Introduction on Inverse Problems.
Description of different approaches.

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Cauchy problem in $\mathbb{R}^d$

Consider the Cauchy Problem (S)

$$\begin{cases} \partial_t u = \Delta u, t > 0, x \in \mathbb{R}^d, \\ u(x,0) = u_0(x), x \in \mathbb{R}^d. \end{cases}$$

From the fundamental solution we get an explicit solution of (S)

Theorem

Let $u_0 \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$*. If*

$$u(x,t) = \frac{1}{(\sqrt{4\pi t})^d} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy,$$

then:

- $u \in C^\infty(]0,\infty[\times\mathbb{R}^d)$
- u *checks* $\partial_t u = \Delta u, t > 0, x \in \mathbb{R}^d$,
- $\lim_{t\to 0} u(t,x_0) = u_0(x_0)$,
- $\int_{\mathbb{R}^d} u(t,x)dx = \int_{\mathbb{R}^d} u_0(x)dx$.

Remarks

- Mass conservation
- Infinite speed of propagation, if u_0 is bounded, continuous and positive, even if u_0 is compactly supported. It is false in the case of porous media $\partial_t u = \Delta u^m, m > 1$.
- Regularization effect (u_0 is assumed to be continuous and bounded, but as soon as $t > 0$, then $u \in C^\infty(\mathbb{R}_*^+ \times \mathbb{R}^d)$).

(D18)

Cauchy problem in a bounded set Ω

The problem

$$\begin{cases} \partial_t u = \Delta u, t > 0, x \in \Omega, \\ \text{Initial condition } u(0,x) = u_0(x), x \in \Omega, \\ + \text{ boundary condition on } \partial\Omega \text{ for } t \geq 0 \end{cases} \tag{1}$$

The more classical boundary conditions:

- Dirichlet condition (absorbing condition):
  $u(t,x) = 0, t > 0, x \in \partial\Omega$, like a precipice
- Neumann condition (reflecting condition)
  $\frac{\partial u}{\partial \nu}(x,t) = \nabla u(t,x).\nu(x) = 0, t > 0, x \in \partial\Omega$ where $\nu(x)$ is the outward normal at the point x at $\partial\Omega$.
- Robin conditions: $\alpha(t,x)\frac{\partial u}{\partial \nu} + \beta(t,x)u = 0, t > 0, x \in \partial\Omega$ with $\alpha^2 + \beta^2 > 0$.

Existence, uniqueness and regularity of the solution

We are interested here in the strong (or classical) solutions

Theorem

1. *Dirichlet case: If* $u_0 \in C^{2,\alpha}(\overline{\Omega}), \alpha > 0$ *and* $u_0 = \Delta u_0 = 0$ *on* $\partial\Omega$*, there exists a unique solution* u *of* (1) *with* $u \in C^2_1(\mathbb{R}_+ \times \overline{\Omega})$ *and* $u \in C^\infty(R^*_+ \times \overline{\Omega})$*.*
2. *Neumann case: If* $u_0 \in C^{2,\alpha}(\overline{\Omega}), \alpha > 0$ *and* $\frac{\partial u_0}{\partial \nu} = 0$ *on* $\partial\Omega$*, there exists a unique solution* u *of* (1) *with* $u \in C^2_1(\mathbb{R}_+ \times \overline{\Omega})$ *and* $u \in C^\infty(R^*_+ \times \overline{\Omega})$*.*

- Here, $C^{2,\alpha}(\overline{\Omega})$ is the Holder space of the function $C^2(\overline{\Omega})$ which second derivative is Holder.

g is $\alpha-$holderienne if $[g]_\alpha = \sup_{x,y\in\overline{\Omega},x\neq y} \dfrac{|g(x)-g(y)|}{|x-y|^\alpha}$ is finite.

- $C^2_1(\mathbb{R}_+ \times \overline{\Omega})$ corresponds to continuous functions on $\mathbb{R}_+$ which space derivative until order 2 and the time derivative are continuous on $\mathbb{R}_+ \times \overline{\Omega}$.

The diffusion-reaction models

Consider

$$\begin{cases} \partial_t u = D\Delta u + f(x,u), t \geq 0, x \in \Omega, \\ \text{Initial condition } u(0,x) = u_0(x), x \in \Omega, \\ + \text{ Dirichlet or Neumann boundary condition on } \partial\Omega \text{ for } t \geq 0. \end{cases} \tag{2}$$

The reaction term $f(x,u)$ allows to give a more precise modeling of the problem studied, e.g. in case of population model it can describe the death and birth events.

e.g. the F-KPP models.

(D21)

The diffusion-reaction models: existence, uniqueness

Theorem

Assume $f = f(x, u)$ *verifies* $f, \partial_u f \in C(\overline{\Omega} \times \mathbb{R})$ *and is Lipschitz in* x*. Assume* $u_0 \in C^{2,\alpha}(\overline{\Omega})$ *and* u_0 *verifies compatibility condition. Then the problem* (2) *admits an unique solution in* $C_1^2(R_+ \times \overline{\Omega})$*.*

Parabolic equations:

We can generalize the previous results to a more general formulation of a parabolic problem as an initial/boundary value problem as follows:

$$\begin{cases} \partial_t u + Lu = f(x,t), t \in (0,T], x \in \Omega, \\ \text{Initial condition } u(0,x) = u_0(x), x \in \Omega, \\ + \text{ Dirichlet or Neumann boundary condition on } \partial\Omega \text{ for } t \in [0,T], \end{cases} \tag{3}$$

for $T > 0$ and where L denotes for each time t a second order partial differential operator having either the divergence form

$$Lu = -\sum_{i,j=1}^{n} (a_{ij}(x,t)u_{x_j})_{x_i} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} + c(x,t)u$$

or else the non divergence form

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} + c(x,t)u$$

Parabolic equations:

Definition

We say that the partial differential operator $\frac{\partial}{\partial t} + L$ is (uniformly) parabolic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2,$$

for all $(x,t) \in \Omega \times (0,T]$, and $\xi \in \mathbb{R}^n$.

Improved regularity: weak solutions for (3)

Some of the methods tackled in this talk need some regularity assumptions. For this we study the generous regularities properties that we can get for the solution of a parabolic equation.

We assume that
$a_{ij}, b_i, c \in L^\infty(\Omega \times (0,T])$, $a_{ij} = a_{ji}$ and $u_0 \in L^2(\Omega), f \in L^2(\Omega \times (0,T])$, and for simplicity, we assume homogeneous Dirichlet boundary condition. We define the bilinear form:

$$b(u,v,t) := \int_\Omega \sum_{i,j=1}^{n} a_{ij}(.,t) u_{x_i} v_{x_j} + \sum_{i=1}^{n} b_i(.,t) u_{x_i} v + c(.,t) uv \, dx,$$

for $u, v \in H_0^1(\Omega)$ and a.e. $0 \leq t \leq T$.

Improved regularity: weak solutions for (3)

Definition

We say that a function

$$u \in L^2(0, T; H_0^1(\Omega)), \text{ with } u' \in L^2(0, T; H^{-1}(\Omega))$$

is a weak solution of the problem (3) provided

- $< u', v > + b(u, v, t) = (f, v)$

  for each $v \in H_0^1(\Omega)$ and a.e. $0 \leq t \leq T$, and
- $u(0) = u_0$

Improved regularity: weak solutions for (3)

The strategy:

First we prove the existence and the uniqueness of weak solution for (3). This first step involves the construction of Galerkin approximations and we carry out solutions from standard theory for ordinary differential equations then we derive energy estimates.

In a second step, basing ourselves on these results we prove existence of strong solutions and get results of higher regularity for the solution of (3).
(D21-1,2)

Improved regularity: formal calculation

We want to have an idea of the level of regularity we can hope for the weak solution of (3). At this point we assume that this solution is sufficiently smooth and tends to 0 when $|x| \to \infty$ to carry out some computations and derive estimates for the L^2 norms for

$$u_t, |Du|, |D^2 u| \text{ and } |Du_t|$$

in terms of the L^2 norms of

$$f, f_t, |Du_0| \text{ and } D^2 u_0.$$

(D21-3,4,5)

Improved regularity for weak solution - Part I

Theorem

- *Assume*

$$u_0 \in H_0^1(\Omega), f \in L^2(0,T;L^2(\Omega))$$

suppose also $u \in L^2(0,T;H_0^1(\Omega))$, *with* $u' \in L^2(0,T;H^{-1}(\Omega))$, *is the weak solution of* (3). *Then we have*

$$u \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)), u' \in L^2(0,T;L^2(\Omega)),$$

and we have the estimate

$$ess \sup_{0\leq t\leq T} \|u(t)\|_{H_0^1(\Omega)} + \|u\|_{L^2(0,T;H^2(\Omega))} + \|u'\|_{L^2(0,T;L^2(\Omega))},$$

$$\leq C\left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|_{H_0^1(\Omega)}\right).$$

Where C *depends only on* Ω, T *and the coefficients.*

Improved regularity for weak solution -Part II

Theorem (following)

- *If in addition,*

$$u_0 \in H^2(\Omega), f' \in L^2(0,T;L^2(\Omega))$$

then

$$u \in L^\infty(0,T;H^2(\Omega))), u' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)),$$

$$u'' \in L^2(0,T;H^{-1}(\Omega))$$

and we have the estimate

$$ess \sup_{0\leq t\leq T} \left(\|u(t)\|_{H^2(\Omega)} + \|u'\|_{L^2(\Omega)}\right) + \|u'\|_{L^2(0,T;H_0^1(\Omega))}$$

$$+\|u''\|_{L^2(0,T;H^{-1}(\Omega))} \leq C\left(\|f\|_{H^1(0,T;L^2(\Omega))} + \|u_0\|_{H^2(\Omega)}\right).$$

Higher regularity (D21-7,8,9,10)

Theorem

Assume

$$u_0 \in H^{2m+1}(\Omega), \frac{\partial^k f}{\partial t^k} \in L^2(0,T;H^{2m-2k}(\Omega)); \ (k = 0, \dots, m),$$

assume also that all the order compatibility conditions hold. Then

$$\frac{\partial^k u}{\partial t^k} \in L^2(0,T;H^{2m+2-2k}(\Omega)), (k = 0, \dots, m+1)$$

and we have the estimate

$$\sum_{k=0}^{m+1} \| \frac{\partial^k u}{\partial t^k} \|_{L^2(0,T;H^{2m+2-2k}(\Omega))}$$

$$\leq C \left(\sum_{k=0}^{m} \| \frac{\partial^k f}{\partial t^k} \|_{L^2(0,T;H^{2m-2k}(\Omega))} + \| u_0 \|_{H^{2m+1}(\Omega)} \right).$$

Where C depends only on m, Ω, T and the coefficients of L.

Infinite differentiability (D21-11)

Theorem

Assume

$$u_0 \in C^\infty(\bar{\Omega}), f \in C^\infty(\overline{\Omega \times (0, T]})$$

assume also that all the $m^{th}-$ *order compatibility conditions hold for* $m = 0, 1 \ldots.$
Then the problem (3) *has a unique solution*

$$u \in C^\infty(\overline{\Omega \times (0, T]})$$

Two interesting models: I- Fisher-Kolmogorov Petrovsky Piskunov

The more classical reaction-diffusion term (Fisher-KPP) is in the form:

$$f(x,u) = u(r(x) - \gamma(x)u), x \in \Omega, u \in \mathbb{R}.$$

The term $r(x)$ corresponds to the intrinsic growth rate of a population, it can be positive or negative.
The term $\gamma(x)$ corresponds to the intraspecific coefficient which is positive.

It appears in several fields of applications such as physics, in combustion flame propagation models, in chemistry, in ecology, to study the dynamics of a population as well as in population genetics.

Two interesting models: II- Lotka-Volterra

Consider a single-species model in 1D:

$$\frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} + r_1(x)\, u - a_{11}(x)\, u^2, \text{for } t > 0,\ x \in (a, b) \subset \mathbb{R},$$

where $D_1 > 0$, r_1 is the intrinsic growth rate and $a_{11} > 0$ is the intraspecific competition coefficient.

Assume that a second species v enters in competition with species u then, the two-species system can be modeled by the Lotka-Volterra competition model:

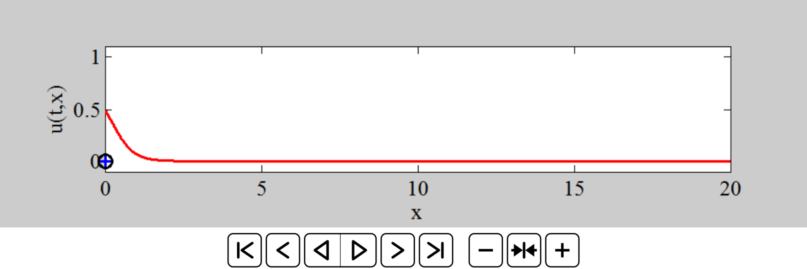
$$\begin{cases} \dfrac{\partial u}{\partial t} = D_1 \dfrac{\partial^2 u}{\partial x^2} + r_1\, u - a_{11}\, u^2 - a_{12}\, uv, \\ \dfrac{\partial v}{\partial t} = D_2 \dfrac{\partial^2 v}{\partial x^2} + r_2\, v - a_{21}\, uv - a_{22}\, v^2, \end{cases} \quad \text{for } t > 0,\ x \in (a, b) \subset \mathbb{R}.$$

with $D_2 > 0$, r_2 is the 2nd species intrinsic growth rate and $a_{22} > 0$ corresponds to the 2nd species intraspecific competition coefficient. a_{12} and a_{21} respectively measure the impact of species 2 upon species 1 (resp. of species 1 upon species 2).

Others specific parabolic equations

and some original associated inverse problems

1. Degenerate (D22)
2. Memory term (D23)
3. Fractional derivative (D24 25)
4. Kernel of dispersion (D26)
5. General diffusion equation (D27)
6. Unbounded domain (D28, 28-1)


u(t,x)
1
0.5
0
0
5
10
15
20
x

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A basic inverse problem solved with powerful tools

We are interested by the determination (meaning uniqueness or (and) reconstruction) of the $x-$dependent potential q in the following parabolic problem :

$$\begin{aligned} &\partial_t u(t,x) - \Delta u(t,x) + q(x)u(t,x) = 0, \quad \text{for } x \in \Omega, t \in (0,T), \\ &u|_{\partial\Omega \times (0,T)} = 0. \end{aligned} \tag{4}$$

using as less as possible of measurements.

Remark: this problem is similar to the reconstruction of a space dependent source term in the form: $f(x)R(x,t)$ where $R(x,t)$ is assumed to be known.