Introduction on Inverse Problems.
Description of different approaches.

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Chalmers
December 2019

Five parts

I. General introduction on Inverse Problems

II. The parabolic operators

III. The Dirichlet to Neumann approach

IV. The Carleman estimates approach

V. The pointwise method approach

Part I

General introduction on Inverse Problems.

Inverse problem !!! what is this ?

A tentative of definition:
The inverse map of a direct problem !!!
D01

Mandatory The direct problem must be well posed (in the sense of Hadamard : Existence, Uniqueness and continuity (stability) of solutions with respect to data).
Exemple:

$$u \mapsto u'$$

is the inverse problem associated to the direct problem

$$u \mapsto \int u,$$

but no the reverse definition.

D1

Inverse problem : a physical definition

Direct problem: Determine the physical state u produced by the knowledge of environmental parameters $(\alpha, \ldots)$ and constraints.

Inverse problem: Determine some physical parameter $\alpha \in U$ from measurements $y_u \in V$ related to the physical state u.
In most of the situation the forward map

$$A : U \to V; \alpha \mapsto y_u$$

is well defined injective and continuous.

Inverse problem : ill-posed problems !!!

Roughly speaking:

Measurements space is mostly L^2 functions and the closure of the range of operator A is compactly embedded in L^2. Then

$$A^{-1} : Range\, A \subset V \to U \text{ cannot be continuous}$$

$$\Rightarrow \text{ the inverse problem } y \mapsto \alpha \text{ is unstable.}$$

D1&D1bis&D2

Inverse problem : a wide range of domains

A non exhaustive list

1. Medical imaging (ultrasound, tomography (EIT, TAT), scanners, X rays, magnetic resonance imaging (MRI) . . .)
2. petroleum engineering
3. chemistry (determination of the constant of reaction)
4. radars (shape determination of an obstacle)
5. submarine acoustic (shape determination of an obstacle)
6. Quantum mechanic (determination of potential, . . .)
7. image processing (restoration of blurred images)
8. non destructive testing
9. . . .

Objectives

Three goals

- Uniqueness !!!
- stability !!
- reconstruction !

with as less as possible of observations ...

Theoretical/Numerical approaches

These two approaches are complementaries and in a lot of cases (at this time) only the numerical approach gives some results.
Several methods among others:

- regularization (”light” transformation of the equation studied into a new problem with better properties)
- minimization of a functional via least square method or more sophisticated one (optimisation methods)
- Bayesian methods (based on a prior information and specific calculations)
- . . .

See the numerical part of this course for nice calculations

Part II

The parabolic operators.

Diffusion equation (or Heat equation): elementary formulation

This PD equation models (e.g.) the dispersion of a population along the time:

$$\partial_t u(x,t) = D\Delta u(x,t), x \in \mathbb{R}^d, t > 0,$$

where $u(x,t)$ corresponds to the density of population and $D > 0$.

Diffusion equation: stochastic approach in 1D

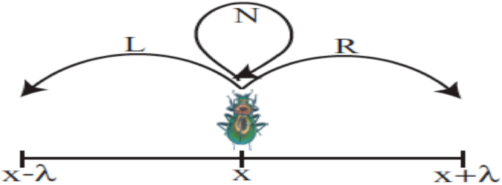
- Assuming that an individual is at point x_0 at time 0 what is the probability that the individual is at point $x \in \mathbb{R}$ at time $t > 0$?

- If a group of individuals starts at point x_0 at time 0 what is the density of population at point $x \in \mathbb{R}$ at time $t > 0$?

These questions are linked and we will carry out a PDE describing the evolution of the density of probability associated to the position of an individual.

D3

1D Random walk (D4&5, 6, 7, 8, 9)



Diffusion equation: Flux approach in 1D

- This approach is based on the Fick law which links the flux J of particles through a surface with the concentration of these particles $u(x,t)$ on this surface: $J = -D(x,t)\partial_x u(x,t)$ with $D(x,t)$ the diffusivity coefficient.

- This model fit with physical phenomena such propagation of **microscopic particles.**

D10&11

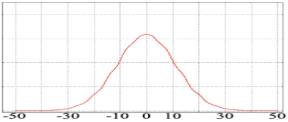
Multidimensional cases

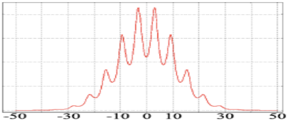
Two versions, with $D(x,t) > 0$ a real valued function:

1. Fokker-Planck (macroscopic model):
   $\partial_t u(x,t) = \Delta(D(x,t)u(x,t)), x \in \mathbb{R}^d, t > 0,$ where D is the mobility coefficient
2. Fick (heat diffusion):
   $\partial_t u(x,t) = div(D(x,t)\nabla u(x,t)), x \in \mathbb{R}^d, t > 0,$ where D is the diffusivity coefficient

Commentaries:
- Recall : $\Delta(Du) = div(\nabla[Du]) = div(D\nabla u) + div(u\nabla D)$.
- If D is constant, Fick and Fokker-Planck are similar models, but in heterogeneous media the Fick model homogenizes and on the other hand the Fokker-Planck model concentrates the solution in the domain where D is small.

Fick versus Fokker-Planck (D12)





Fundamental solution

Consider the diffusion equation

$$\partial_t u = \Delta u, t > 0, x \in \mathbb{R}^d. \quad (1)$$

We are looking for a fundamental solution of (1) which will allow us to write a solution of the Cauchy problem associated to (1), see next slide.

Lemma

A fundamental solution of (1) *is*

$$\varphi(x,t) = \frac{1}{(\sqrt{4\pi t})^d} e^{-\frac{|x|^2}{4t}},$$

and

$$\forall t > 0, \int_{\mathbb{R}^d} \varphi(x,t)dx = 1.$$