Introduction to inverse and ill-posed problems:

Methods of regularization of inverse problems

Lecture 3

Quasi-Solution

The concept of quasi-solutions was originally proposed by V. K. Ivanov. It is designed to provide a rather general method for solving the ill-posed problem $F(x) = y, x \in G$ on a compact set $G \subset B_1$.

Example

Let the solution is parametrized, i.e.

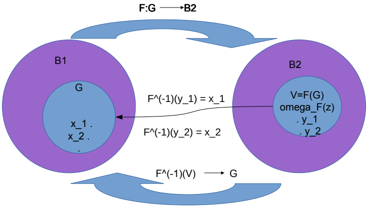
$$F(x, a) = \sum_{i=1}^{N} a_i \varphi_i(x),$$

where elements $\{\varphi_i(x)\}$ are a part of an orthonormal basis in a Hilbert space, the number N is fixed and coefficients $\{a_i\}_{n=1}^{N}$ are unknown. This problem appears in the solution of least squares problem for measured data y: find $\min_a \|F(x, a) - y\|_2^2$. One is seeking numbers $\{a_i\}_{n=1}^{N} \subset G$, where $G \subset \mathbb{R}^N$ is *a priori* chosen closed bounded set.
Since the right hand side y of equation

$$F(x, a) = y, \ x \in G. \tag{1}$$

is given with an error, it is unlikely that y belongs to the range of the operator F, and thus, the problem (2) is ill-posed. The question is: *Since the right hand side y of equation (2) most likely does not belong to the range $F(G)$ of the operator F, then what is the practical meaning of solving this equation on the compact set G, as required by Tikhonov's Theorem ?*

Quasi-Solution

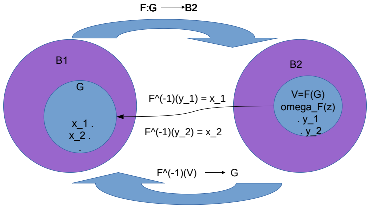


Suppose that the problem

$$F(x) = y, \; x \in G. \tag{2}$$

is conditionally well-posed and let $G \subset B_1$ be a compact set. Then the set $F(G) \subset B_2$ is also a compact set. We have $\|y - y^*\|_{B_2} \leq \delta$.

Quasi-Solution

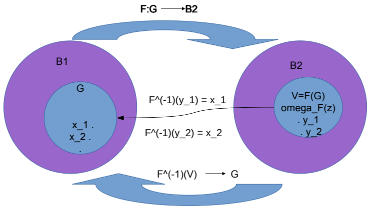


Consider the minimization problem,

$$\min_{x \in G} J(x), \text{ where } J(x) = ||F(x) - y||^2_{B_2} \tag{3}$$

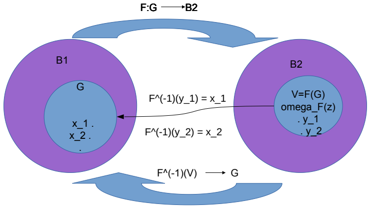
Since G is a compact set, then there exists a point $x = x(y_\delta) \in G$ at which the minimum in (3) is achieved (one can have many points $x(y_\delta)$).
Definition Any point $x = x(y) \in G$ of the minimum of the functional $J(x)$ in (3) is called *quasi-solution* of equation in (2) on the compact set G.

Quasi-Solution



A question is: *How far is the quasi-solution from the exact solution x^*?* Since by Tikhonov's Theorem the operator $F^{-1}: F(G) \to G$ is continuous and the set $F(G)$ is compact, then one of classical results of Real Analysis implies that there exists the modulus of the continuity $\omega_F(z)$ of the operator F^{-1} on the set $F(G)$.

Quasi-Solution



The function $\omega_F(z)$ satisfies the following four conditions:

- 1. $\omega_F(z)$ is defined for $z \geq 0$.
- 2. $\omega_F(z) > 0$ for $z > 0$, $\omega_F(0) = 0$ and $\lim_{z \to 0^+} \omega_F(z) = 0$.
- 3. The function $\omega_F(z)$ is monotonically increasing for $z > 0$.
- 4. For any two points $y_1, y_2 \in F(G)$ the following estimate holds

$$||F^{-1}(y_1) - F^{-1}(y_2)||_{B_1} \leq \omega_F ||y_1 - y_2||_{B_2}.$$

Quasi-Solution

The following theorem states that the problem of finding a quasi-solution is stable and two quasi-solutions are close to each other as long as the error in the data is small.

Theorem *Let B_1 and B_2 be two Banach spaces, $G \subset B_1$ be a compact set and $F : G \to B_2$ be a continuous one-to-one operator. Consider equation (2). Let $\|y^* - y_\delta\|_{B_2} \leq \delta$. Let $F(x^*) = y^*$. Let x_δ^q be a quasi-solution of equation (2), i.e.*

$$J(x_\delta^q) = \min_{x \in G} ||F(x) - y_\delta||_{B_2}^2. \tag{4}$$

Let $\omega_F(z), z \geq 0$ be the modulus of the continuity of the operator $F^{-1} : F(G) \to G$. Then the following error estimate holds

$$\|x_\delta^q - x^*\|_{B_1} \leq 2\delta\omega_F. \tag{5}$$

Quasi-Solution

Proof. Since $\|y^* - y_\delta\|_{B_2} \leq \delta$ then

$$J(x^*) = ||F(x^*) - y_\delta||^2_{B_2} = \|y^* - y_\delta\|^2_{B_2} \leq \delta^2.$$

Since the minimal value of the functional $J(x^*)$ is achieved at the point x^q_δ, then

$$J(x^*) \leq J(x^q_\delta) \leq \delta^2.$$

Since $J(x^q_\delta) = ||F(x^q_\delta) - y_\delta||^2_{B_2} \leq \delta^2$, then $||F(x^q_\delta) - y_\delta||_{B_2} \leq \delta$ and

$$\|y^q_\delta - y^*\| = ||F(x^q_\delta) - F(x^*)||_{B_2} \leq ||F(x^q_\delta) - y_\delta||_{B_2} + \|y_\delta - F(x^*)\|_{B_2}$$

$$= ||F(x^q_\delta) - y_\delta||_{B_2} + \|y_\delta - y^*\|_{B_2} \leq 2\delta.$$

Quasi-Solution

Since by the definition of the modulus of the continuity of the operator F^{-1} we have

$$\|x_1 - x_2\| = ||F^{-1}(y_1) - F^{-1}(y_2)||_{B_1} \leq \omega_F ||y_1 - y_2||_{B_2}.$$

Then applying this definition for $x_1 = x_\delta^q, x_2 = x^*, y_1 = y_\delta^q, y_2 = y^*$ we get

$$\|x_\delta^q - x^*\| = ||F^{-1}(y_\delta^q) - F^{-1}(y^*)||_{B_1} \leq \omega_F \underbrace{||y_\delta^q - y^*||_{B_2}}_{\leq 2\delta},$$

and thus,

$$\|x_\delta^q - x^*\|_{B_1} \leq 2\delta\omega_F. \tag{6}$$

□

Regularization

To solve ill-posed problems, regularization methods should be used. In this section we present main ideas of the regularization.

Definition Let B_1 and B_2 be two Banach spaces and $G \subset B_1$ be a set. Let the operator $F : G \to B_2$ be one-to-one. Consider the equation

$$F(x) = y. \tag{7}$$

Let y^* be the ideal noiseless right hand side of equation (8) and x^* be the ideal noiseless solution corresponding to $y^*, F(x^*) = y^*$. For every $\delta \in (0, \delta_0)\,,\ \delta_0 \in (0, 1)$ denote

$$K_\delta(y^*) = \left\{ z \in B_2 : \|z - y^*\|_{B_2} \leq \delta \right\}.$$

Regularization

Let $\alpha > 0$ be a parameter and $R_\alpha : K_{\delta_0}(y^*) \to G$ be a continuous operator depending on the parameter α. The operator R_α is called the *regularization operator* for

$$F(x) = y \tag{8}$$

if there exists a function $\alpha(\delta)$ defined for $\delta \in (0, \delta_0)$ such that

$$\lim_{\delta \to 0} \left\| R_{\alpha(\delta)}(y_\delta) - x^* \right\|_{B_1} = 0.$$

The parameter α is called the regularization parameter. The procedure of constructing the approximate solution $x_{\alpha(\delta)} = R_{\alpha(\delta)}(y_\delta)$ is called the *regularization procedure*, or simply *regularization*.
There might be several regularization procedures for the same problem. In the case of CIPs, usually $\alpha(\delta)$ is a vector of regularization parameters, such as, e.g. the number of iterations, the truncation value of the parameter of the Laplace transform, the number of finite elements, etc..

The Tikhonov Regularization Functional

Let B_1 and B_2 be two Banach spaces. Let Q be another space, $Q \subset B_1$ as a set and $\overline{Q} = B_1$. In addition, we assume that Q is compactly embedded in B_1. Let $G \subset B_1$ be the closure of an open set. Consider a continuous one-to-one operator $F : G \to B_2$. Our goal is to solve

$$F(x) = y, \quad x \in G. \tag{9}$$

Let y^* be the ideal noiseless right hand side corresponding to the ideal exact solution x^*,

$$F(x^*) = y^*, \quad \|y - y^*\|_{B_2} < \delta. \tag{10}$$

To find an approximate solution of equation (9), we minimize the Tikhonov regularization functional $J_\alpha(x)$,

$$J_\alpha(x) = \frac{1}{2}\|F(x) - y\|_{B_2}^2 + \frac{\alpha}{2}\|x - x_0\|_Q^2, \tag{11}$$

$$J_\alpha : G \to \mathbb{R}, \quad x_0 \in G$$

where $\alpha = \alpha(\delta) > 0$ is a small regularization parameter and the point $x_0 \in Q$.

The Tikhonov Regularization Functional

Tikhonov regularization functional $J_{\alpha}(x)$

$$J_{\alpha}(x) = \frac{1}{2}\|F(x) - y\|_{B_2}^2 + \frac{\alpha}{2}\|x - x_0\|_Q^2, \qquad (12)$$

- Usually x_0 is a good first approximation for the exact solution x^*, it is sometimes called the *first guess* or the *first approximation*.
- The term $\alpha\|x - x_0\|_Q^2$ is called the *Tikhonov regularization term* or simply the *regularization term*.
- Consider a sequence $\{\delta_k\}_{k=1}^{\infty}$ such that $\delta_k > 0, \lim_{k\to\infty}\delta_k = 0$. Our goal is to construct sequences $\{\alpha(\delta_k)\}, \{x_{\alpha(\delta_k)}\}$ in a stable way such that

$$\lim_{k\to\infty}\left\|x_{\alpha(\delta_k)} - x^*\right\|_{B_1} = 0.$$

The Tikhonov Regularization Functional

Using (10) and (12), we obtain

$$J_\alpha\left(x^*\right) \leq \frac{1}{2}\left\|F(x^*) - y\right\|_{B_2}^2 + \frac{\alpha}{2}\left\|x^* - x_0\right\|_Q^2 \tag{13}$$

$$\leq \frac{\delta^2}{2} + \frac{\alpha}{2}\left\|x^* - x_0\right\|_Q^2 . \tag{14}$$

Let

$$m_{\alpha(\delta_k)} = \inf_G J_{\alpha(\delta_k)}\left(x\right).$$

By (14)

$$m_{\alpha(\delta_k)} \leq \frac{\delta_k^2}{2} + \frac{\alpha\left(\delta_k\right)}{2}\left\|x^* - x_0\right\|_Q^2 .$$

Hence, there exists a point $x_{\alpha(\delta_k)} \in G$ such that

$$m_{\alpha(\delta_k)} \leq J_{\alpha(\delta_k)}\left(x_{\alpha(\delta_k)}\right) \leq \frac{\delta_k^2}{2} + \frac{\alpha\left(\delta_k\right)}{2}\left\|x^* - x_0\right\|_Q^2 . \tag{15}$$

The Tikhonov Regularization Functional

By the definition of Tikhonov's functional,

$$\frac{1}{2}\left\|F(x_{\alpha(\delta_k)})-y\right\|_{B_2}^2+\frac{\alpha\left(\delta_k\right)}{2}\left\|x_{\alpha(\delta_k)}-x_0\right\|_Q^2=J_\alpha\left(x_{\alpha(\delta_k)}\right) \quad (16)$$

$$\leq\frac{\delta_k^2}{2}+\frac{\alpha\left(\delta_k\right)}{2}\left\|x^*-x_0\right\|_Q^2, \quad (17)$$

and thus

$$\frac{1}{2}\left\|F(x_{\alpha(\delta_k)})-y\right\|_{B_2}^2\leq\frac{\delta_k^2}{2}+\frac{\alpha\left(\delta_k\right)}{2}\left\|x^*-x_0\right\|_Q^2, \quad (18)$$

$$\frac{\alpha\left(\delta_k\right)}{2}\left\|x_{\alpha(\delta_k)}-x_0\right\|_Q^2\leq\frac{\delta_k^2}{2}+\frac{\alpha\left(\delta_k\right)}{2}\left\|x^*-x_0\right\|_Q^2. \quad (19)$$

The Tikhonov Regularization Functional

From the second inequality of (19) we get

$$\left\|x_{\alpha(\delta_k)} - x_0\right\|_Q^2 \leq \frac{2}{\alpha(\delta_k)} J_\alpha\left(x_{\alpha(\delta_k)}\right) = \frac{2}{\alpha(\delta_k)} \cdot \left[\frac{\delta_k^2}{2} + \frac{\alpha(\delta_k)}{2}\left\|x^* - x_0\right\|_Q^2\right] \tag{20}$$

$$\left\|x_{\alpha(\delta_k)} - x_0\right\|_Q^2 \leq \frac{\delta_k^2}{\alpha(\delta_k)} + \left\|x^* - x_0\right\|_Q^2 . \tag{21}$$

Suppose that

$$\lim_{k\to\infty} \alpha(\delta_k) = 0 \text{ and } \lim_{k\to\infty} \frac{\delta_k^2}{\alpha(\delta_k)} = 0. \tag{22}$$

Then (22) implies that the sequence $\left\{x_{\alpha(\delta_k)}\right\} \subset G \subseteq Q$ is bounded in the norm of the space Q. Since Q is compactly embedded in B_1, then there exists a subsequence of the sequence $\left\{x_{\alpha(\delta_k)}\right\}$ which converges in the norm of the space B_1.

The Tikhonov Regularization Functional

We assume that the sequence $\left\{x_{\alpha(\delta_k)}\right\}$ itself converges to a point $\overline{x} \in B_1$,

$$\lim_{k\to\infty} \left\|x_{\alpha(\delta_k)} - \overline{x}\right\|_{B_1} = 0.$$

Then (15) and (23) imply that $\lim_{k\to\infty} J_{\alpha(\delta_k)}\left(x_{\alpha(\delta_k)}\right) = 0$. On the other hand, by the definition of Tikhonov's functional,

$$\begin{aligned}\lim_{k\to\infty} J_{\alpha(\delta_k)}\left(x_{\alpha(\delta_k)}\right) &= \frac{1}{2}\lim_{k\to\infty}\left[\left\|F\left(x_{\alpha(\delta_k)}\right) - y\right\|_{B_2}^2 + \alpha\left(\delta_k\right)\left\|x_{\alpha(\delta_k)} - x_0\right\|_Q^2\right] \\ &= \frac{1}{2}\left\|F\left(\overline{x}\right) - y\right\|_{B_2}^2.\end{aligned}$$

Hence, $\left\|F\left(\overline{x}\right) - y\right\|_{B_2} = 0$, which means that $F\left(\overline{x}\right) = y$. Thus, we have constructed the sequence of regularization parameters $\left\{\alpha\left(\delta_k\right)\right\}_{k=1}^{\infty}$ and the sequence $\left\{x_{\alpha(\delta_k)}\right\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty}\left\|x_{\alpha(\delta_k)} - \bar{x}\right\|_{B_1} = 0$.

The Tikhonov Regularization Functional

- We should choose $\alpha(\delta_k)$ such that following conditions are fulfilled:

$$\lim_{k\to\infty} \alpha(\delta_k) = 0 \text{ and } \lim_{k\to\infty} \frac{\delta_k^2}{\alpha(\delta_k)} = 0. \quad (23)$$

  one can choose, for example $\alpha(\delta_k) = C\delta_k^{\mu}, \mu \in (0,2)$.

- The sequence $\{\alpha(\delta_k\}_{k=1}^{\infty}$ is called *regularizing sequence*.
- The sequence $\{x_{\alpha(\delta_k)}\}_{k=1}^{\infty}$ is called *minimizing sequence*.

Regularized Solution

- The considered process of the construction of the regularized sequence does not guarantee that the functional $J_\alpha(x)$ indeed achieves it minimal value.
- Suppose now that the functional $J_\alpha(x)$ does achieve its minimal value, $J_\alpha(x_\alpha) = \min_{x \in G} J_\alpha(x), \alpha = \alpha(\delta)$. Then $x_{\alpha(\delta)}$ is called a *regularized solution* of equation (9) for this specific value $\alpha = \alpha(\delta)$ of the regularization parameter.
- Let $\delta_0 > 0$ be a sufficiently small number. Suppose that for each $\delta \in (0, \delta_0)$ there exists an $x_{\alpha(\delta)}$ such that $J_{\alpha(\delta)}\left(x_{\alpha(\delta)}\right) = \min_{x \in G} J_{\alpha(\delta)}(x)$.
- Thus, one might have several points $x_{\alpha(\delta)}$, we select a single one of them for each $\alpha = \alpha(\delta)$.