Introduction on Inverse Problems.

Description of different approaches.

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Five parts

I. General introduction on Inverse Problems

II. The parabolic operators

III. The Dirichlet to Neumann approach

IV. The Carleman estimates approach

V. The pointwise method approach

Part I

General introduction on Inverse Problems.

Part II

The parabolic operators.

Part III

The Dirichlet to Neumann approach

The inverse toy problem

We consider

$$\begin{aligned} &\partial_t u(x,t) - \Delta u(x,t) + q(x,t)u(x,t) = 0, \quad \text{for } x \in \Omega, t \in (0,T), \\ &+ \text{boundary conditions on } \partial\Omega \times (0,T). \end{aligned} \tag{1}$$

Let $\Gamma = \partial\Omega$ and set $\Sigma = \Gamma \times (0,T)$, then we can define the so-called bounded DtN map Λ_q:

$$\Lambda_q : u_{|\Sigma} = \varphi \to \partial_\nu u \text{ on } \Sigma,$$

where $\nu(x)$ is the outward unit normal to $\partial\Omega$ at x.

Our goal :

Determine $\boxed{\text{q(x,t)}}$ for $(x,t) \in \Omega \times (0,T)$ by the knowledge of a partial DtN map.

Solutions "optic geometric "

The key **words** are :

- **optic geometric** solutions that means perturbations of exponential harmonics in the form

$$e^{-i(x.\xi+\tau.t)}$$

- **density** of product of solutions

The strategy

The proof consists in the following steps :

- we carry out an ”optic geometric” solution in Theorem (1)
- we deduce from this existence of ”optic geometric” solution a density result in Theorem (2)
- we end the proof via the main uniqueness result of Theorem (3).

Solutions "optic geometric "

General settings:

$Q = \Omega \times (0, T)$ where Ω is a bounded open set of $\mathbb{R}^n$.
$D_j = -i\partial_j$ and for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$.
We consider the differential operator with constant coefficients

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$$

where $m \in \mathbb{N}$ and $a_\alpha \in \mathbb{C}$. We state for $a \in \mathbb{C}^n$:

$$P_a^{\pm} := \pm\partial_t - ia.\nabla_x - \Delta_x.$$

The symbol of $P_a^{\pm}$ is

$$P_a^{\pm}(\xi, \tau) = \pm i\tau + a.\xi + \xi.\xi, \quad (\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}$$

Solutions "optic geometric "

General settings:

$$H^{2,1}(Q) = \{u \in L^2(Q); \partial_x^\alpha \partial_t^{\alpha_{n+1}} u \in L^2(Q), |\alpha| + \alpha_{n+1} \leq 2\},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index with $\alpha_j \in \mathbb{N} \cup \{0\}$ and $\alpha_{n+1} \in \mathbb{N} \cup \{0\}$. In the following, $\nu(x)$ is the outward unit normal to $\partial\Omega$ at x and we denote by $|\nabla_{x,t}| = (|\partial_t|^2 + |\partial_x|^2)^{\frac{1}{2}}$.

Solutions "optic geometric "

We assume the following result :

Lemma (0)

If $u \in L^2(Q)$ *verifies* $P_a^{\pm} u \in L^2(Q)$ *then* $u \in H^{2,1}(Q)$.

We set for $q \in L^\infty(Q)$

$$S_q^{\pm} = \{u \in H^{2,1}(Q); \pm u_t - \Delta u + qu = 0 \text{ in } Q\}$$

Solutions "optic geometric "

Then we prove the following existence result

Theorem (1)

Let $q \in L^{\infty}(Q), \|q\| \leq M$. *Then exists a constant* C *depending on* Q, n, M *such that for all* $(\xi, \tau) \in \mathbb{C}^n \times \mathbb{C}$ *such that* $\pm i\tau + \xi.\xi = 0$ *and* $|Im(\xi)| > C$ *there exist* $w_{\xi}^{\pm} \in H^{2,1}(Q)$ *verifying*

$$\|w_{\xi}^{\pm}\|_{L^2(Q)} \leq \frac{C}{|Im(\xi)| - C}$$

and

$$u^{\pm} = e^{-i(\xi.x+\tau.t)}(1 + w_{\xi}^{\pm}) \in S_q^{\pm}.$$

Proof:
D1,2, 3, 4, 5

Density result

Thanks to the optic geometric solutions we get

Theorem (2)

Let $(p, q) \in L^\infty(Q)$, *then*

$$F = vect\{uv, u \in S_p^+, v \in S_q^-\} \text{ is dense in } L^1(Q).$$

For the proof we use the

Lemma (1)

Let $(k, l) \in \mathbb{R}^n \times \mathbb{R}, k \neq 0$. *We can find a non negative constant* R_0 *such that if* $R \geq R_0$ *there exist* $(\xi_\pm, \tau_\pm) \in \mathbb{C}^n \times \mathbb{C}$ *verifying:*

$$|Im(\xi_\pm)| \geq R, \pm i\tau_\pm + \xi_\pm.\xi_\pm = 0, (\xi_+, \tau_+) + (\xi_-, \tau_-) = (k, l)$$

Proofs
D 6, 7 & D 8, 9

Uniqueness: the settings

Consider Ω an open bounded set of $\mathbb{R}^n$, and

$$\partial\Omega = \Gamma, Q = \Omega \times (0,T), \Sigma_0 = \Omega \times \{0\}, \Sigma = \Gamma \times (0,T).$$

We define: $H^{\frac{3}{2},\frac{3}{4},0}(\Sigma) = \{\psi \in H^{\frac{3}{2},\frac{3}{4}}(\Sigma); \psi(.,0) = 0 \text{ on } \Gamma\}$ where

$$H^{2p,p}(\Sigma) = L^2(0,T,H^{2p}(\Gamma)) \cap H^p(0,T,L^2(\Gamma)),$$

Uniqueness result: the settings

If $q \in \boxed{L^\infty(Q)}$ and $\varphi \in H^{\frac{3}{2},\frac{3}{4},0}(\Sigma)$ then $u_{q,\varphi}$ is the unique solution in $H^{2,1}(Q)$ of the boundary problem

$$(P) \quad \left\{ \begin{array}{r} \partial_t u - \Delta u + qu = 0 \text{ in } Q \\ u(.,0) = 0 \text{ in } \Omega \\ u = \varphi \text{ on } \Sigma, \end{array} \right.$$

such that

$$\|u_{q,\varphi}\|_{H^{2,1}(Q)} \leq C\|\varphi\|_{H^{\frac{3}{2},\frac{3}{4},0}(\Sigma)} \text{ for } C \text{ independant of } \varphi.$$

Now, consider Γ' a subset of Γ and set $\Sigma' = \Gamma' \times (0,T)$, then we can define the bounded DtN map :

$$\begin{aligned} \Lambda_q : H^{\frac{3}{2},\frac{3}{4},0}(\Sigma) &\to H^{\frac{1}{2},\frac{1}{4}}(\Sigma') \\ \varphi &\to \partial_\nu u_{q,\varphi}, \end{aligned}$$

Main Theorem

We define $D_0 = \{\varphi \in H^{\frac{3}{2},\frac{3}{4},0}(\Sigma), \varphi = 0 \text{ outside } \Sigma'\}$

Theorem (3)

The application : $q \in \boxed{L^\infty(Q)} \to \Lambda_{q|D_0}$ *is one to one*

Proof of the uniqueness result

First we need to prove :

Lemma (2)

Let $q \in L^\infty(Q)$, $\omega \subset Q$ *an open set such that* $\overline{\omega} \subset Q$ *and* $f \in L^2(\omega)$. *If*

$$\int_Q fudxdt = 0, u \in S_q^0, (resp. S_q^T)$$

then

$$\int_Q fudxdt = 0, u \in S_q^+, (resp. S_q^-)$$

where:

$S_q^\pm = \{u \in H^{2,1}(Q), \pm u_t - \Delta u + qu = 0 \ in \ Q\}$,
$S_q^0 = \{u, u = u_{q,\varphi} \ for \ \varphi \in D_0\}$,
$S_q^T = \{v, v(., T) = u(., T-t) \ for \ u \in S_q^0\}$

D 11, 12, 13

Proof of the uniqueness result

Density result:

Lemma (3)

Let $p, q \in l^{\infty}(Q)$. *Then,*

$$F_{p,q} = \{uv, u \in S_p^0, v \in S_q^T\}$$

is dense in $L^1(Q)$.

recall that:

$S_q^0 = \{u, u = u_{q,\varphi} \text{ for } \varphi \in D_0\}$

$S_q^T = \{v, v(., T) = u(., T - t) \text{ for } u \in S_q^0\}$

D 13

Proof of the uniqueness result

Then using Previous Lemma and the following result

Lemma (4)

Let $p, q \in l^\infty(Q)$. *If* $\Lambda_{p|D_0} = \Lambda_{q|D_0}$ *then*

$$\int_Q (p-q) z dx dt = 0, \quad z \in F_{p,q},$$

we get the desired uniqueness.

D 14,15

A stability result

A (logarithmic) stability inequality is available assuming final overdetermination plus more regularity on the potential.

We keep the same notations:

$$Q = \Omega \times (0, T), \Sigma_{.} = \Omega \times \{.\}, \Sigma = \Gamma \times (0, T).$$

We define:

$$\chi = \{(u_0, g) \in H^1(\Omega) \times H^{\frac{3}{2}, \frac{3}{4}}(\Sigma); u_{0|\Gamma} = g(., 0)\}.$$

A stability result

We know that If $q \in \boxed{L^\infty(Q)}$ and $f = (u_0, g) \in \chi$ then

$$(P) \quad \begin{cases} \partial_t u - \Delta u + qu = 0 \text{ in } Q \\ u = u_0 \text{ on } \Sigma_0 \\ u = g \text{ on } \Sigma, \end{cases}$$

admits an unique solution $u = u(q, f) \in H^{2,1}(\Omega)$

such that

$$\|(u(q, f)\|_{H^{2,1}(Q)} \leq C\|f\|_\chi.$$

A stability result

then we can state that the linear operator Λ_q

$$\Lambda_q = (\Lambda_q^1, \Lambda_q^2) : \chi \to H^1(\Omega) \times H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$$
$$(u_0, g) = f \to (u(q,f)_{|\Sigma_T}, \partial_\nu u(q,f)_{|\Sigma}),$$

is bounded.

We plan to prove the next stability result

A stability result

Theorem (4)

For $i = 1, 2$ *let* $q_i \in L^\infty(Q)$ *such that* $q_1 - q_2 \in H_0^1(Q)$ *and*

$$\|q_i\|_{H^1(Q)}, \|q_i\|_{L^\infty(Q)} \leq M,$$

for $M > 0$.
Then

$$\|q_1 - q_2\|_{L^2(Q)} \leq C \left[\log \frac{1}{\delta \|\Lambda_{q_1} - \Lambda_{q_2}\|}\right]^{-\frac{1}{n+3}},$$

if $\|\Lambda_{q_1} - \Lambda_{q_2}\|$ *is sufficiently small, where* C, δ *are two positive constants depending on* M.

D 16,17

A stability result

Lemma (5)

For $i = 1, 2$ *let* $q_i \in L^\infty(Q)$ *and* $\|q_i\|_{L^\infty(Q)} \leq M$. *Then exist* r_0 *such that for all* $\zeta \in \mathbb{R}^{n+1}$ *and* $r \geq r_0$, *we find* $u^+ \in S^+_{q_1}$ *and* $v^- \in S^-_{q_2}$ *verifying*

- $u^+ = e^{-\zeta_+ .(x,t)}(1 + w_{\zeta_+}), v^- = e^{-\zeta_- .(x,t)}(1 + w_{\zeta_-}),$ *with* $\zeta_+, \zeta_- \in \mathbb{C}^{n+1}$ *and* $\zeta_+ + \zeta_- = \zeta$.
- $\|w_{\zeta_+}\|_{L^2(Q)}, \|w_{\zeta_-}\|_{L^2(Q)} \leq \frac{C}{r}$,
- $\|u^+\|_{H^{2,1}(Q)}, \|v^-\|_{H^{2,1}(Q)} \leq Ce^{\delta(r+|\zeta|^2)}$,

where C, δ *are two positive constants depending on* M *and* Q.

D 17,18,19,20

References

Main reference for this part [in French ;)]

- M. Choulli "Une introduction aux problemes inverses elliptiques et paraboliques "Springer (2009).

Very few results for inverse parabolic problems by DtN map :

- Nakamura " A note on uniqueness in an inverse problem for a semilinear parabolic equation" Nihonkai Math. J. (2001)
- Isakov Topical review : "Inverse obstacle problems", Inverse problems (2009)

What to improve by this method ?

How to arrange the infinite measurements needed by DtN method ?