The Finite Element Method for the Wave Equation

The Wave Equation

We consider the scalar wave equation modelling acoustic wave propagation in a bounded domain Ω^3, with boundary Γ :

$$
\begin{aligned}
\frac{1}{c(\mathrm{x})^2}\frac{\partial^2 u}{\partial t^2} - \triangle u = 0, &\qquad \text{in } \Omega \times (0,T), \\
u(\cdot,0) = 0, &\quad \frac{\partial u}{\partial t}(\cdot,0) = f, \text{ in } \Omega, \\
\partial_n u\big|_{\Gamma_1} = 0, &\quad \text{on } \Gamma \times (0,T),
\end{aligned}
\tag{1}
$$

where $u(\mathrm{x},t)$ is the pressure and $c(\mathrm{x})$ is the wave speed depending on $\mathrm{x} = (x_1, x_2, x_3) \in \Omega$, t is the time variable, T is a final time, and f is a load vector.

Finite Element Discretization

We now formulate a finite element method for (1) based on using continuous piecewise linear functions in space and time. We discretise $\Omega \times (0,T)$ in the usual way denoting by $K_h = \{K\}$ a partition of the domain Ω into tetrahedra K ($h = h(\mathrm{x})$ being a mesh function representing the local diameter of the elements), and we let $J_k = \{J\}$ be a partition of the time interval $(0,T)$ into time intervals $J = (t_{k-1}, t_k]$ of uniform length $\tau = t_k - t_{k-1}, k = 1, ..., N$.

To formulate the finite element method for (1) we introduce the finite element space W_h^u defined by :

$$\begin{aligned} W^u &:= \{u \in H^1(\Omega \times J) : u(\cdot, 0) = 0,\ \partial_n u|_\Gamma = 0\}, \\ W_h^u &:= \{u \in W^u : u|_{K \times J} \in P_1(K) \times P_1(J), \forall K \in K_h, \forall J \in J_k\}, \end{aligned} \tag{2}$$

where $P_1(K)$ and $P_1(J)$ are the set of piecewise linear functions on K and J, respectively.

Fully discrete scheme

To obtain discrete scheme we apply CG(1) in space and time and seek a discrete solution in the space W_h^u for u spanned by the functions

$$u(x,t) = \sum_{l=0}^{N} \sum_{i=1}^{M} u_i^l \varphi_i(x) \psi_l(t),$$

where $\varphi_i(x)$ and $\psi_l(t)$ are standard continuous piecewise linear functions in space and time, respectively.

Substituting this into (1), we obtain the following system of linear equations:

$$M(\mathbf{u}^{k+1}-2\mathbf{u}^{k}+\mathbf{u}^{k-1}) = \tau^2 F^k - \tau^2 A(\frac{1}{6}\mathbf{u}^{k-1}+\frac{2}{3}\mathbf{u}^{k}+\frac{1}{6}\mathbf{u}^{k+1}), \quad k = 1, ..., N-1, \tag{3}$$

with initial conditions :

$$u(0) = \frac{\partial u}{\partial t}\Big|_{t=0} = 0. \tag{4}$$

Here, M is the mass matrix in space, A is the stiffness matrix, F^k is the load vector at time level $t_k, k = 1, 2, 3 \ldots,$ $\mathbf{u}$ is the unknown discrete field values of u, and τ is the time step.

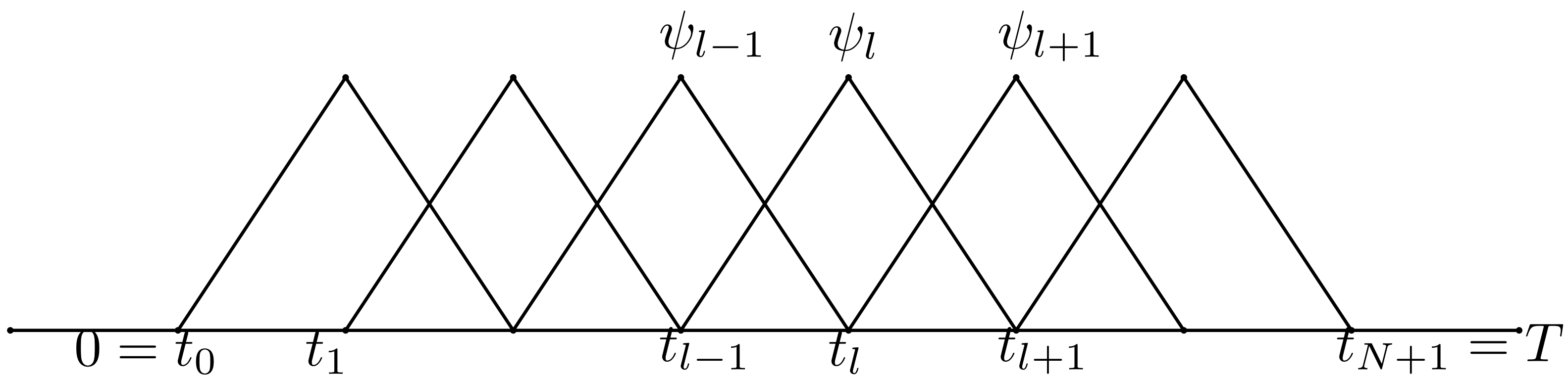
The explicit formulas for the entries in system (3) at each element e can be given as:

$$\begin{aligned} M^e_{i,j} &= (\frac{1}{c^2}\varphi_i, \varphi_j)_e, \\ K^e_{i,j} &= (\bigtriangledown\varphi_i, \bigtriangledown\varphi_j)_e, \\ F^e_j &= (f, \varphi_j)_e, \end{aligned} \tag{5}$$

where $(.,.)_e$ denotes the $L_2(e)$ scalar product. The matrix M_e is the contribution from element e to the global assembled matrix in space M, K^e is the contribution from element e to the global assembled matrix K.

W_h^u is a finite dimensional vector space, $\dim W_h^u = M \times N$, with a basis consisting of the standard continuous piecewise linear functions$\{\varphi_i\}_{i=1}^M$ in space and hat functions $\{\psi_l\}_{l=1}^N$ in time, where:

$$\psi_l(t) = \begin{cases} 0, & \text{if } t \notin [t_{l-1}, t_{l+1}], \\ \frac{t-t_{l-1}}{t_i-t_{l-1}}, & \text{if } t \in [t_{l-1}, t_l], \\ \frac{t_{l+1}-t}{t_{l+1}-t_l}, & \text{if } t \in [t_l, t_{l+1}]; \end{cases} \tag{6}$$



The discrete system of equations

Using basis of functions $\{\varphi_i\}_{i=1}^{M}$ in space and $\{\psi_l\}_{l=1}^{N}$ in time we have: $u_h = \sum_{l=0}^{N} \sum_{i=1}^{M} {u_h}_i^l \varphi_i(x)\psi_l(t)$. Substituting into (1) we get:

$$
\begin{aligned}
& - \sum_{l=0}^{N} \sum_{i=1}^{M} {u_h}_i^l \int_\Omega \frac{1}{c^2} \varphi_i(x) \int_{t_{l-1}}^{t_{l+1}} \frac{\partial \psi_l(t)}{\partial t} \frac{\partial v(x,t)}{\partial t} \, dxdt \\
& + \sum_{l=0}^{N} \sum_{i=1}^{M} {u_h}_i^l \int_\Omega \int_{t_{l-1}}^{t_{l+1}} \nabla \varphi_i(x) \nabla v(x,t) \psi_l(t) \, dxdt \qquad (7) \\
& = \int_\Omega \int_0^T f(x) v(x,t) \, dxdt \quad \forall v \in W_h^u.
\end{aligned}
$$

We take $v(x,t) = \varphi_j(x)\psi_m(t)$ and get:

$$
\begin{aligned}
&-\sum_{l=0}^{N}\sum_{i=1}^{M} {u_h}_i^l \int_\Omega \frac{1}{c^2}\varphi_i(x)\varphi_j(x)\int_{t_{l-1}}^{t_{l+1}} \frac{\partial\psi_l(t)}{\partial t}\frac{\partial\psi_m(t)}{\partial t}\,dxdt \\
&+\sum_{l=0}^{N}\sum_{i=1}^{M} {u_h}_i^l \int_\Omega \nabla\varphi_i(x)\nabla\varphi_j(x)\int_{t_{l-1}}^{t_{l+1}} \psi_l(t)\psi_m(t)\,dxdt \\
&= \int_\Omega\int_0^T f(x)v(x,t)\,dxdt \quad \forall v \in W_h^u.
\end{aligned}
\tag{8}
$$

Let $U = {u_h}_i^l$ denote the vector of unknown coefficients,
$M = (m_{ij}), A = (a_{ij})$ are mass and stiffness matrices $M \times M$ in space, correspondingly, with coefficients

$$\begin{aligned} m_{ij} &= \int_\Omega \varphi_i(x)\varphi_j(x)\,dx, \\ a_{ij} &= \int_\Omega \nabla\varphi_i(x)\nabla\varphi_j(x)\,dx, \end{aligned} \tag{9}$$

$P = (p_{lm}), K = (k_{lm})$ are stiffness and mass matrices $N \times N$ in time with coefficients

$$\begin{aligned} k_{lm} &= \int_{t_{l-1}}^{t_{l+1}} \frac{\partial\psi_l(t)}{\partial t}\frac{\partial\psi_m(t)}{\partial t}\,dt, \\ k_{lm} &= \int_{t_{l-1}}^{t_{l+1}} \psi_l(t)\psi_m(t)\,dt, \end{aligned} \tag{10}$$

and the load vector $b = (b_{jm})$ with coefficients

$$b_{jm} = \int_\Omega \int_{t_{l-1}}^{t_{l+1}} f(x)\varphi_j(x)\psi_m(t)\,dxdt. \tag{11}$$

First, we compute $K = (k_{lm})$ and $P = (p_{lm})$. Note, that $k_{lm} = 0, p_{lm} = 0$ unless $l = m-1, l = m, l = m+1$. Using the definition of test functions (6), we compute first diagonal elements k_{ll}:

$$\begin{aligned}
k_{ll} &= \int_{t_{l-1}}^{t_{l+1}} \psi_l'(t)\psi_l'(t)\, dt \\
&= \int_{t_{l-1}}^{t_l} \left(\frac{1}{\tau}\right)^2 dt + \int_{t_l}^{t_{l+1}} \left(\frac{-1}{\tau}\right)^2 dt = \frac{2}{\tau}, \\
p_{ll} &= \int_{t_{l-1}}^{t_{l+1}} \psi_l(t)\psi_l(t)\, dt \\
&= \int_{t_{l-1}}^{t_l} \left(\frac{t - t_{l-1}}{\tau}\right)^2 dt + \int_{t_l}^{t_{l+1}} \left(\frac{t_{l+1} - t}{\tau}\right)^2 dt = \frac{2}{3}\tau,
\end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned}
k_{l,l+1} &= \int_{t_{l-1}}^{t_{l+1}} \psi_l'(t)\psi_{l+1}'(t)\,dt \\
&= \int_{t_l}^{t_{l+1}} \frac{-1}{\tau}\frac{1}{\tau}\,dt \\
&= -\frac{1}{\tau}, \\
k_{l-1,l} &= \int_{t_{l-1}}^{t_{l+1}} \psi_{l-1}'(t)\psi_l'(t)\,dt \\
&= \int_{t_{l-1}}^{t_l} \frac{-1}{\tau}\frac{1}{\tau}\,dt \\
&= -\frac{1}{\tau}.
\end{aligned} \tag{13}$$

Verification: write (6) for ψ_{l+1} and ψ_{l-1} and insert then into (14).

$$
\begin{aligned}
p_{l,l+1} &= \int_{t_{l-1}}^{t_{l+1}} \psi_l(t)\psi_{l+1}(t)\,dt \\
&= \int_{t_l-1}^{t_l} \left(\frac{t-t_{l-1}}{\tau}\right)\cdot\left(\frac{t-t_l}{\tau}\right)dt \\
&= \frac{1}{6}\tau, \\
p_{l-1,l} &= \int_{t_{l-1}}^{t_{l+1}} \psi_{l-1}(t)\psi_l(t)\,dt \\
&= \int_{t_l}^{t_{l+1}} \left(\frac{t_{l+1}-t}{\tau}\right)\cdot\left(\frac{t_l-t}{\tau}\right)dt \\
&= \frac{1}{6}\tau.
\end{aligned}
\tag{14}
$$

Verification: write (6) for ψ_{l+1} and ψ_{l-1} and insert then into (14).

We compute coefficients of b in the same way to get:

$$\begin{aligned} b_{jm} &= \int_{\Omega} \int_{t_{m-1}}^{t_{m+1}} f(x)\varphi_j(x)\psi_m(t)\, dxdt \\ &= \int_{\Omega} f(x)\varphi_j(x) \int_{t_{m-1}}^{t_m} \frac{t - t_{m-1}}{\tau}\, dxdt \\ &+ \int_{\Omega} f(x)\varphi_j(x) \int_{t_m}^{t_{m+1}} \frac{t_{m+1} - t}{\tau}\, dxdt \\ &\approx \tau \int_{\Omega} f(x_j)\varphi_j(x)\, dx. \end{aligned} \tag{15}$$

We substitude computed coefficients to (8) and get the system of linear equations (3):

$$\begin{aligned} M(\mathbf{u}^{k+1} - 2\mathbf{u}^k + \mathbf{u}^{k-1}) &= \tau^2 F^k \\ &- \tau^2 A(\frac{1}{6}\mathbf{u}^{k-1} + \frac{2}{3}\mathbf{u}^k + \frac{1}{6}\mathbf{u}^{k+1}), \quad k = 1, ..., N-1. \end{aligned} \tag{16}$$

To obtain an explicit scheme we approximate M with the lumped mass matrix M^L in space, the diagonal approximation obtained by taking the row sum of M, as well use mass lumping in time by replacing the terms $\frac{1}{6}\mathbf{u}^{k-1} + \frac{2}{3}\mathbf{u}^k + \frac{1}{6}\mathbf{u}^{k+1}$ by $\mathbf{u}^k$. By multiplying (16) with $(M^L)^{-1}$ we obtain an efficient explicit formulation:

$$\mathbf{u}^{k+1} = \tau^2 F^k + 2\mathbf{u}^k - \tau^2 (M^L)^{-1} K \mathbf{u}^k - \mathbf{u}^{k-1}, \quad k = 1, ..., N-1, \tag{17}$$