{l-1} \) and insert then into (14).
\[ p_{l, l+1} = \int_{t_{l-1}}^{t_{l+1}} \psi_l(t) \psi_{l+1}(t) \, dt \]

\[ = \int_{t_{l-1}}^{t_{l}} \left( \frac{t - t_{l-1}}{\tau} \right) \cdot \left( \frac{t - t_l}{\tau} \right) \, dt \]

\[ = \frac{1}{6} \tau, \]

\[ p_{l-1, l} = \int_{t_{l-1}}^{t_{l+1}} \psi_{l-1}(t) \psi_l(t) \, dt \]

\[ = \int_{t_{l}}^{t_{l+1}} \left( \frac{t_{l+1} - t}{\tau} \right) \cdot \left( \frac{t_l - t}{\tau} \right) \, dt \]

\[ = \frac{1}{6} \tau. \]

Verification: write (6) for \( \psi_{l+1} \) and \( \psi_{l-1} \) and insert then into (14).
We compute coefficients of $b$ in the same way to get:

$$b_{jm} = \int_{\Omega} \int_{t_{m-1}}^{t_{m+1}} f(x) \varphi_j(x) \psi_m(t) \, dx \, dt$$

$$= \int_{\Omega} f(x) \varphi_j(x) \int_{t_{m-1}}^{t_m} \frac{t - t_{m-1}}{\tau} \, dx \, dt$$

$$+ \int_{\Omega} f(x) \varphi_j(x) \int_{t_m}^{t_{m+1}} \frac{t_{m+1} - t}{\tau} \, dx \, dt$$

$$\approx \tau \int_{\Omega} f(x) \varphi_j(x) \, dx.$$
We substitute computed coefficients to (8) and get the system of linear equations (3):

\[ M(u^{k+1} - 2u^k + u^{k-1}) = \tau^2 F^k \]

\[ - \tau^2 A\left(\frac{1}{6}u^{k-1} + \frac{2}{3}u^k + \frac{1}{6}u^{k+1}\right), \quad k = 1, ..., N - 1. \]  \hspace{1cm} (16)

To obtain an explicit scheme we approximate \( M \) with the lumped mass matrix \( M^L \) in space, the diagonal approximation obtained by taking the row sum of \( M \), as well use mass lumping in time by replacing the terms \( \frac{1}{6}u^{k-1} + \frac{2}{3}u^k + \frac{1}{6}u^{k+1} \) by \( u^k \). By multiplying (16) with \((M^L)^{-1}\) we obtain an efficient explicit formulation:

\[ u^{k+1} = \tau^2 F^k + 2u^k - \tau^2 (M^L)^{-1} Ku^k - u^{k-1}, \quad k = 1, ..., N - 1, \]  \hspace{1cm} (17)