Picosecond scale experimental verification of a globally convergent algorithm for a coefficient inverse problem

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Wave equation

$$\epsilon_r(x)u_{tt} - \Delta u = 0, \quad x \in \mathbb{R}^3, t \in (0, \infty),$$

$$u(x,0) = 0, u_t(x,0) = \delta(x-x_0),$$

- $\varepsilon_r(x)$  is the spatially distributed dielectric constant (i.e., the relative dielectric permittivity function).
- This PDE cannot be derived from the Maxwell's system
- Nevertheless, we have made this equation working for our case
- Therefore, this is a simplified mathematical model

**Classic Forward Problem.** Let the function  $\varepsilon_r(x)$  be known. Determine the function u(x, t)

**Coefficient Inverse Problem.** Let  $\Omega$  be a bounded domain with the boundary  $\partial\Omega$  and the source  $x_0$  is outside of  $\overline{\Omega}$ . Suppose that  $\varepsilon_r(x) = 1$  outside of  $\Omega$  and  $\varepsilon_r(x)$  is unknown inside of  $\Omega$ . Determine  $\varepsilon_r(x)$  inside of  $\Omega$ , assuming that the following function p is known on the boundary

$$u\mid_{\partial\Omega\times(0,T)}=g(x,t)$$

In the traditional experiments people measure the refractive index n = √ε<sub>r</sub> by invasive methods, e.g. Wave Guide Method.
Let c<sub>0</sub> be the speed of light in the vacuum and c be the speed of light in the inclusion. Then

$$n=rac{c_0}{c}$$

• The refractive index shows how the electric wave slows down when propagating through the inclusion.

• To measure *n* invasively, the following formula is commonly used

$$\Delta t=\frac{d}{c_0}\left(n-1\right),$$

where d is the thickness of the inclusion and  $\Delta t$  is the time delay of the signal due to the inclusion presence.

- Two well established methods were used in our case to measure *n* for self-checking **after blind** images of *n* were obtained:
- The Wave Guide Method and the Oscilloscope Method.
- "Semi Blind" means that we knew locations of inclusion.
- However, we did not know values of refractive indexes in inclusions.
- Semi blind study was carried from the **very beginning**, i.e. without any preliminary adaptation.
- Our algorithm does not assume any a priori knowledge of:
- refractive indexes of inclusions;
- locations of inclusions;
- values of refractive indexes outside of inclusions.
- Only value of those indexes in a "far away" zone are assumed to be known

- Refractive indexes were measured only after imaging results were obtained.
- $\cdot$  We have completely blind about values of refractive indices
- Subsequent comparison of computational results with measurements of n was made.
- This comparison with both methods has demonstrated an **excellent accuracy** of our imaging results.

# Schematic diagram of the source/detectors configuration



Figure 1. a) The prism depicts our computational domain  $\Omega$ . This domain is a part of another prism, which was our holder made out of Styrofoam. Only a single source location was used. Tomographic measurements of the scattered time resolved EM wave were conducted on the bottom side of this prism. b) Schematic diagram of locations of detectors (probes) on the bottom side of the prism  $\Omega$ . The distance between neighboring probes was 10 mm.

• Solution of any PDE depends nonlinearly on its coefficients.

$$y' - ay = 0 \rightarrow y(a, t) = Ce^{at}.$$

- Any coefficient inverse problem is nonlinear.
- Two major challenges in numerical solution of any coefficient inverse problem: NONLINEARITY and III-POSEDNESS.
- Local minima of objective functionals.
- Locally convergent methods: linearizaton, Newton-like and gradient-like methods.

• **GOAL: QUANTIFIABLE** imaging of spatially distributed dielectric constants

• The main tool: The Beilina-Klibanov globally convergent inverse algorithm (SIAM J. Scientific Computing, 2008)

- Radiation sources:
- microwave
- acoustical
- $\bullet \ infrared/thermo$
- THZ (once the technology becomes available)

#### 1. MEDICINE

- a. medical optical imaging;
- b. acoustic imaging.

#### 2. MILITARY

a. humanitarian demining, i.e. cleaning former battlefields from land mines; checking out the baggage in airports and sea portsb. detecting targets covered by smog or flames on the battlefield (via diffuse optics).

# The Laplace transform

$$w(x,s) = \int_{0}^{\infty} u(x,t)e^{-st}dt = \int_{0}^{\infty} \widetilde{u}(x,t)e^{-s^{2}t}dt$$
$$\Delta w - s^{2}\varepsilon_{r}(x)w = -\delta(x - x_{0}),$$
$$\forall s > s_{0} = const. > 0.$$
$$\lim_{|x| \to \infty} w(x,s) = 0, \forall s > s_{0} = const. > 0.$$
$$w(x,s) > 0, \forall s > s_{0}.$$

# THE TRANSFORMATION PROCEDURE

• First, we make the Liouville transform

$$v = \frac{\ln w}{s^2},$$
$$\Delta v + s^2 |\nabla v|^2 = \varepsilon_r(x).$$

• The asymptotic behavior of the function v is

$$D_x^{\alpha} D_s^{\beta} v = D_x^{\alpha} D_s^{\beta} \left[ -\frac{l(x, x_0)}{s} + O\left(\frac{1}{s^2}\right) \right], \ s \to \infty$$

•  $I(x, x_0)$  is the length of the geodesic line connecting points x and  $x_0$ .

• Eliminate the unknown coefficient  $\varepsilon_r$  via the differentiation:  $\partial_s \varepsilon_r \equiv 0$ 

$$q(x,s) = \partial_{s}v(x,s),$$
$$v(x,s) = -\int_{s}^{\infty} q(x,\tau) d\tau \approx -\int_{s}^{\overline{s}} q(x,\tau) d\tau.$$

- We call the parameter s > 0 pseudo frequency.
- $\overline{s}$  is the truncation pseudo frequency.
- This truncation is similar with truncation of high frequencies which is routinely performed in engineering and everything still works.

• Approximate nonlinear Dirichlet boundary value problem for the function  $\boldsymbol{q}$  is

$$\Delta q - 2s^2 \nabla q \cdot \int_{s}^{\overline{s}} \nabla q(x,\tau) d\tau + 2s \left[ \int_{s}^{\overline{s}} \nabla q(x,\tau) d\tau \right]^2 = 0.$$
$$q(x,s) = \psi(x,s), \ \forall (x,s) \in \partial \Omega \times [\underline{s},\overline{s}].$$

• Backwards calculations

$$\varepsilon_r(x) = \Delta v + \underline{s}^2 (\nabla v)^2,$$

#### Approximation of the function q

• Layer stripping with respect to the pseudo frequency *s*.

• On each step the Dirichlet boundary value problem is solved for an elliptic equation for the function  $q_n$ .

$$\underline{s} = s_{N} < s_{N-1} < \dots < s_{1} < s_{0} = \overline{s}, s_{i-1} - s_{i} = h$$
$$q(x, s) = q_{n}(x) \text{ for } s \in (s_{n}, s_{n-1}].$$
$$\int_{s}^{\overline{s}} \nabla q(x, \tau) d\tau = (s_{n-1} - s) \nabla q_{n}(x) + h \sum_{j=1}^{n-1} \nabla q_{j}(x), s \in (s_{n}, s_{n-1}].$$

• Dirichlet boundary condition:

$$q_{n}(x) = \overline{\psi}_{n}(x) = \frac{1}{h} \int_{s_{n}}^{s_{n-1}} \psi(x,s) \, ds, \ x \in \partial \Omega$$

Hence,

$$egin{aligned} \widetilde{L}_n\left(q_n
ight) &:= \Delta q_n - 2\left(s^2 - 2s\left(s_{n-1} - s
ight)
ight) \left(h\sum_{j=1}^{n-1} 
abla q_j\left(x
ight)
ight) \cdot 
abla q_n - arepsilon q_n \ \end{aligned}$$
 $&= 2\left(s_{n-1} - s
ight)\left[s^2 - s\left(s_{n-1} - s
ight)
ight](
abla q_n)^2 - 2sh^2\left(\sum_{j=1}^{n-1} 
abla q_j\left(x
ight)
ight)^2, \ s \in (s_{n-1}, s_n] \end{aligned}$ 

Introduce the *s*-dependent Carleman Weight Function  $C_{n\mu}(s)$  by

$$\mathcal{C}_{n\mu}(s) = \exp\left[\mu\left(s-s_{n-1}\right)\right], s \in (s_n, s_{n-1}],$$

where  $\mu >> 1$  is a parameter.

• Multiply the equation by  $\mathcal{C}_{n\mu}(s)$  and integrate with respect to  $s \in [s_n, s_{n-1}]$ .

$$L_{n}(q_{n}) := \Delta q_{n} - A_{1n}(\mu, h) \left(h \sum_{i=1}^{n-1} \nabla q_{i}(x)\right) \cdot \nabla q_{n} - \kappa q_{n}$$

$$=2\frac{I_{1n}(\mu,h)}{I_0(\mu,h)}(\nabla q_n)^2-A_{2n}(\mu,h)h^2\left(\sum_{i=1}^{n-1}\nabla q_i(x)\right)^2,$$

where

$$I_{0}(\mu,h)=\int_{s_{n}}^{s_{n-1}}\mathcal{C}_{n\mu}(s)\,ds=\frac{1-e^{-\mu h}}{\mu},$$

$$I_{1n}(\mu, h) = \int_{s_n}^{s_{n-1}} (s_{n-1} - s) \left[ s^2 - s (s_{n-1} - s) \right] C_{n\mu}(s) \, ds,$$

$$A_{1n}(\mu,h) = \frac{2}{I_0(\mu,h)} \int_{s_n}^{s_{n-1}} (s^2 - 2s(s_{n-1} - s)) C_{n\mu}(s) ds,$$

$$A_{2n}(\mu,h)=\frac{2}{l_0(\mu,h)}\int\limits_{s_n}^{s_{n-1}}s\mathcal{C}_{n\mu}(s)\,ds.$$

• Important observation:

$$\frac{|I_{1n}\left(\mu,h\right)|}{I_{0}\left(\mu,h\right)} \leq \frac{4\bar{s}^{2}}{\mu}, \text{ for } \mu h > 1.$$

• When approximating functions  $q_n$ , we modify equations for them via introducing the so-called tail functions  $V_{n,i}$ 

• First, we choose  $V_{1,0}(x)$ . We can either choose  $V_{1,0}(x) \equiv 0$ , or we can choose

$$V_{1,0}(x) = \frac{\ln w_{1,0}(x,\overline{s})}{\overline{s}^2} = -\frac{|x-x_0|}{\overline{s}} - \frac{\ln (4\pi |x-x_0|)}{\overline{s}^2}$$

• Both choices work well, although the second one provides a faster convergence

•  $w_{1,0}(x,\overline{s})$  is the solution of the elliptic forward problem for w with  $\varepsilon_r = 1$ , i.e. the same as the value of  $\varepsilon_r$  outside of the domain of interest  $\Omega$ 

• 
$$q_{1,1}^0 := 0$$

• Step  $n_1, n \ge 1$ . First, iterate with respect to the nonlinear term. Suppose that functions  $q_1, ..., q_{n-1}, q_{n,1}^0 := q_{n-1} \in C^{2+\alpha}(\overline{\Omega})$  and the tail function  $V_{n,0}(x, \overline{s}) \in C^{2+\alpha}(\overline{\Omega})$  are constructed. Then we solve iteratively the following Dirichlet boundary value problems, k = 1, 2, ...

$$\begin{split} \Delta q_{n,1}^{k} - A_{1n} \left( h \sum_{j=1}^{n-1} \nabla q_{j} \right) \cdot \nabla q_{n,1}^{k} - \varepsilon q_{n,1}^{k} + A_{1n} \nabla q_{n,1}^{k} \cdot \nabla V_{n,0} \\ &= 2 \frac{l_{1n}}{l_{0}} \left( \nabla q_{n,1}^{k-1} \right)^{2} - A_{2n} h^{2} \left( \sum_{j=1}^{n-1} \nabla q_{j} \left( x \right) \right)^{2} \\ &+ 2 A_{2n} \nabla V_{n,0} \cdot \left( h \sum_{j=1}^{n-1} \nabla q_{j} \left( x \right) \right) - A_{2n} \left( \nabla V_{n,0} \right)^{2}, \\ q_{n,1}^{k} = \overline{\psi}_{n} \left( x \right), \ x \in \partial \Omega \end{split}$$

•  $q_{n,1} := \lim_{k o \infty} q_{n,1}^k$  in the  $C^{2+lpha}(\overline{\Omega})$  norm

- Calculate  $\varepsilon_r^{(n,1)}(x)$  via backwards calculations
- Solve the hyperbolic forward problem with  $\varepsilon_r(x) := \varepsilon_r^{(n,1)}(x)$ , calculate the Laplace transform and obtain the function  $w_{n,1}(x, \overline{s})$
- Find a new approximation for the tail function

$$V_{n,1}(x) = \frac{\ln w_{n,1}(x,\overline{s})}{\overline{s}^2}$$

• Step  $n_i$ ,  $i \ge 2, n \ge 1$ . We now iterate with respect to the tails. Suppose that functions  $q_{n,i-1}, V_{n,i-1}(x, \overline{s}) \in C^{2+\alpha}(\overline{\Omega})$  are constructed. Then solve the boundary value problem

$$\begin{split} \Delta q_{n,i} &- A_{1n} \left( h \sum_{j=1}^{n-1} \nabla q_j \right) \cdot \nabla q_{n,i} - \kappa q_{n,i} + A_{1n} \nabla q_{n,i} \cdot \nabla V_{n,i-1} \\ &= 2 \frac{l_{1n}}{l_0} \left( \nabla q_{n,i-1} \right)^2 - A_{2n} h^2 \left( \sum_{j=1}^{n-1} \nabla q_j \left( x \right) \right)^2 \\ &+ 2 A_{2n} \nabla V_{n,i-1} \cdot \left( h \sum_{j=1}^{n-1} \nabla q_j \left( x \right) \right) - A_{2n} \left( \nabla V_{n,i-1} \right)^2, \end{split}$$

 $q_{n,i}(x) = \overline{\psi}_n(x), x \in \partial \Omega$ 

• Calculate  $\varepsilon_r^{(n,i)}(x)$  via backwards calculations

- Solve the hyperbolic forward problem with  $\varepsilon_r(x) := \varepsilon_r^{(n,i)}(x)$ , calculate the Laplace transform and obtain the function  $w_{n,1}(x, \overline{s})$
- Find a new approximation for the tail function

$$V_{n,i}(x) = \frac{\ln w_{n,i}(x,\overline{s})}{\overline{s}^2}$$

- Iterate with respect to i until a certain convergence criterion is satisfied at  $i := m_n$
- Then set

$$q_{n} := q_{n,m_{n}}, \varepsilon_{r}^{(n)}(x) := \varepsilon_{r}^{(n,m_{n})}(x), V_{n+1,0}(x) := \frac{\ln w_{n,m_{n}}(x,\overline{s})}{\overline{s}^{2}}$$

- Proceed with  $q_{n+1}$  until a certain stopping rule is reached
- While convergence with respect to the nonlinear term can be analytically proven, convergence with respect to tail is established only numerically
- The stopping rule is in an agreement with our global convergence theorem

- Below we assume that  $1 \leq \varepsilon_r(x) \leq d$ , where the number d > 1 is known
- $\bullet$  It is important that a smallness condition is NOT imposed on the number d-1
- $K = K(d, \overline{s}, \Omega) \ge 1$  is a constant depending on the an upper bound d of an addmissible set of parameters, on the regularization parameter  $\overline{s}$  and on the domain  $\Omega$
- The next is a brief version of the most recent (February 2010) and the strongest formulation of our global convergence theorem

**Theorem (global convergence)** Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain with the boundary  $\partial \Omega \in \mathbb{C}^3$ . Denote  $\varepsilon_r^*(x) \in \mathbb{C}^2(\mathbb{R}^3)$  the exact coefficient (it is unknown) with the properties:  $1 \leq \varepsilon_r^*(x) \leq d$  in  $\Omega$ ,  $\varepsilon_r^*(x) = 1$  outside of  $\Omega$ . Let the small number h be the step size of our layer stripping procedure and the small number  $\sigma > 0$ be the level of error in the available data. Let  $\beta = \overline{s} - \underline{s}$  be the length of the s-interval we consider. Assume that

$$\overline{s}^2 \beta \leq d$$
 and  $h + \sigma < \frac{1}{K}$ .

Assume that the first approximation for the tail function is  $V_{1,0}(x) \equiv 0$ . Let  $\varepsilon_r^{(n)}(x)$  be the approximation for our coefficient due to the above iterative process. Then the following convergence estimate holds

$$\max_{\overline{\Omega}} \left| \varepsilon_r^{(n)} - \varepsilon_r^* \right| \le (Knh)^n \left( h + \sigma \right).$$

• It follows from this convergence estimate that, for sufficiently small *h*, the difference between exact  $\varepsilon_r^*(x)$  and the approximate  $\varepsilon_r^{(n)}(x)$  solutions **decreases** on first few iterations with *n* such that

$$1 \le n \le n_0 :\approx rac{1}{h} \cdot rac{1}{Ke}$$

- Next, this difference **increases** with iterations for  $n > n_0$
- Therefore one should stop iterations at  $n \approx n_0$
- This is **exactly** what we observe in computations and it is in the **full** agreement with the regularization theory
- This theorem is verified numerically for both experimental and computationally simulated data



Figure 2. Imaging example from computationally simulated data. a) Exact image. b) Computed image. Locations of inclusions as well as the value of  $\varepsilon_r(x) = 4$  in them are accurately imaged. The value  $\varepsilon_r(x) = 1$  outside of inclusions is also imaged accurately.

## Scheme of the experiment



Figure 3. a) Picosecond Pulse Generator 10070A and b) Tektronix DSA70000 Series Real Time Oscilloscope

- A radically new data pre-processing procedure was developed
- The pre-processed data were used as an input for Dirichlet boundary conditions  $\overline{\psi}_n(x), x \in \partial \Omega$

• The standard Fast Fourier Transform was only a preliminary step for data pre-processing



Figure 4. Samples of curves for measured signals: reference medium (top) and the medium with inclusion (bottom). Both on the same location of the detector. The signal before the burst is a parasitic one. The time step in measuremens was 20 picoseconds=0.02 nanosecond. The total burst takes 1200 picoseconds=1.2 nanoseconds, and this is our measured input data, which we pre-process before using it as boundary conditions for elliptic equations of our globally convergent method.



Figure 5. Samples of experimental curves after cleaning some noise via the Fourier transform: reference medium (top) and the medium with inclusion present (bottom). Both for the same location of the detector.



Figure 6. This figure explains the idea of the immersing procedure in the time domain. a) Resulting superimposed experimental curves obtained from curves on Figures 4-a), b). The red curve is for the reference signal and the blue curve is for the signal with a dielectric inclusion present, both at the same location  $x_m \in P$  of the probe number m. b) The red curve displays computationally simulated data  $u_{ref}(x_m, t)$ . The blue curve  $u_{incl}(x_m, t) = u_{ref}(x_m, t - \Delta t^m) K_{exp}^m / M_{exp}^m$  represents a sample of the immersed experimental data in the time domain at the same probe location  $x_m \in P$ . It is only the blue curve with which we work further. The red curve is displayed for the illustration purpose only.

## Further Data Pre-Processing in the Laplace Domain



Figure 7. Let  $w_{incl}(x, s)$  be the Laplace transform of the pre-processed data in time domain with inclusion present and  $\widetilde{w}_{incl}(x, s) = -(\ln w_{incl}(x, s))/s^2$ . a) The function  $\widetilde{w}_{incl}(x, \overline{s}), \overline{s} = 7.5$ . b) The function  $-(\ln w_{sim}(x, \overline{s}))/\overline{s}^2$  is depicted, where  $w_{sim}(x, \overline{s})$  is the Laplace transform (7) of the function  $u_{sim}(x, \overline{s})$  is the Laplace transform (7) of the function  $u_{sim}(x, t)$  for a computationally simulated data. Figure 7-b) is given only for the sake of comparison with Figure 7-a). c) The function  $\widetilde{w}_{smooth}(x, \overline{s})$  resulting from fitting of a) by the Lowess Fitting procedure in the 2-D case, see MATLABR 2009a. d) The final function  $\widetilde{w}_{immers}(x, \overline{s})$ . Values of  $\widetilde{w}_{immers}(x, s)$  are used to produce the Dirichlet boundary conditions  $\psi_n(x)$  for our elliptic PDEs of the globally convergent algorithm.

• Our goal was to image: (1) locations of inclusions and (2) most importantly, the value of the refractive index  $n = \sqrt{\varepsilon_r}$  in them.

• Therefore shapes of imaged inclusions are **NOT** of our concern.

• The 4 cm wooden cube (inclusion #1) was sequentially placed in three different positions: (a) on the center line connecting the tip of the EM wave generator with the center of the bottom side of the prism, (b) a little bit off the center line and (c) very much off the center line.

• The 6 cm wooden cube (inclusion #2) was sequentially placed in three different positions: (a) on the center line connecting the tip of the EM wave generator with the center of the bottom side of the prism and (b) a little bit off the center line.

#### Results of reconstruction: inclusion number 1.



a) n = 2.16
b) n = 2.27
c) n = 2.15
d) n = 2.0
Figure 8. Computed images of several locations of the 4 cm cube in semi-blind testing (see above for details). The cube is: a) on the center line, b) on the center line, but measured on the second day of experiments, c) a small shift off the center line, d) a large shift of the center line. Imaged values of the refractive index are displayed. Note that in d) the wave amplitude has decayed quite significantly at the inclusion location compared with the center line. Hence, the energy of the signal there was much less than on the center line. See Tables 1 and 2 for the accuracy of this imaging.

### Results of reconstruction: inclusion number 2.



a) n = 1.73 b) n = 1.79

Figure 9. Computed images of two locations of the 6 cm cube in semi-blind testing (see above for details). The cube is: a) on the center line and b) a small shift off the center line. Imaged values of the refractive index are displayed. See Tables 1 and 2 for the accuracy of this imaging. Table 1. Comparison of Blind Imaging Results of the Refractive Index n With Measurements by the Waveguide Method.

Cube	Blindly imaged n	Measured <i>n</i>	Imaging error	Measurement error
4 cm, Fig. 8a	2.16	2.07	4.3%	11%
4cm, Fig. 8b	2.27	2.07	10%	11%
4cm, Fig. 8c	2.15	2.07	3.9%	11%
4 cm, Fig. 8d	2	2.07	3.5%	11%
6 cm, Fig. 9a	1.73	1.71	1%	3.5%
6 cm, Fig. 9b	1.79	1.71	5%	3.5%

Table 2. Comparison of Blind Imaging Results of the Refractive Index n With Measurements by the Oscilloscope Method.

Cube	Blindly imaged n	Measured <i>n</i>	Imaging error	Measurement error
4 cm, Fig 8a	2.16	2.17	0.5%	6%
4 cm, Fig. 8b	2.27	2.17	4.6%	6%
4 cm, Fig. 8c	2.15	2.17	1%	6%
4 cm, Fig. 8d	2	2.17	7.8%	6%
6 cm, Fig. 9a	1.73	1.78	2.8%	6%
6 cm, Fig. 9b	1.79	1.78	0.56%	6%

We had four sources of error in the data for our algorithm:

- Our Partial Differential Equation cannot be derived from the Maxwell system
- Our theory does not work for the case of discontinuous function  $\varepsilon_r(x)$
- The natural measurement noise
- The modeling error due to the data pre-processing
- Thus, the accuracy of imaging results points towards a high degree of robustness of our globally convergent algorithm

• An excellent accuracy of our blind testing completely validates the above globally convergent algorithm

• A radically new and very effective data pre-processing procedure was invented and successfully applied to get a suitable input data for that numerical method.

• Although the measured refractive indexes in two cubes differ by 21% only (2.07/1.71-1), our algorithm has confidently differentiated between them in six our of six available experimental cases (100%).

• The case of back reflected data is duable by a modification of the globally convergent method. Mr. A. Kuzhuget, a doctoral student of Klibanov, has obtained some promising results already, although not from experimental data yet.

- Back reflected data are more practical.
- Back reflected data should lead to imaging of both dielectric and metallic inclusions, including semi-metallic ones.
- Radiation sources: microwave, acoustical, infrared/thermo, THZ (once the technology becomes available).
- Potential applications are listed above.

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