A globally convergent numerical method and adaptivity
for an inverse problem via Carleman estimates

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Globally convergent method

A numerical method X is globally convergent if:

1. A theorem is proved claiming convergence to a good approximation for the correct solution regardless on a priori availability of a good guess

2. This theorem is confirmed by numerical experiments for at least one applied problem

Challenges in solution of CIP

- Solution of any PDE depends nonlinearly on its coefficients.

$$y' - ay = 0 \rightarrow y(a,t) = Ce^{at}.$$

- Any coefficient inverse problem is nonlinear.
- Two major challenges in numerical solution of any coefficient inverse problem: NONLINEARITY and Ill-POSEDNESS.
- Local minima of objective functionals.
- Locally convergent methods: linearizaton, Newton-like and gradient-like methods.

A hyperbolic equation

$$c(x)\,u_{tt} = \Delta u - a(x)u \text{ in } \mathbb{R}^n \times (0,\infty)\,, n = 2, 3,$$

$$u(x,0) = 0, u_t(x,0) = \delta(x - x_0)\,.$$

INVERSE PROBLEM. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let one of coefficients $c(x)$ or $a(x)$ be unknown in Ω but it is a given constant outside of Ω. Determine this coefficient in Ω, given the function $g(x,t)$,

$$u(x,t) = g(x,t)\,, x \in \partial\Omega, t \in (0,\infty)$$

Similarly for the parabolic equation

$$c(x)\,\widetilde{u}_t = \Delta\widetilde{u} - a(x)\widetilde{u} \text{ in } \mathbb{R}^n \times (0,\infty)\,,$$

$$\widetilde{u}(x,0) = \delta(x - x_0)\,.$$

Applications

1. **MEDICINE**

a. medical optical imaging;

b. acoustic imaging.

2. **MILITARY**

a. identification of hidden targets, like, e.g. landmines; improvised explosive devices via electric or acoustic sensing.

b. detecting targets covered by smog or flames on the battlefield (via diffuse optics).

Laplace transform:

$$w(x,s) = \int\limits_0^\infty u(x,t)e^{-st}dt = \int\limits_0^\infty \widetilde{u}(x,t)e^{-s^2 t}dt$$

$$\Delta w - \left[s^2 c\left(x\right) + a(x)\right] w = -\delta\left(x - x_0\right), \tag{1}$$

$$\forall s > s_0 = const. > 0.$$

$$\lim_{|x| \to \infty} w(x,s) = 0, \forall s > s_0 = const. > 0. \tag{2}$$

$$w(x,s) > 0, \forall s > s_0.$$

THE TRANSFORMATION PROCEDURE.

First, we eliminate the unknown coefficient from the equation:

$$v = \ln w.$$

$$\Delta v + |\nabla v|^2 = s^2 c(x) + a(x) \text{ in } \Omega,$$

• Let, for example $c(x) =$? For simplicity let $a(x) = 0$. It follows from works of V.G. Romanov that

$$D_x^\alpha D_s^\beta (v) = D_x^\alpha D_s^\beta \left[-\frac{sl(x, x_0)}{g(x, x_0)} \left(1 + O\left(\frac{1}{s}\right)\right)\right], s \to \infty.$$

• Introduce a new function

$$\widetilde{v} = \frac{v}{s^2}$$

. Then

$$\widetilde{v}(x, s) = O\left(\frac{1}{s}\right), s \to \infty.$$

• Eliminate the unknown coefficient $c\,(x)$ via the differentiation:
$\partial_s c(x) \equiv 0$

$$q\,(x,s) = \partial_s \widetilde{v}\,(x,s)\,,$$

$$\widetilde{v}\,(x,s) = -\int\limits_{s}^{\infty} q\,(x,\tau)\,d\tau \approx -\int\limits_{s}^{\overline{s}} q\,(x,\tau)\,d\tau + V\,(x,\overline{s})\,.$$

• $V\,(x,\overline{s})$ is the tail function, $V\,(x,\overline{s}) \approx 0$. But still we iterate with respect to the tail.

• This truncation is similar to the truncation of high frequencies.

- Obtain Dirichlet boundary value problem for the nonlinear equation

$$\Delta q - 2s^2 \nabla q \cdot \int_{s}^{\overline{s}} \nabla q(x,\tau)\, d\tau + 2s \left[\int_{s}^{\overline{s}} \nabla q(x,\tau)\, d\tau \right]^2 \tag{3}$$

$$+2s^2 \nabla q \nabla V - 2s \nabla V \cdot \int_{s}^{\overline{s}} \nabla q(x,\tau)\, d\tau + 2s (\nabla V)^2 = 0,$$

$$q(x,s) = \psi(x,s),\ \forall (x,s) \in \partial\Omega \times [\underline{s}, \overline{s}]. \tag{4}$$

- Backwards calculations

$$c(x) = \Delta \widetilde{v} + \underline{s}^2 (\nabla \widetilde{v})^2,$$

How To Solve the Problem (3), (4)?

• Layer stripping with respect to the pseudo frequency s.

• On each step the Dirichlet boundary value problem is solved for an elliptic equation.

$$\underline{s} = s_N < s_{N-1} < ... < s_1 < s_0 = \overline{s}, s_{i-1} - s_i = h$$

$$q(x,s) = q_n(x) \text{ for } s \in (s_n, s_{n-1}].$$

$$\int\limits_s^{\overline{s}} \nabla q(x,\tau)\, d\tau = (s_{n-1} - s)\nabla q_n(x) + h \sum_{j=1}^{n-1} \nabla q_j(x), s \in (s_n, s_{n-1}].$$

• Dirichlet boundary condition:

$$q_n(x) = \overline{\psi}_n(x), x \in \partial\Omega,$$

$$\overline{\psi}_n(x) = \frac{1}{h} \int\limits_{s_n}^{s_{n-1}} \psi(x, s)\, ds.$$

Hence,

$$\widetilde{L}_n\left(q_n\right) := \Delta q_n - 2\left(s^2 - 2s\left(s_{n-1} - s\right)\right)\left(h\sum_{j=1}^{n-1}\nabla q_j\left(x\right)\right)\cdot\nabla q_n$$

$$+2\left(s^2 - 2s\left(s_{n-1} - s\right)\right)\nabla q_n\cdot\nabla V\left(x,\overline{s}\right) - \varepsilon q_n$$

$$= 2\left(s_{n-1} - s\right)\left[s^2 - s\left(s_{n-1} - s\right)\right]\left(\nabla q_n\right)^2 - 2sh^2\left(\sum_{j=1}^{n-1}\nabla q_j\left(x\right)\right)^2$$

$$+4s\nabla V\left(x,\overline{s}\right)\cdot\left(h\sum_{j=1}^{n-1}\nabla q_j\left(x\right)\right) - 2s\left[\nabla V\left(x,\overline{s}\right)\right]^2, s\in\left(s_{n-1}, s_n\right]$$

Introduce the s-dependent Carleman Weight Function $\mathcal{C}_{n\mu}\left(s\right)$ by

$$\mathcal{C}_{n\mu}\left(s\right) = \exp\left[\mu\left(s - s_{n-1}\right)\right], s\in\left(s_n, s_{n-1}\right],$$

where $\mu >> 1$ is a parameter.

• Multiply the equation by $\mathcal{C}_{n\mu}(s)$ and integrate with respect to $s \in [s_n, s_{n-1}]$.

$$L_n(q_n) := \Delta q_n - A_{1n}(\mu,h)\left(h\sum_{i=1}^{n-1}\nabla q_i(x)\right)\cdot\nabla q_n - \varepsilon q_n$$

$$= 2\frac{I_{1n}(\mu,h)}{I_0(\mu,h)}(\nabla q_n)^2 - A_{2n}(\mu,h)\,h^2\left(\sum_{i=1}^{n-1}\nabla q_i(x)\right)^2$$

$$+2A_{1n}(\mu,h)\,\nabla V(x,\overline{s})\cdot\left(h\sum_{i=1}^{n-1}\nabla q_i(x)\right)$$

$$-A_{2n}(\mu,h)\,\nabla q_n\cdot\nabla V(x,\overline{s}) - A_{2n}(\mu,h)\left[\nabla V(x,\overline{s})\right]^2,$$

where

$$I_0(\mu,h) = \int_{s_n}^{s_{n-1}}\mathcal{C}_{n\mu}(s)\,ds = \frac{1-e^{-\mu h}}{\mu},$$

$$I_{1n}(\mu,h) = \int\limits_{s_n}^{s_{n-1}} (s_{n-1}-s)\left[s^2 - s(s_{n-1}-s)\right]\mathcal{C}_{n\mu}(s)\,ds,$$

$$A_{1n}(\mu,h) = \frac{2}{I_0(\mu,h)}\int\limits_{s_n}^{s_{n-1}} \left(s^2 - 2s(s_{n-1}-s)\right)\mathcal{C}_{n\mu}(s)\,ds,$$

$$A_{2n}(\mu,h) = \frac{2}{I_0(\mu,h)}\int\limits_{s_n}^{s_{n-1}} s\mathcal{C}_{n\mu}(s)\,ds.$$

• Important observation:

$$\frac{|I_{1n}(\mu,h)|}{I_0(\mu,h)} \leq \frac{4\bar{s}^2}{\mu},\ \text{for } \mu h > 1.$$

• Iterative solution for every q_n

$$\Delta q_{nk}^{i} - A_{1n}\left(h\sum_{j=1}^{n-1}\nabla q_j\right)\cdot\nabla q_{nk}^{i} - \varepsilon q_{nk}^{i} + A_{1n}\nabla q_{nk}^{i}\cdot\nabla V_n^{i} =$$

$$2\frac{I_{1n}\left(\mu,h\right)}{I_0\left(\mu,h\right)}\left(\nabla q_{n(k-1)}^{i}\right)^2 - A_{2n}h^2\left(\sum_{j=1}^{n-1}\nabla q_j\left(x\right)\right)^2$$

$$+2A_{2n}\nabla V_n^{i}\cdot\left(h\sum_{j=1}^{n-1}\nabla q_j\left(x\right)\right) - A_{2n}\left(\nabla V_n^{i}\right)^2, k\geq 1,$$

$$q_{nk}^{i}\left(x\right) = \overline{\psi}_n\left(x\right), x\in\partial\Omega$$

• Hence, we obtain the function

$$q_n^{i} = \lim_{k\to\infty} q_{nk}^{i},\ \text{in}\ C^{2+\alpha}\left(\overline{\Omega}\right).$$

CONVERGENCE THEOREM.

• First, Schauder Theorem. Consider the Dirichlet boundary value problem

$$\Delta u+\sum_{j=1}^{3}b_j(x)u_{x_j}-m(x)u=f\left(x\right),x\in\Omega,$$

$$u\mid_{\partial\Omega}=g\left(x\right)\in C^{2+\alpha}\left(\partial\Omega\right).$$

Let

$$b_j,m,f\in C^{\alpha}\left(\overline{\Omega}\right),d\left(x\right)\geq 0;\ \max\left(\left|b_j\right|_{\alpha},\left|m\right|_{\alpha}\right)\leq 1,$$

Then

$$\left|u\right|_{2+\alpha}\leq K\left[\left\|g\right\|_{C^{2+\alpha}(\partial\Omega)}+\left|f\right|_{\alpha}\right],$$

where $K=K\left(\Omega\right)=const.\geq 1.$

Global Convergence Theorem. *Let* $\Omega \subset \mathbb{R}^3$ *be a convex bounded domain with the boundary* $\partial\Omega \in C^3$. *Let the exact coefficient* $c^*(x) \in C^2(\mathbb{R}^3), c^* \in [2d_1, 2d_2]$ *and* $c^*(x) = 2d_1$ *for* $x \in \mathbb{R}^3 \diagdown \Omega$, *where numbers* $d_1, d_2 > 0$ *are given. For any function* $c(x) \in C^\alpha\left(\mathbb{R}^3\right)$ *such that* $c(x) \geq d_1$ *in* Ω *and* $c(x) = 2d_1$ *in* $\mathbb{R}^3 \diagdown \Omega$ *consider the solution* $u_c(x,t)$ *of the original Cauchy problem. Let* $C^* = const. \geq 1$ *be a constant bounding certaon functions associated with the solution of this Cauchy problem. Let* $w_c(x,s) \in C^3\left(\mathbb{R}^3 \diagdown \{|x - x_0| < \gamma\}\right), \forall \gamma > 0$ *be the Laplace transform of* $u_c(x,t)$ *and* $V_c(x) = \overline{s}^{-2} \ln w_c(x, \overline{s}) \in C^{2+\alpha}\left(\overline{\Omega}\right)$ *be the corresponding tail function. Suppose that the cut-off pseudo frequency* $\overline{s}$ *is so large that for any such function* $c(x)$ *the following estimates hold*

$$|V^*|_{2+\alpha} \leq \xi, |V_c|_{2+\alpha} \leq \xi,$$

where $\xi \in (0,1)$ *is a sufficiently small number.*

Let $V_{1,1}(x,\overline{s}) \in C^{2+\alpha}\left(\overline{\Omega}\right)$ *be the initial tail function and let*

$$\left|V_{1,1}\right|_{2+\alpha} \leq \xi.$$

Denote $\eta := 2\left(h + \sigma + \xi + \varepsilon\right)$. *Let* $\overline{N} \leq N$ *be the total number of functions* q_n *calculated by the algorithm of section 5. Suppose that the number* $\overline{N} = \overline{N}(h)$ *is connected with the step size* h *via* $\overline{N}(h)\, h = \beta$, *where the constant* $\beta > 0$ *is independent on* h. *Let* β *be so small that*

$$\beta \leq \frac{1}{384KC^{*}\overline{s}^{2}}.$$

In addition, let the number η *and the parameter* μ *of the CWF satisfy the following estimates*

$$\eta \leq \eta_0\left(K, C^*, d_1, \overline{s}\right) = \min\left(\frac{1}{16KM^*}, \frac{3}{8}d_1\right) = \min\left(\frac{1}{256KC^*\overline{s}^2}, \frac{3}{8}d_1\right),$$

$$\mu \geq \mu_0\left(C^*, K, \overline{s}, \eta\right) = \max\left(\frac{\left(C^*\right)^2}{4}, 48KC^*\overline{s}^2, \frac{1}{\eta^2}\right).$$

Then for each appropriate n the sequence $\left\{ q_{n,1}^{k} \right\}_{k=1}^{\infty}$ converges in $C^{2+\alpha}\left(\overline{\Omega}\right)$ and the following estimates hold

$$\left| q_n - q_n^* \right|_{2+\alpha} \leq 2KM^* \left(\frac{1}{\sqrt{\mu}} + 3\eta \right), n \in \left[1, \overline{N} \right],$$

$$\left| q_n \right|_{2+\alpha} \leq 2C^*,\ n \in \left[1, \overline{N} \right],$$

$$\left| c_n - c^* \right|_{\alpha} \leq \frac{\eta}{2 \cdot 9^{n-1}} + \frac{23}{8}\eta,\ n \in \left[2, \overline{N} \right]. \tag{5}$$

In addition, functions $c_{n,k}(x) \geq d_1$ in Ω and $c_{n,k}(x) = 2d_1$ outside of Ω.

Brief Outline of the Proof

$$\widetilde{q}_{n,1}^{k} = q_{n,1}^{k} - q_{n}^{*}, \;\; \widetilde{q}_{n,i} = q_{n,i} - q_{n}^{*},$$

$$\widetilde{V}_{n,k} = V_{n,k} - V^{*}, \;\; \widetilde{c}_{n,k} = c_{n,k} - c^{*}, \widetilde{\psi}_{n} = \overline{\psi}_{n} - \overline{\psi}_{n}^{*}$$

$$\widetilde{H}_{n,i}\left(x\right) = H_{n,i}\left(x\right) - H^{*}\left(x, s_{n}\right), \;\; \widetilde{H}_{n}\left(x\right) = H_{n}\left(x\right) - H^{*}\left(x, s_{n}\right),$$

Sequentially estimate norms $\left|\widetilde{q}_{n,1}^{k}\right|_{2+\alpha}, \left|\widetilde{q}_{n,i}\right|_{2+\alpha}$ from the above using Schauder theorem. Subtracting the equation for q_{1}^{*} from the equation for $q_{1,1}^{k}$, we obtain for $x \in \Omega$

$$\Delta \widetilde{q}_{1,1}^{k} - \varepsilon \widetilde{q}_{1,1}^{k} + A_{1,1} \nabla V_{1,1} \nabla \widetilde{q}_{1,1}^{k} = 2 \frac{I_{1,1}}{I_{0}} \nabla \widetilde{q}_{1,1}^{k-1} \left(\nabla q_{1,1}^{k-1} + \nabla q_{1}^{*}\right)$$

$$- A_{1,1} \nabla \widetilde{V}_{1,1} \nabla q_{1}^{*} - A_{2,1} \nabla \widetilde{V}_{1,1} \left(\nabla V_{1,1} + \nabla V^{*}\right) + \varepsilon q_{1}^{*} - F_{1},$$

$$\widetilde{q}_{1,1}^{1}\left(x\right) = \widetilde{\psi}_{1}\left(x\right), x \in \partial\Omega.$$

$$\left| 2\frac{I_{1,1}}{I_0} \right| \leq \frac{C}{\mu} << 1.$$

- Thus the term responsible for the nonlinearity is small.

WHY THE FINITE ELEMENT ADAPTIVE METHOD SHOULD BE NEXT?

$$|c_n - c^*|_\alpha \leq \frac{\eta}{2 \cdot 9^{n-1}} + \frac{23}{8}\eta, \; n \in \left[2, \overline{N}\right]. \tag{5}$$

• The estimate (5) is typical for ill-posed problems.

• (5) tells us that our globally convergent numerical method can be categorized as the so-called "stabilizing method".

• The notion of stabilizing numerical methods was introduced in the field of ill-posed problems by one of classics Dr. Anatoly B. Bakushinskii (Moscow, Russia) in *Computational Mathematics and Mathematical Physics*, 1998, 2000.

• A numerical method for an ill-posed problem is called *stabilizing* if

$$\overline{\lim}_{n\to\infty} \|x_n - x^*\| = O\left(\sigma + \Delta\right),$$

where $\sigma > 0$ is an error in the data and $\Delta > 0$ is a parameter which can be chosen small in a ”smart” choice.

• In our case $\Delta = h + \xi$

• As soon as the procedure (5) is stabilized, we have a good approximation $c_{\overline{N}}$ for the exact solution c^*.

• Thus, on the FIRST globally convergent stage of our procedure we got a good first guess for the solution.

IDEA:

- Use a locally convergent numerical method on the SECOND stage.
- This method should be independent on the parameter $\Delta = h + \xi$.
- This method should take the the function $c_{\overline{N}}$ as the first guess.
- We have chosen Finite Element Adaptive Method for the second stage.

Two step procedure.

STEP 1. To get the first approximation using the globally convergent method.

The first approximation is exactly what a locally convergent method needs.

STEP 2. To use the adaptivity technique to improve the first approximation.

The solution taken from the globally convergent method would be a first guess.

The adaptivity does not depend on the tail.

THE ADAPTIVITY AS THE SECOND STAGE

OF THE 2-STAGE GLOBALLY CONVERGENT PROCEDURE

Denote $Q_T = \Omega \times (0,T)\,, S_T = \partial\Omega \times (0,T)\,.$

- Functional spaces

$$
\begin{aligned}
H_u^2\,(Q_T) &= \{f \in H^2(Q_T) : f(x,0) = f_t(x,0) = 0\}, \\
H_u^1(Q_T) &= \{f \in H^1(Q_T) : f(x,0) = 0\}, \\
H_\varphi^2(Q_T) &= \{f \in H^2(Q_T) : f(x,T) = f_t(x,T) = 0\}, \\
H_\varphi^1(Q_T) &= \{f \in H^1(Q_T) : f(x,T) = 0\}, \\
U &= H_u^2(Q_T) \times H_\varphi^2(Q_T) \times C^2(\overline{\Omega}), \\
\overline{U} &= H_u^1(Q_T) \times H_\varphi^1(Q_T) \times L_2(\Omega), \\
\overline{U}^1 &= L_2\,(Q_T) \times L_2\,(Q_T) \times L_2\,(\Omega)\,.
\end{aligned}
$$

• Finite dimensional subspaces of finite elements

$$W_h^u \subset H_u^1\left(Q_T\right), W_h^{\varphi} \subset H_{\varphi}^1\left(Q_T\right), V_h \subset L_2\left(\Omega\right),$$

$$U_h \subset \overline{U}, U_h = W_h^u \times W_h^{\varphi} \times V_h.$$

• Since all norms in finite dimensional spaces are equivalent, set for convenience $\left\|\cdot\right\|_{U_h} := \left\|\cdot\right\|_{\overline{U}^1}$.

• Tikhonov regularization functional:

$$E(c) = \frac{1}{2}\int\limits_{S_T}(u\mid_{S_T} - g(x,t))^2 d\sigma dt + \frac{1}{2}\gamma\int\limits_{\Omega}(c - c_{glob})^2 \ dx.$$

• c_{glob} is the solution obtained on the globally convergent stage.

• $\gamma \in (0,1)$ is the regularization parameter.

• To calculate the Frechet derivative of $E(c)$, introduce the Lagrangian.

• To be 100% rigorous, we need to assume in the Lagrangian that variations of state and adjoint operators actually depend on c and depend on each other. This would make things more complicated. We currently are working on this extension.

• However, to simplify things, we assume in this presentation that these functions are mutually independent. In particular, we assume that $E(c) := E(u, c)$, where functions u and c can be varied independently on each other.

• Many authors also use this kind of assumption.

• Given the data $g = u \mid_{S_T}$ for the inverse problem, one can uniquely determine the normal derivative $p\left(x,t\right)$,

$$\frac{\partial u}{\partial n} \mid_{S_T} = p\left(x,t\right).$$

Let $v = (c, u, \varphi)$. Then we define the Lagrangian as

$$L(v) = E(u,c) + \int\limits_{Q_T} \varphi \cdot \left(c u_{tt} - \Delta u\right) dxdt, \forall \varphi \in H_{\varphi}^{2}\left(Q_T\right).$$

Clearly

$$L(v) = E(u,c).$$

The integration by parts leads to

$$L(v) = E(u,c) - \int\limits_{Q_T} c(x) u_t \varphi_t dxdt + \int\limits_{Q_T} \nabla u \nabla \varphi dxdt - \int\limits_{S_T} p\varphi dSdt.$$

We search for a stationary point of the functional $L(v), v \in U$ satisfying

$$L'(v)\left(\overline{v}\right) = 0, \quad \forall \bar{v} = (\bar{u}, \bar{\varphi}, \bar{c}) \in \overline{U}$$

where $L'(v)(\cdot)$ is the Frechet derivative of L at the point v.

$$L'(v)\left(\overline{v}\right) = \int\limits_{\Omega} \bar{c}\left[\gamma\left(c - c_0\right) - \int\limits_{0}^{T} u_t \varphi_t dt\right] dx - \int\limits_{Q_T} c(x)\left(\varphi_t \overline{u}_t + u_t \overline{\varphi}_t\right) dx dt$$

$$+ \int\limits_{Q_T} \left(\nabla u \nabla \overline{\varphi} + \nabla \overline{u} \nabla \varphi\right) - \int\limits_{S_T} p \overline{\varphi} d\sigma dt + \int\limits_{S_T} \left(u|_{S_T} - g\right) \overline{u} d\sigma dt = 0,$$

$$\forall \bar{v} = \left(\overline{u}, \overline{\varphi}, \overline{c}\right) \in \overline{U}.$$

Integration by parts leads to

$$L'(v)\left(\overline{v}\right) = \int\limits_{\Omega} \bar{c}\left[\gamma\left(c - c_0\right) - \int\limits_0^T u_t \varphi_t dt\right] dx + \int\limits_{Q_T} \bar{\varphi}\left(c u_{tt} - \Delta u\right) dx dt$$

$$+ \int\limits_{Q_T} \bar{u}\left(c \varphi_{tt} - \Delta \varphi\right) dx dt + \int\limits_{S_T} \bar{\varphi}\left[\partial_n u - p\right] d\sigma dt$$

$$+ \int\limits_{S_T} \left(\left(u|_{S_T} - g\right) + \partial_n \varphi\right) \overline{u} d\sigma dt,$$

$$\forall \bar{v} = \left(\overline{u}, \overline{\varphi}, \overline{c}\right) \in \overline{U}.$$

We obtain for the minimizer $v = (c, u, \varphi)$:

• The state problem is:

$$cu_{tt} - \triangle u = 0, (x,t) \in Q_T,$$

$$u(x,0) = u_t(x,0) = 0,$$

$$\partial_n u \mid_{S_T} = p\,(x,t)\,.$$

• The adjoint problem, which should be solved backwards in time, is:

$$c\varphi_{tt} - \triangle \varphi = 0, (x,t) \in Q_T,$$

$$\varphi(x,T) = \varphi_t(x,T) = 0,$$

$$\frac{\partial \varphi}{\partial n} \mid_{S_T} = (g-u)\,(x,t)\,, (x,t) \in S_T.$$

And the gradient with respect to the unknown coefficient c should be equal to zero:

$$\gamma(c - c_{glob}) - \int_0^T u_t \varphi_t \, dt = 0, x \in \Omega. \qquad (6)$$

How to Find the Minimizer Which Would Approximately Guarantee (6)?

• We solve (6) iteratively.

• Let $u_n = u\left(x,t,c_n\right), \varphi_n = \varphi\left(x,t,c_n\right)$ be solutions of state and adjoint problems with $c := c_n$.

• Set

$$c_0 := c_{glob},$$

$$c_n = \frac{1}{\gamma} \int_0^T \partial_t u_{n-1} \cdot \partial_t \varphi_{n-1} \, dt + c_{glob}, x \in \Omega.$$

• We have computationally observed convergence of this procedure in terms of a stabilizing procedure introduced above

A POSTERIORI ERROR ESTIMATE FOR THE LAGRANGIAN

• Let $v \in U$ and $v_h \in U_h$ be the local minimizers of L on the spaces $\overline{U}$ and U_h respectively (recall that $U_h \subset \overline{U}$ as a set),

$$\|v - v^*\|_{\overline{U}}, \|v_h - v^*\|_{\overline{U}} \leq \delta << 1,$$

where v^* is the exact solution of our inverse problem.

• We assume that such local minimizers v, v_h exist

• For any vector $w \in \overline{U}^1$ let w_h^I be the interpolant of w via finite elements of U_h.

• Using the Galerkin orthogonality with the splitting $v - v_h = (v - v_h^I) + (v_h^I - v_h)$, we obtain the following error representation:

$$L(v) - L(v_h) \approx L'(v_h)(v - v_h^I),$$

involving the residual $L'(v_h)(\cdot)$ with $v - v_h^I$ appearing as the interpolation error.

• It turns out that an approximate error estimate from the above for the Lagrangian is

$$|L(v) - L(v_h)| \approx \left|L'(v_h)(v - v_h^I)\right|$$

$$\leq V(\Omega) \max|[c_h]| \cdot \left[\gamma \max_{\overline{\Omega}} |c - c_{glob}| + \int_0^T \max_{\overline{\Omega}} |u_{ht}| |\varphi_{ht}| \, dt\right]. \quad (7)$$

• Thus, we refine the mesh in regions where

$$\gamma |c - c_{glob}| (x) + \int_0^T |u_{ht}| |\varphi_{ht}| (x,t) \, dt$$

$$\geq \beta \left[\gamma \max_{\overline{\Omega}} |c - c_{glob}| + \int_0^T \max_{\overline{\Omega}} |u_{ht}| |\varphi_{ht}| \, dt\right],$$

where $\beta = const. \in (0,1)$ is a parameter which we choose in computational experiments.

• We have chosen in our computations:

$$\beta = \left\{ \begin{array}{c} 0.1 \text{ on the coarse mesh,} \\ 0.2 \text{ on first two refinements,} \\ 0.6 \text{ on the refinement } n \geq 3. \end{array} \right\}.$$

A POSTERIORI ERROR ESTIMATE FOR THE UNKNOWN COEFFICIENT

• In a paper

L. Beilina and C. Johnson " A posteriori error estimation in computational inverse scattering", *Math. Models and Methods in Applied Sciences* , V. 15, pp. 23-37, 2005.

a posteriori error estimate for the unknown coefficient in the adaptivity was introduced.

• This estimate was based on the so-called "error estimator", which was denoted as $\psi(x)$.

• The meaning of $\psi(x)$ was not explained analytically and we are unaware about other references where this meaning would be explained for inverse problems.

• We provide this explanation below.

• Let $((\cdot,\cdot))$ be the inner product in $\overline{U}^1$.

• Let $L''(v_h)(\overline{v},\widetilde{v})$ be the Hessian, i.e., the second Frechet derivative of the Lagrangian L, at the point v_h, where v_h is the local minimizer of L on the space U_h.

• Consider solution $\widetilde{v}$ of the following so-called "Hessian problem"

$$-L''(v_h)(\overline{v},\widetilde{v}) = ((\psi,\overline{v})), \forall\overline{v} \in U_h.$$

• $\psi \in \overline{U}$ is a function of our choice.

• Suppose that for any $\psi \in \overline{U}$ there exists such a solution $\widetilde{v} = \widetilde{v}_\psi \in \overline{U}$ that $\|\widetilde{v}_\psi - v^*\|_{\overline{U}} \leq \delta$.

• Then

$$((\psi, v - v_h)) = -L''(v_h)(v - v_h, \widetilde{v}_\psi)$$
$$= -L'(v)(\widetilde{v}_\psi) + L'(v_h)(\widetilde{v}_\psi) + R = L'\left(v_h\right)(\widetilde{v}_\psi) + R,$$

where $R \approx 0$ is the reminder term, which is of the second order of smallness with respect to $\|v - v_h\|_{\overline{U}}$. Thus, we ignore R.

• Splitting: $\widetilde{v}_\psi = \widetilde{v}_\psi^I + \left(\widetilde{v}_\psi - \widetilde{v}_\psi^I\right), L'\left(v_h\right)\left(\widetilde{v}_\psi^I\right) = 0.$

• Thus, we have obtained the following analog of a posteriori error estimate for the error in the Lagrangian

$$((\psi, v - v_h)) \approx L'(v_h)(\widetilde{v}_\psi - \widetilde{v}_{\psi h}^I). \tag{8}$$

• We conclude, that the concrete form of the estimate (8) is the same as one for the Lagrangian $L(v)$ with only $v - v_h^I$ replaced with $\widetilde{v}_\psi - \widetilde{v}_{\psi h}^I$.

• Let $\{\psi_k\}_{k=1}^{M} \subset U_h$ be an orthonormal basis in the finite dimensional space U_h.

• Let $P_h : \overline{U}^1 \to U_h$ be the operator of the orthogonal projection of the space $\bar{U}^1$ on the subspace U_h. Represent $\bar{U}^1 = U_h + G$, where the subspace G is orthogonal to U_h. We have
$v - v_h = (P_h v - v_h) + (v - P_h v)$, where $v - P_h v \in G$ and $P_h v - v_h \in U_h$. Therefore $((\psi_k, v - P_h v)) = 0$.

• Hence, numbers $((\psi_k, v - v_h)) = ((\psi_k, P_h v - v_h))$ are Fourier coefficients of the vector function with respect to the orthonormal basis $\{\psi_k\}_{k=1}^{M}$ in the space U_h. Thus,

$$[P_h v - v_h]^2 = \sum_{k=1}^{M} |((\psi_k, v - v_h))|^2 \leq \sum_{k=1}^{M} \left| L'(v_h)(\widetilde{v}_{\psi_k} - \widetilde{v}_{\psi_k h}^{I}) \right|^2,$$

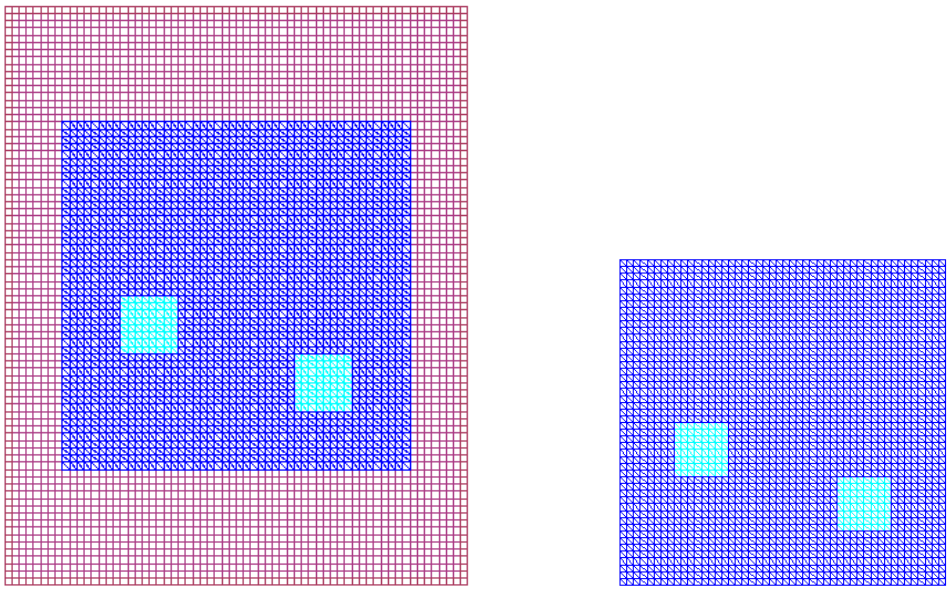
$$[P_h v - v_h] \leq \left(\sum_{k=1}^{M} \left| L'(v_h)(\widetilde{v}_{\psi_k} - \widetilde{v}_{\psi_k h}^{I}) \right|^2 \right)^{1/2}.$$

• **In summary**, estimates $\left|L'(v_h)(\widetilde{v}_{\psi_k} - \widetilde{v}^{I}_{\psi_k h})\right|$ from the above for all $k = 1, ..., M$ would provide us with an estimate of the difference between the $\overline{U}^1$-projection (i.e., L_2-like projection) of our target minimizer of the Lagrangian on the subspace of finite elements and the minimizer of this Lagrangian on the subspace U_h, which will be found in computations.

• Thus, assuming the existence of the solution of the Hessian problem, and using (7), we obtain the following approximate error estimate for the unknown coefficient

$$\|P_h c - c_h\| \leq MCV\,(\Omega) \max |[\tilde{c}_h]| \cdot \left[\gamma \max_{\overline{\Omega}} |c - c_{glob}| + \int\limits_0^T \max_{\overline{\Omega}} |u_{ht}|\,|\varphi_{ht}|\,dt\right].$$

A globally convergent numerical method and adaptivity in 2-d

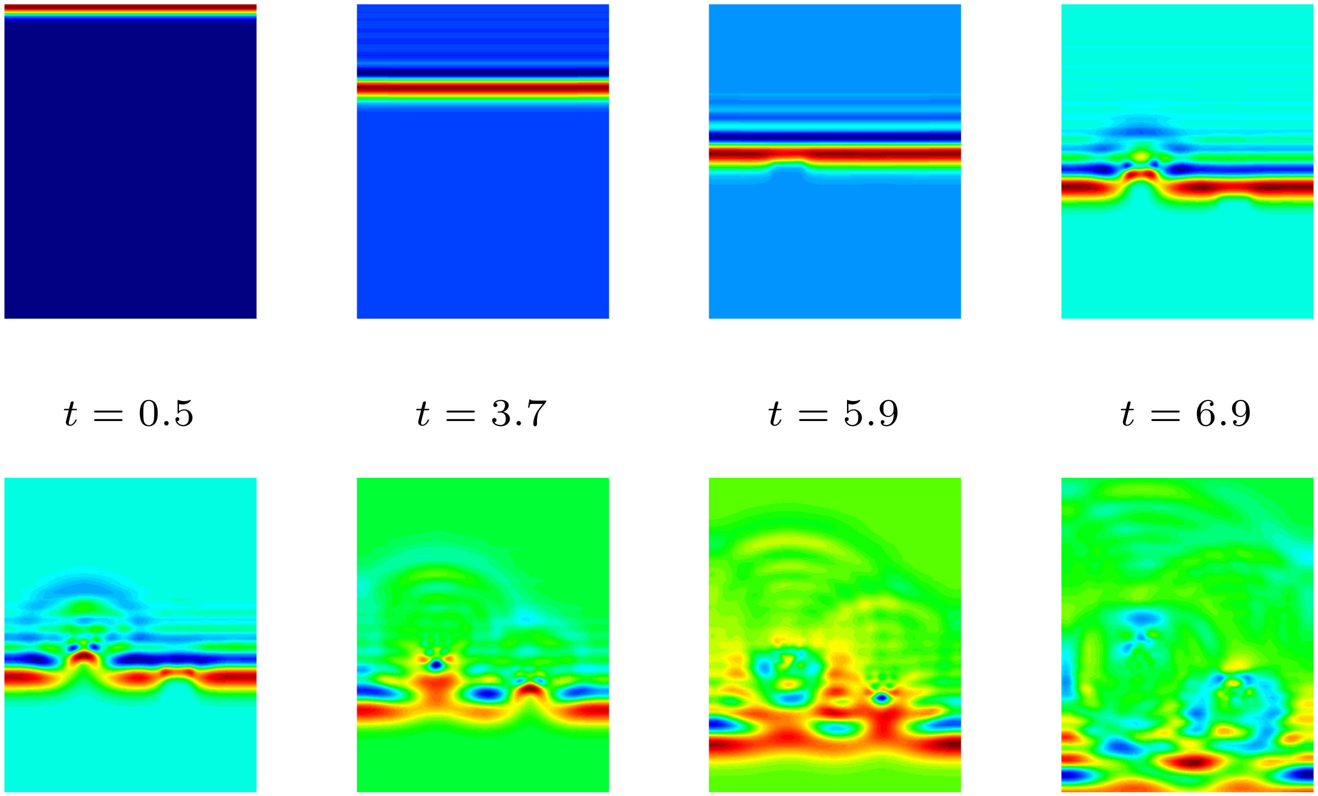


(a) $G = G_{FEM} \cup G_{FDM}$ (b) $G_{FEM} = \Omega$

Figure 1:

1-a. The forward problem for $c(x)u_{tt} = \Delta u, c\,(x) \geq const. > 0$ is solved in the bigger rectangle to generate the boundary data for the inverse problem. The data for the inverse problem are generated at the boundary of the smaller square. 1-b. The correct image. The unknown coefficient $c(x) = 1$ in the background and $c(x) = 4$ in two inclusions. A priori knowledge of neither background nor inclusions nor values of the unknown coefficient $c(x)$ is not assumed and the coefficient $c(x)$ is the target of solution by the globally convergent numerical method. Applications: Imaging of antipersonnel land mines in which case $c(x) := \varepsilon(x)$, the spatially distributed dielectric permittivity; also acoustical imaging of land mines, in which case $1/\sqrt{c(x)}$ is the speed of sound.

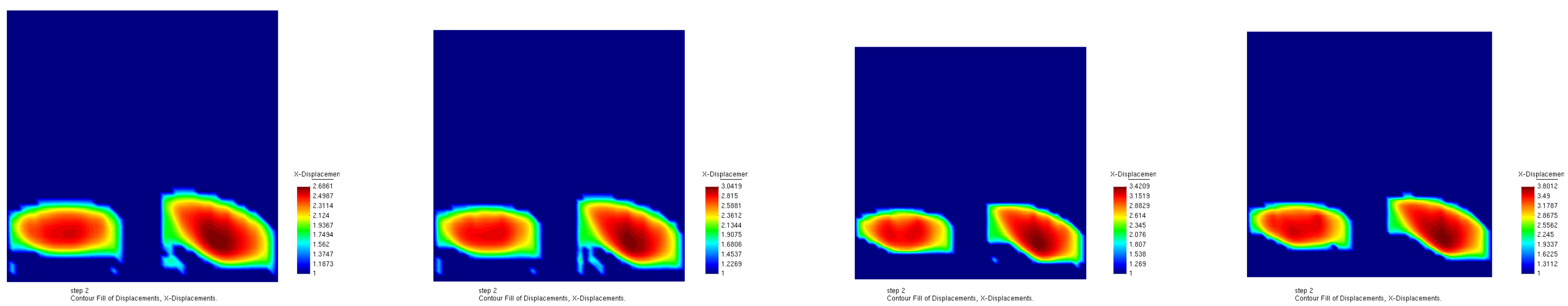
Forward problem solution



$t = 0.5$ $t = 3.7$ $t = 5.9$ $t = 6.9$

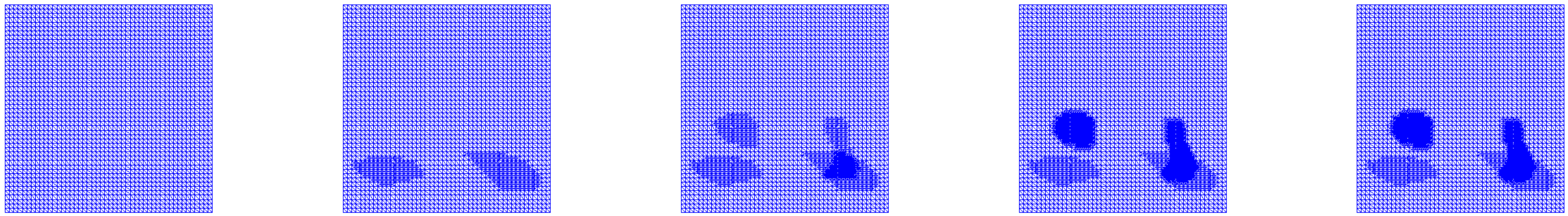
$t = 7.5$ $t = 8.5$ $t = 9.6$ $t = 11.2$

Figure 2: Isosurfaces of the simulated exact solution to the forward problem with a plane wave initialized at the top boundary.

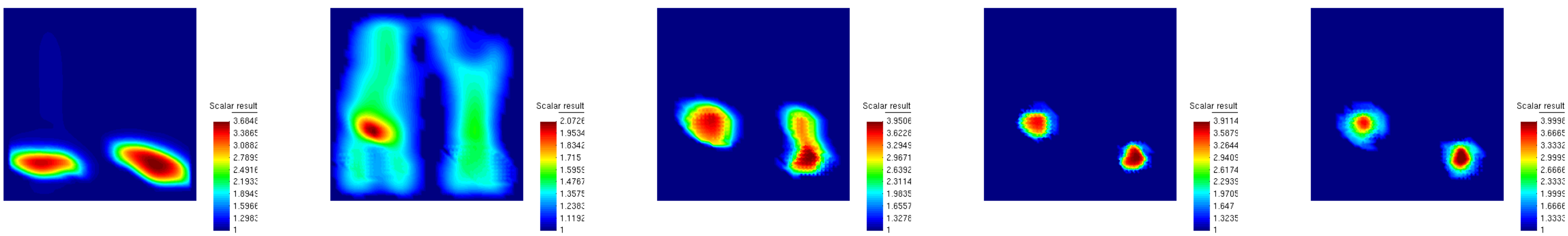


a) $c_{9,2}$ b) $c_{10,2}$ c) $c_{11,2}$ d) $c_{12,2}$

Figure 3: Spatial distribution of c_h after computing $q_{n,k}$; $n = 9, 10, 11, 12$, where n is number of the computed function q for the case of Fig. 1b. We have incorporated 5% random noise in the data. While values of the unknown coefficient $c(x)$ are correctly reconstructed both inside and outside inclusions, locations of inclusions need to be enhanced. Thus, using our globally convergent method, we got a good first approximation for the solution of the inverse problem. And now we need to enhance it using a locally convergent adaptivity technique. The resulting method is a two-stage procedure: global convergence on the first stage and a more detailed enhancement on the second.



a) 4608 el. b) 5340 el. c) 6356 el. d) 10058 el. e) 14586 el.

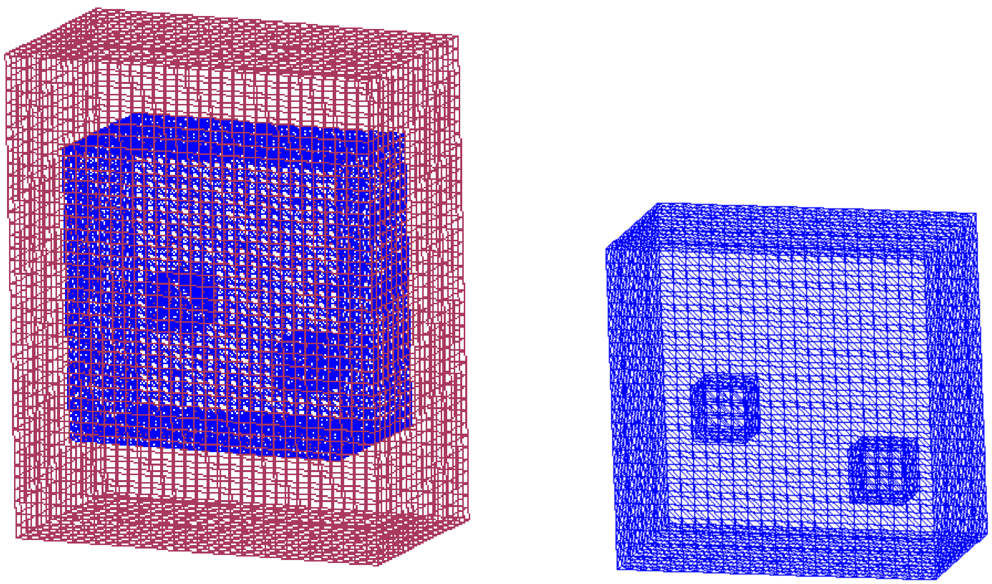


f)4608 el. g) 5340 el. h) 6356 el. i) 10058 el. j) 14586 el.

Figure 4:

Adaptively refined computational meshes: with $\sigma = 5\%$ - on a),b),c),d),e), and correspondingly spatial distribution of the parameter c_h: with $\sigma = 5\%$ - on f),g),h),i),j) when the first guess was taken from the globally convergent numerical method (Fig. 3). Upper figures represent refined meshes and lower figures represent corresponding images. The final image (j) displays correctly located inclusions and the function c both inside and outside of them.

A globally convergent numerical method and adaptivity in 3-d



(a) $G = G_{FEM} \cup G_{FDM}$ (b) $G_{FEM} = \Omega$

Figure 5:

The forward problem is solved in the larger rectangular prizm depicted on Fig. 5a. The plane wave is falling from the top.

A globally convergent numerical method in 3-d

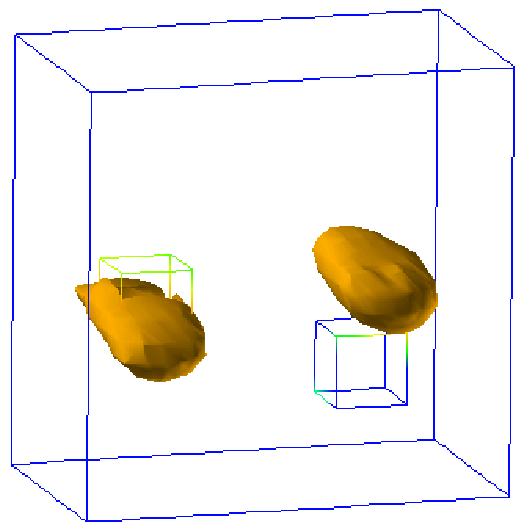


Figure 6:

The image of Fig. 6 reconstructed by the globally convergent numerical method. This image corresponds to the function q_{12} in the globally convergent method. The maximal computed value of the coefficient $c(x) = 3.66$ inside of inclusions depicted here and $c\,(x) = 1$ outside. Recall that correct values are $c\,(x) = 4$ inside of inclusions and $c\,(x) = 1$ outside. Both locations of inclusions and values of the unknown coefficient $c\,(x)$ inside of them need to be enhanced by the adaptivity technique.

it. n	i=1	i=2	i=3
1	0.0522995	0.0522995	
2	0.0523043	0.0521772	
3	0.0535235	0.053353	
4	0.0516891	0.0556757	
5	0.0467661	0.091598	
6	0.0466467	0.0440336	0.0464053

Table 1: Test 1. Computed L_2 norms of the $F_{n,i} = ||q_{n,i}|_{\partial\Omega} - \overline{\psi}_n||_{L_2(\partial\Omega)}$ with $\mu = 100$.

it. n	i=1	i=2	i=3
7	0.0486575	0.0657632	
8	0.0631762	0.0892608	
9	0.0852419	0.111969	
10	0.0914603	0.106285	
11	0.090428	0.104433	
12	0.11104	0.133783	

Table 2: Test 1. Computed L_2 norms of the $F_{n,i} = || q_{n,i}|_{\partial\Omega} - \overline{\psi}_n ||_{L_2(\partial\Omega)}$ with $\mu = 100$.

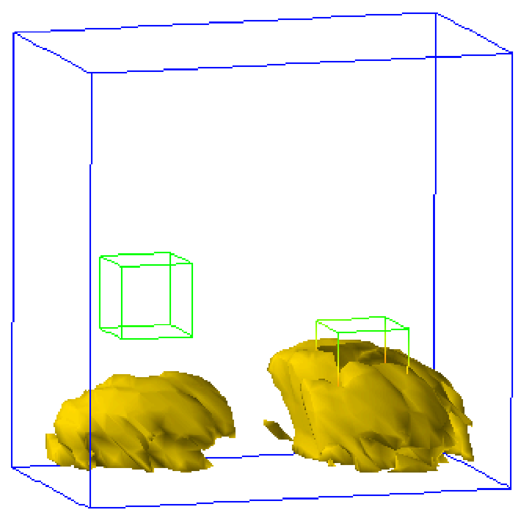
Conclusion: we should stop at $n \leq 7$, because norms start to grow at n=8. This is one of the key ideas of the stopping criterion for stablizing algorithms in ill-posed problems.

Adaptivity in 3-d

Since the adaptivity is a locally convergent numerical method, we take the starting point on the coarse mesh from the results of Test 1 of the globally convergent method and with the plane wave initialized at the top boundary of the computational domain G. More precisely, we present two set of tests where the starting point for the coefficient $c(x)$ in the adaptive algorithm on the coarse mesh is $c_{4,2}$, and $c_{7,2}$, correspondingly.

At the boundary data $g = u \mid_{\partial\Omega}$ we use three noise levels: $0\%, 3\%$, and 5% correspondingly. In all tests let Γ be the side of the cube Ω, opposite to the side from which the plane wave is launched and $\Gamma_T = \Gamma \times (0, T)$.

Test 1.



$c_{4,2} \approx 1.2$

Figure 7:

The starting point for the coefficient $c(x)$ in the adaptive algorithm.

$\boxed{\sigma = 3\%}$

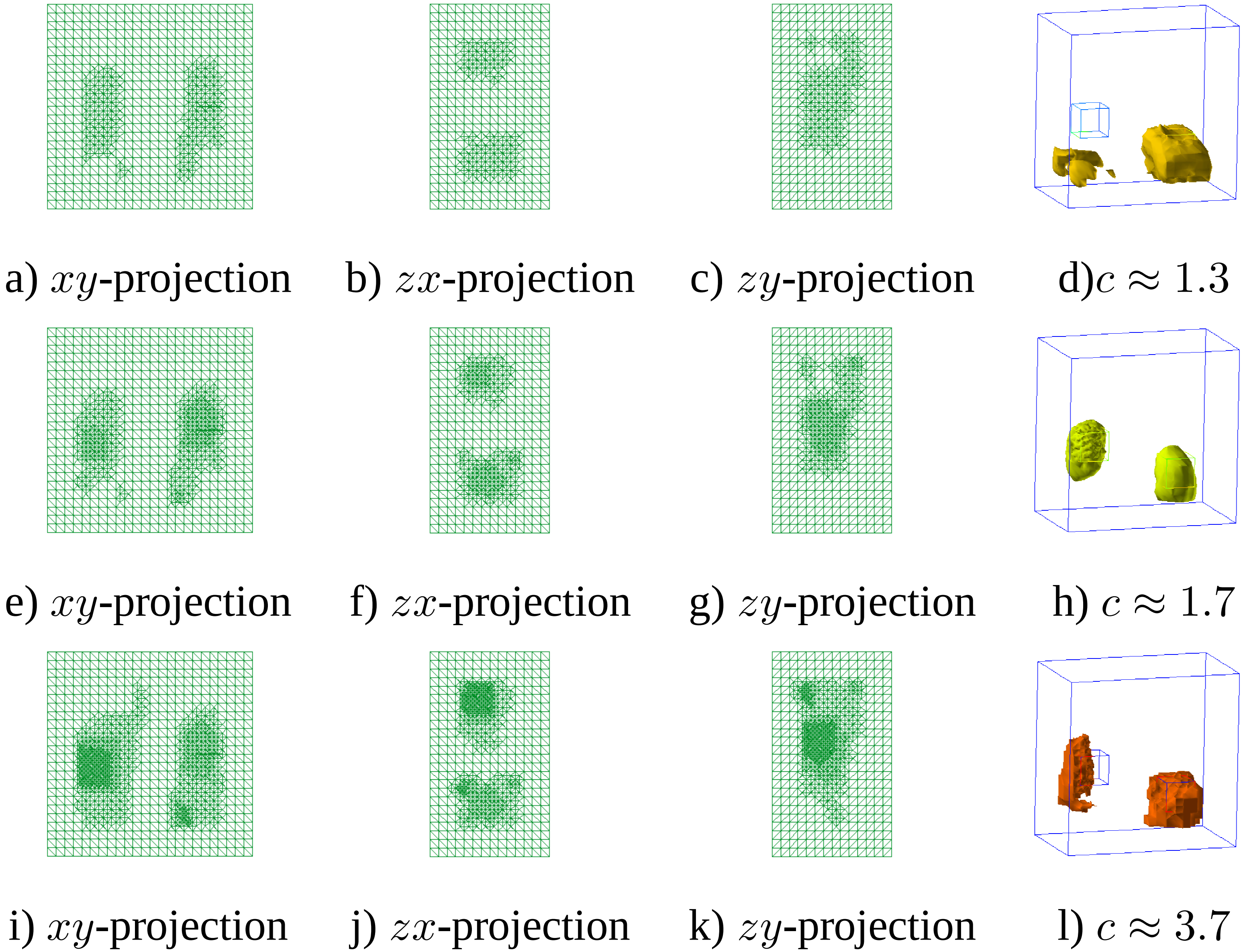
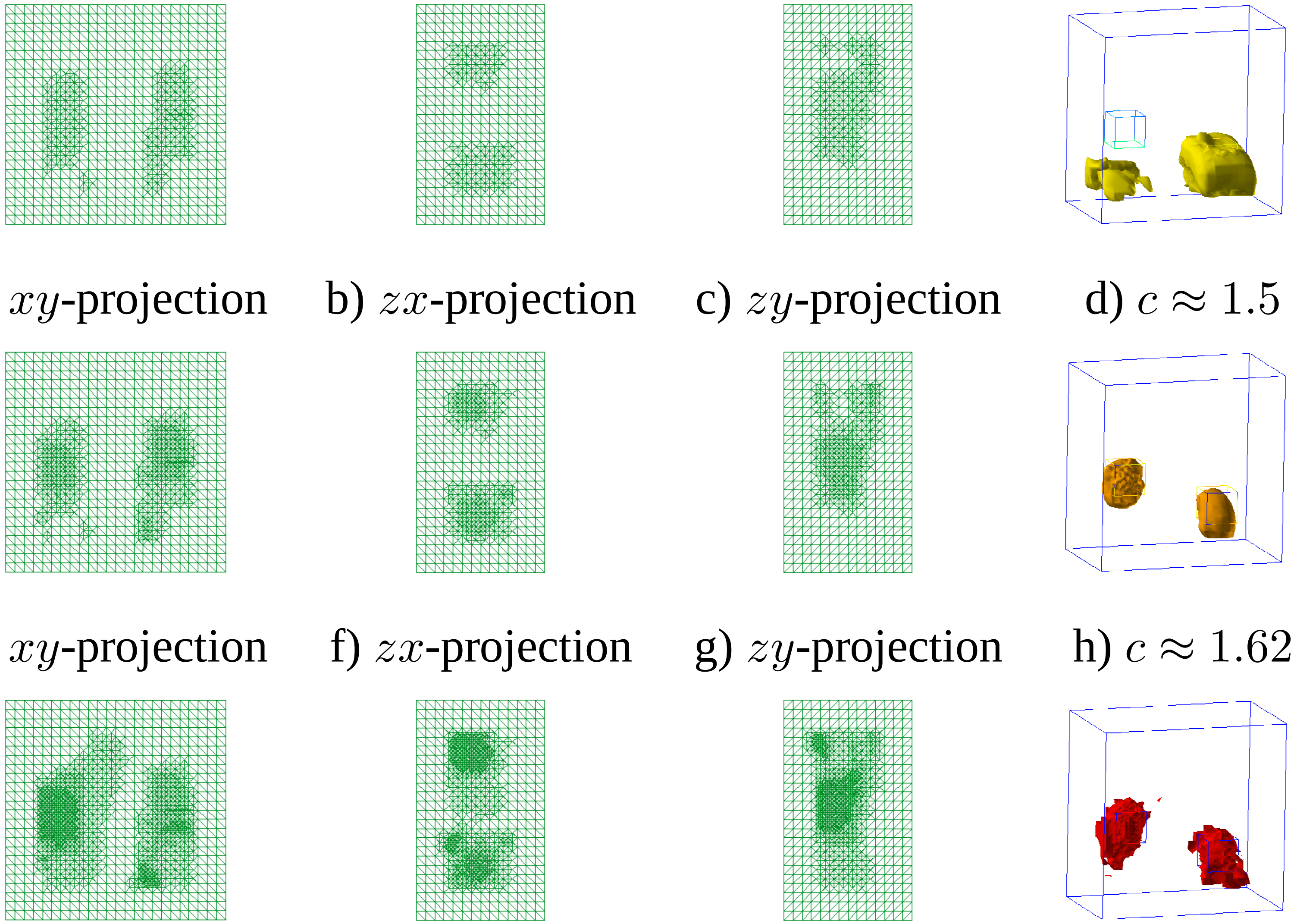

a) xy-projection
b) zx-projection
c) zy-projection
d)$c \approx 1.3$
e) xy-projection
f) zx-projection
g) zy-projection
h) $c \approx 1.7$
i) xy-projection
j) zx-projection
k) zy-projection
l) $c \approx 3.7$

Figure 8:

$\boxed{\sigma = 5\%}$



a) xy-projection b) zx-projection c) zy-projection d) $c \approx 1.5$

e) xy-projection f) zx-projection g) zy-projection h) $c \approx 1.62$

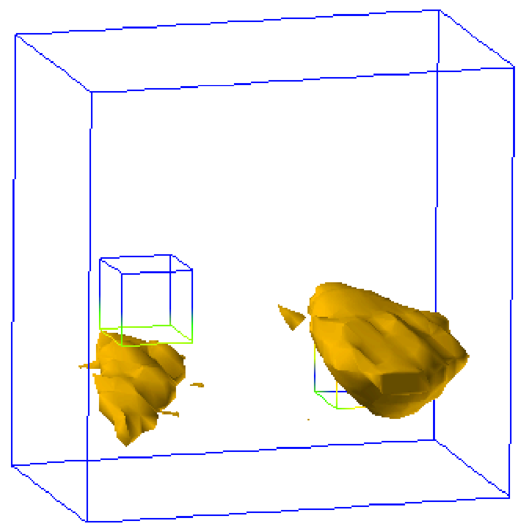
i) xy-projection j) zx-projection k) zy-projection l) $c \approx 4.0$

Figure 9:

Mesh	$\sigma = 3\%$	q.N.it.	CPU time (s)	min CPU time/node (s)
9375	0.030811	3	26.2	0.0028
10564	0.029154	3	29.08	0.0028
12001	0.035018	3	32.91	0.0027
16598	0.034	8	46.49	0.0028
Mesh	$\sigma = 5\%$	q.N.it.	CPU time (s)	min CPU time/node (s)
9375	0.0345013	3	26.53	0.0028
10600	0.0324908	3	29.78	0.0028
12370	0.03923	2	34.88	0.0028
19821	0.0277991	8	53.12	0.0027

Table 3: Test 2.2: $||u\ |_{\Gamma_T}\ -g||_{L_2(\Gamma_T)}$ on adaptively refined meshes with different noise level σ in data.

Test2



$c_{7,2} \approx 1.8$

Figure 10:

The starting point for the coefficient $c(x)$ in the adapt. algorithm.

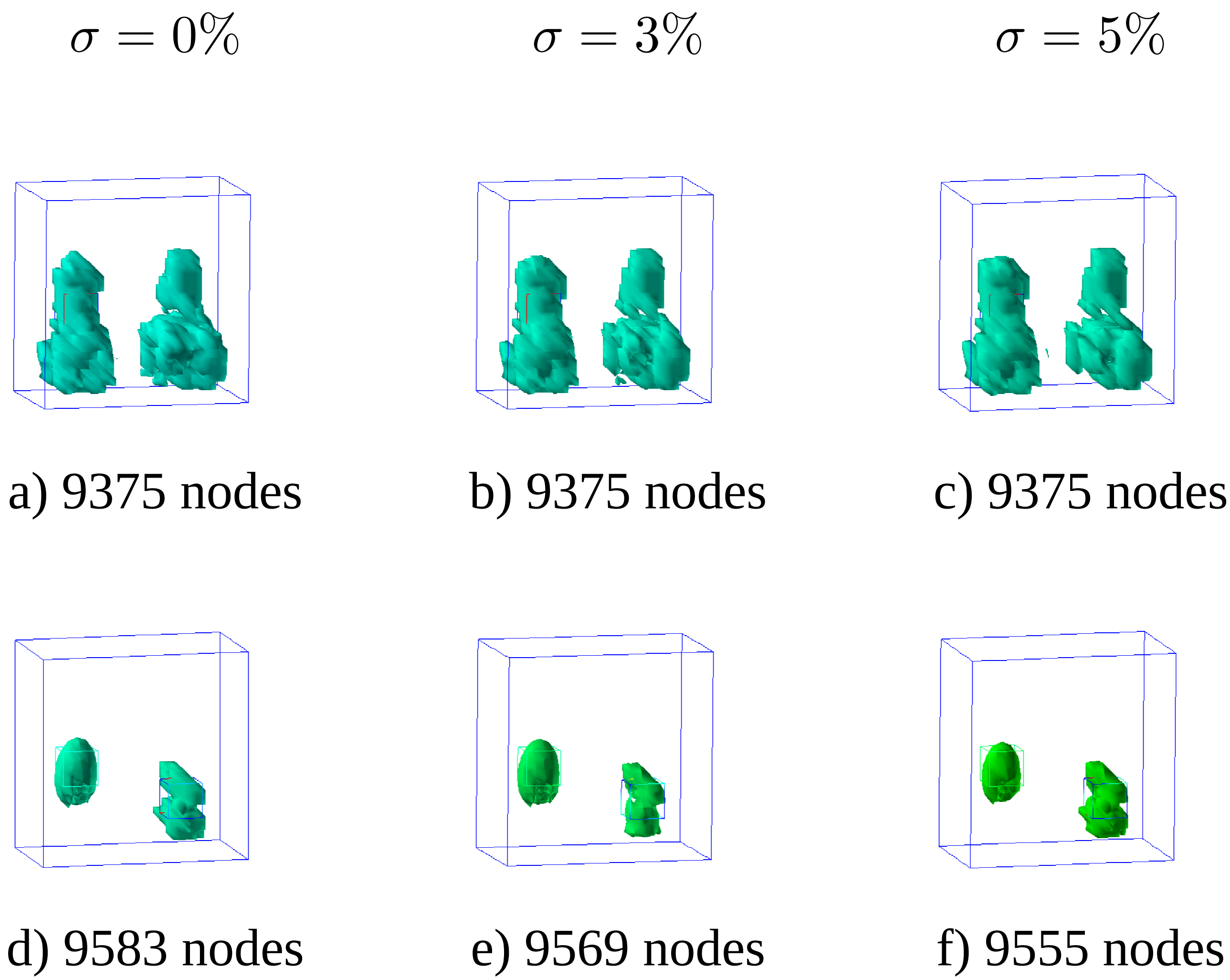


Figure 11: Test 2.3: reconstruction parameter on different adaptively refined meshes.

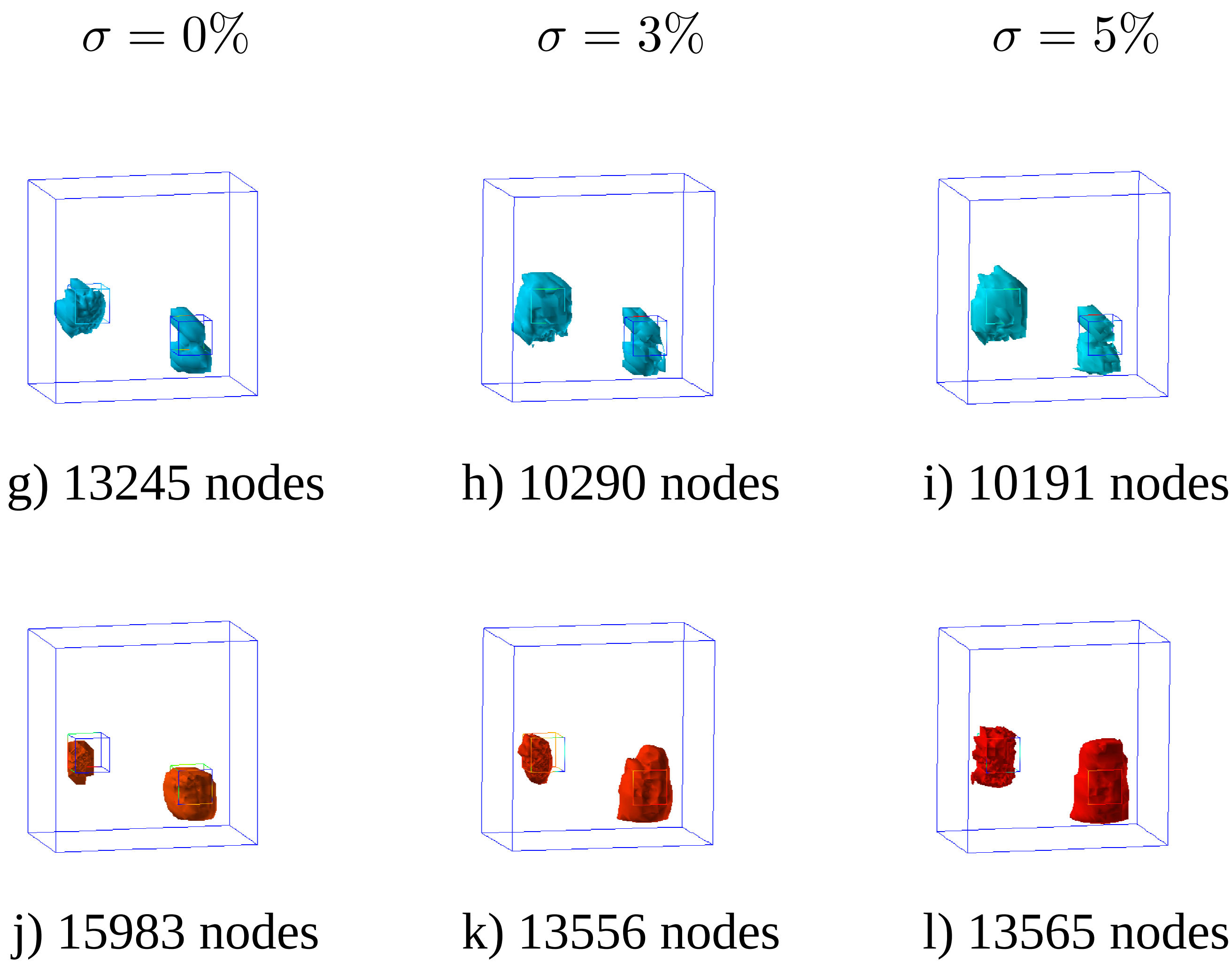


Figure 12: Test 2.3: reconstruction parameter on different adaptively refined meshes.

Work in progress

We eliminate two assumptions of our adaptivity technique. Rather, we prove them now: These are:

1. The assumption of the existence of the minimizer.

2. We now can estimate the accuracy of the reconstruction of the coefficient without using a Hessian but rather via a new idea.

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2. L. Beilina and M. V. Klibanov, A globally convergent numerical method and adaptivity for a hyperbolic coefficient inverse problem, submitted for publication in February 2009, available on-line at *Chalmers Preprint Series* ISSN 1652-9715, 2009 and at http://www.ma.utexas.edu/mp_arc.

3. L. Beilina and M. V. Klibanov, Synthesis of global convergence and adaptivity for a hyperbolic coefficient inverse problem in 3D, submitted for publication in March 2009, available on-line at *Chalmers Preprint Series* ISSN 1652-9715, 2009 and at http://www.ma.utexas.edu/mp_arc.

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