Lecture 6. A posteriori error estimates for the adaptivity technique for the Tikhonov functional

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Introduction

Presentation is based on the paper

- L. Beilina, M. V. Klibanov, A posteriori error estimates for the adaptivity technique for the Tikhonov functional and global convergence for a coefficient inverse problem, submitted for publication, 2009, available on-line at *http://www.ma.utexas.edu/mp_arc*, 2009.

- New a posteriori error estimate for the Tikhonov functional.
- Formulate an adaptive algorithm.
- Present efficiency of adaptivity technique for one CIP.

Statement of forward problem

As the forward problem, we consider the Cauchy problem for a hyperbolic PDE

$$\begin{aligned} c(x)\,u_{tt} &= \Delta u \text{ in } \mathbb{R}^3 \times (0,\infty), && (1) \\ u(x,0) &= 0,\, u_t(x,0) = \delta(x - x_0). && (2) \end{aligned}$$

Equation (1) governs propagation of acoustic and electromagnetic waves. In the acoustical case $1/\sqrt{c(x)}$ is the sound speed. In the 2-D case of EM waves propagation, the dimensionless coefficient $c(x) = \varepsilon_r(x)$, where $\varepsilon_r(x)$ is the relative dielectric function of the medium. Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$. Assume that the coefficient $c(x)$ is

$$\begin{aligned} c(x) &\in [1,d],\, d = const. > 1,\, c(x) = 1 \text{ for } x \in \mathbb{R}^3 \setminus \Omega, && (3) \\ c(x) &\in C^2(\mathbb{R}^3). && (4) \end{aligned}$$

Inverse Problem

We consider the following

Inverse Problem. *Suppose that the coefficient $c(x)$ satisfies (3) and (4), where the number $d > 1$ is given. Assume that the function $c(x)$ is unknown in the domain Ω. Determine the function $c(x)$ for $x \in \Omega$, assuming that the following function $g(x,t)$ is known for a single source position $x_0 \notin \overline{\Omega}$*

$$u(x,t) = g(x,t), \forall (x,t) \in \partial\Omega \times (0,\infty). \tag{5}$$

Remarks

- In applications the assumption $c(x) = 1$ for $x \in \mathbb{R}^3 \diagdown \Omega$ means that the target coefficient $c(x)$ has a known constant value outside of the medium of interest Ω.
- The function $g(x,t)$ models time dependent measurements of the wave field at the boundary of the domain of interest. In practice measurements are performed at a number of detectors, of course. In this case the function $g(x,t)$ can be obtained via one of standard interpolation procedures.
- The question of uniqueness of this Inverse Problem is a well known long standing open problem. It is addressed positively only if the function $\delta(x - x_0)$ in (2) is replaced with a function $f(x)$ such that $f(x) \neq 0, \forall x \in \overline{\Omega}$. Corresponding uniqueness theorems were proven via the method of Carleman estimates.

Frechét Derivatives

- The first step of the adaptivity is the calculation of the Frechét derivative of the Tikhonov functional. To do this, we need, to calculate Frechét derivatives of state and adjoint initial boundary value problems.
- To achieve the latter, we need in turn to establish a certain smoothness of solutions of state and adjoint initial boundary value problems.
- This smoothness cannot be guaranteed for the solution of the problem (1), (2) because of the $\delta-$ function in the initial condition.

Hence, we assume that the δ-function in condition (2) is replaced with a regularized one,

$$u(x,0) = 0, u_t(x,0) = \delta_\theta(x - x_0), \tag{6}$$

where

$$\begin{gathered}
\delta_\theta(x - x_0) = \left\{ \begin{array}{c} C_\theta \exp\left(\frac{1}{|x-x_0|^2 - \theta^2}\right), \ \ |x - x_0| < \theta \\ 0, \ \ |x - x_0| \geq \theta \end{array} \right\}, \\
\int\limits_{\mathbb{R}^m} \delta_\theta(x - x_0)\, dx = 1,
\end{gathered} \tag{7}$$

where $\theta > 0$ is so small that $\delta_\theta(x - x_0) = 0$ for $x \in \Omega$.

Introduce the set Y of functions $c(x)$ satisfying the following conditions

$$Y = \left\{ \begin{array}{c} c \in C\left(\mathbb{R}^3\right), c-1 \in H^1\left(\mathbb{R}^3\right), c(x) = 1 \text{ in } \mathbb{R}^3 \diagdown \Omega \\ c_{x_i} \in L_\infty(\Omega), c(x) \in (1-\omega, d+\omega) \text{ for } x \in \overline{\Omega} \end{array} \right\}, \tag{8}$$

where $\omega \in (0,1)$ is a small positive number. Let $T = const. > 0$. It follows from results of Chapter 4 of [Ladizenskaja] that the solution of the problem (1), (6) $u \in C^\infty\left(\mathbb{R}^3 \times [0,T]\right), \forall c \in Y$.

State and adjoint problems and their Frechét derivatives

- Let the function $c \in Y$. Since $c(x) = 1$ outside of the domain Ω, then, given the function g in (5), one can uniquely solve the initial boundary value problem (1), (5), (6) in the domain $\left(\mathbb{R}^3 \diagdown \Omega\right) \times (0, T)$. Thus, we can uniquely find the function u in this domain.
- Let Ω_1 be a convex bounded domain such that $\Omega \subset \Omega_1, \partial\Omega \cap \partial\Omega_1 = \varnothing, \partial\Omega_1 \in C^\infty$ and $\delta_\theta(x - x_0) = 0$ in $\overline{\Omega}_1$. Denote $Q_T = \Omega_1 \times (0, T), S_T = \partial\Omega_1 \times (0, T)$.
- We assume that there exists a function $a \in C^\infty\left(\overline{\Omega}_1\right)$ such that $a\mid_{\partial\Omega} = 0, \partial_n a \mid_{\partial\Omega} = 1.$

Smooth extensions for boundary conditions

Let $\widetilde{g}\,(x,t) = u\mid_{S_T},\ p(x,t) = \partial_n u\mid_{S_T}$. Since the function u can be uniquely determined in $\left(\mathbb{R}^3 \diagdown \Omega\right) \times (0,T)$, then functions $\widetilde{g}, p$ can also be uniquely determined. It turns out that classic theorems about existence of solutions of initial boundary value problems for hyperbolic PDEs require that the boundary condition should have a sufficiently smooth extension inside the domain of interest, see, e.g., [Lad,Evans].

Hence, we assume that there exist two functions F, W such that

$$
\begin{aligned}
F, W &\in H^5\left(Q_T\right), && (9)\\
\partial_n F\mid_{S_T} &= p\left(x,t\right), \partial_n W\mid_{S_T} = \widetilde{g}\left(x,t\right), && (10)\\
F\left(x,t\right) &= W\left(x,t\right) = 0 \text{ for } x \in \Omega, && (11)\\
\partial_t^j F\left(x,0\right) &= 0 \text{ in } \Omega_1, j = 0, ..., 3. && (12)
\end{aligned}
$$

State and adjoint problems

Consider now solutions u and λ of the following initial boundary value problems (we do not use a new notation for u for brevity),

$$
\begin{aligned}
c(x)\,u_{tt} &= \Delta u \text{ in } Q_T, \\
u(x,0) &= u_t(x,0) = 0, \\
\partial_n u\mid_{S_T} &= p(x,t);
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
c(x)\,\lambda_{tt} &= \Delta\lambda \text{ in } Q_T, \\
\lambda(x,T) &= \lambda_t(x,T) = 0, \\
\partial_n\lambda\mid_{S_T} &= (\widetilde{g} - u\mid_{S_T})\,z_\varepsilon(t).
\end{aligned}
\tag{14}
$$

We call these problems the “state problem” and the “adjoint problem”, respectively.

In (14) $z_{\varepsilon}(t)$ is a cut-off function, which is introduced to ensure that compatibility conditions at $\overline{S}_T \cap \{t = T\}$ are satisfied. Here $\varepsilon > 0$ is a small number. So, we choose such a function z_{ε} that

$$z_{\varepsilon} \in C^{\infty}[0, T], \; z_{\varepsilon}(t) = \left\{ \begin{array}{c} 1 \text{ for} \in [0, T - \varepsilon] \\ 0 \text{ for } t \in \left(T - \frac{\varepsilon}{2}, T\right] \\ \text{between 0 and 1 for } t \in \left(T - \varepsilon, T - \frac{\varepsilon}{2}\right) \end{array} \right\}.$$

We now reformulate for our specific needs a result, which follows immediately from Theorems 5 and 6 in section 7.2 of [Evans]. Consider the following initial boundary value problem

$$\begin{aligned} c(x)\, v_{tt} &= \Delta v + f \text{ in } Q_T, \\ v(x,0) &= v_t(x,0) = 0, \\ \partial_n v \mid_{S_T} &= y(x,t) \in L_2(S_T), \end{aligned} \qquad (15)$$

where the function $f \in H^k(Q_T), k \geq 0.$ The weak solution $v \in H^1(Q_T)$ of this problem satisfies the following integral identity for all functions $r \in H^1(Q_T)$ with $r(x,T) = 0$

$$\int\limits_{Q_T} (-c(x)\, v_t r_t + \nabla v \nabla r)\, dxdt - \int\limits_{S_T} yrdxdt - \int\limits_{Q_T} frdxdt = 0. \quad (16)$$

Assume that there exists such an extension $P(x,t)$ of the function $y(x,t)$ from the boundary S_T in the domain Q_T that $P \in H^{k+2}(Q_T)$, $\partial_n P\mid_{S_T} = y(x,t)$, $P(x,t) = 0$ for $x \in \Omega$, and in the case $k \geq 2$ let $\partial_t^j P(x,0) = 0, j = 0, ..., k$ and $\partial_t^i f(x,0) = 0, i = 0, ..., k-2$. Consider the function $v - P$. Dividing both sides of equation (15) by $c(x)$ and using above cited theorems and the formula $c^{-1}\Delta v = \nabla \cdot \left(c^{-1}\nabla v\right) - \nabla\left(c^{-1}\right)\nabla v$, we obtain that actually the weak solution $v \in H^{k+1}(Q_T)$ and the following estimate holds

$$\|v\|_{H^{k+1}(Q_T)} \leq B\left[\|P\|_{H^{k+2}(Q_T)} + \|f\|_{H^k(Q_T)}\right]. \tag{17}$$

Here and below $B = B(Y, Q_T, a(x))$ and $C = C\left(B, z_\varepsilon, \|F\|_{H^5(Q_T)}, \|W\|_{H^5(Q_T)}\right)$ are different positive constants depending on listed parameters.

Consider functions $\widehat{u} = u - F, \widehat{\lambda} = \lambda - (W - a(x) u) z_{\varepsilon}$ and substitute them in (13), (14). Then, using (9)-(12), (15) and (17), we obtain that functions $u, \lambda \in H^4(Q_T)$ and

$$\|u\|_{H^4(Q_T)} \leq B \|F\|_{H^5(Q_T)}, \quad \|\lambda\|_{H^4(Q_T)} \leq B \left(\|F\|_{H^5(Q_T)} + \|W\|_{H^5(Q_T)}\right). \tag{18}$$

Introduce the set Z of functions defined in Ω_1,

$$Z = \left\{ f : f \in C\left(\overline{\Omega}_1\right) \cap H^1\left(\Omega_1\right), \partial_{x_i} f \in L_\infty\left(\Omega_1\right) \right\}.$$

Define the norm in Z as

$$\|f\|_Z := \|f\|_{C\left(\overline{\Omega}_1\right)} + \sum_{i=1}^{3} \|\partial_{x_i} f\|_{L_\infty(\Omega_1)}. \tag{19}$$

Then Z is a Banach space, since convergence in the norm $\|\cdot\|_Z$ implies convergence in both spaces $C\left(\overline{\Omega}_1\right)$ and $H^1\left(\Omega_1\right)$. Let $\widetilde{Y}$ be the set of restrictions of all functions of the set Y on the domain Ω_1. Then it follows from (8) and (19) that $\widetilde{Y}$ is an open set in the space Z and

$$c_1(x) - c_2(x) \in Z' := \left\{ f \in Z : f(x) = 0 \text{ in } \Omega_1 \diagdown \Omega \right\}, \quad \forall c_1, c_2 \in \widetilde{Y}. \tag{20}$$

Frechét derivatives of state and adjoint problems with respect to the coefficient $c(x)$

Theorem 3.1. *Assume that initial conditions (2) are replaced with initial conditions (6), where the function $\delta_\theta(x - x_0)$ is defined in (7). Let domains Ω, Ω_1 and the function $a(x)$ be those specified above and $\delta_\theta(x - x_0) = 0$ in Ω_1. Assume that there exist functions F, W satisfying conditions (9)-(12). Consider the set $\widetilde{Y}$ as an open set in the space Z. Let operators $A_1 : \widetilde{Y} \to H^1(Q_T)$ and $A_2 : \widetilde{Y} \to H^1(Q_T)$ map every function $c \in \widetilde{Y}$ in the weak solution $u(x,t,c)$ of the problem (13) and the weak solution $\lambda(x,t,c)$ of the problem (14) respectively, where in (14) $u\,|_{S_T} := u(x,t,c)\,|_{S_T}$.*

Then in fact functions $u\left(x,t,c\right),\lambda\left(x,t,c\right)\in H^{4}\left(Q_{T}\right)$ *and each of the operators* A_1 *and* A_2 *has the Frechét derivative* $A_1^{\prime}\left(c\right)\left(b\right)=\widetilde{u}\left(x,t,c,b\right)\in H^{1}\left(Q_{T}\right)$ *and* $A_2^{\prime}\left(c\right)\left(b\right)=\widetilde{\lambda}\left(x,t,c,b\right)\in H^{1}\left(Q_{T}\right)$ *at each point* $c\in\widetilde{Y}$, *where* $b(x)\in Z'$ *is an arbitrary function. In fact, functions* $\widetilde{u},\widetilde{\lambda}$ $\in H^{2}\left(Q_{T}\right)$ *and they are solutions of the following initial boundary value problems*

$$\begin{aligned} &c\left(x\right)\widetilde{u}_{tt}=\Delta\widetilde{u}-b\left(x\right)u_{tt}\left(x,t,c\right),\ \text{ in } Q_{T},\\ &\widetilde{u}\left(x,0\right)=\widetilde{u}_{t}\left(x,0\right)=0,\partial_{n}\widetilde{u}\mid_{S_{T}}=0; \end{aligned} \tag{21}$$

$$\begin{aligned} &c\left(x\right)\widetilde{\lambda}_{tt}=\Delta\widetilde{\lambda}-b\left(x\right)\lambda_{tt}\left(x,t,c\right),\ \text{in } Q_T,\\ &\widetilde{\lambda}\left(x,T\right)=\widetilde{\lambda}_t\left(x,T\right)=0,\partial_n\widetilde{\lambda}\mid_{S_T}=-z_\varepsilon\widetilde{u}\mid_{S_T}. \end{aligned} \tag{22}$$

Denote

$$A_3\left(c\right)\left(x\right):=\int\limits_0^T\left(u_t\lambda_t\right)\left(x,t,c\right)dt, x\in\Omega,\forall c\in\widetilde{Y}.$$

Then the operator $A_3:\widetilde{Y}\rightarrow C\left(\overline{\Omega}\right)$.

Proof. The validity of the statement about the smoothness of functions u, λ follows from (18).
In the proof we will use definition of the Frechet derivative:

Definition 1. Let X, Y be two Banach spaces, $M \subseteq X$ be an open set and the operator $F : M \to Y$. We say that this operator has the Frechet derivative $F'(x)$ at the point $x \in M$ if there exists such a linear operator $F'(x) \in \mathcal{L}(X, Y)$ that for all h such that $x + h \in M$

$$F(x+h) - F(x) = F'(x)(h) + \alpha(x,h), \tag{1}$$

where

$$\lim_{\|h\|_X \to 0} \frac{\|\alpha(x,h)\|_Y}{\|h\|_X} = 0. \tag{2}$$

Suppose now that conditions of Definition 1 are satisfied. Suppose that $x \in M$ is such that $\{z : \|x - z\| < \varepsilon\} \subset M$.

Consider vectors $h \in \{h \in X : \|h\| < 1\}$ and let $t \in \mathbb{R}, |t| < \varepsilon$ be a parameter. Then $x + th \in M$ and by (1), (2)

$$F(x + th) - F(x) = F'(x)(th) + \alpha(x, th) \tag{3}$$

and since $\lim_{t \to 0} \|th\| = \|h\| \lim_{t \to 0} |t| = 0$, then by (2)

$$\lim_{t \to 0} \frac{\|\alpha(x, th)\|_Y}{|t| \, \|h\|_X} = 0. \tag{4}$$

Dividing both sides of (3) by t and using (4), we obtain

$$\lim_{t \to 0} \frac{F(x + th) - F(x)}{t} = F'(x)(th)/t = F'(x)(h). \tag{5}$$

Now we apply above described definition for Frechet derivative to our case. Consider an arbitrary function $c \in \widetilde{Y}$. It follows from (8) that there exists a sufficiently small number $\varepsilon_1 \in (0,1)$ such that $1-\omega(1-\varepsilon_1) \leq c(x) \leq d+\omega(1-\varepsilon_1)$. Let the function $b \in Z'$ be such that $\|b\|_{C(\overline{\Omega}_1)} < \varepsilon_1 \omega$, where the set Z' is defined in (20). Then $c+b \in \widetilde{Y}$. By (17)-(21) the function $\widetilde{u} \in H^2(Q_T)$ and

$$\|\widetilde{u}\|_{H^2(Q_T)} \leq B\|F\|_{H^5(Q_T)} \cdot \|b\|_Z . \tag{23}$$

Denote

$$\begin{aligned} u^{c+b}(x,t) &:= u(x,t,c+b), u^c(x,t) := u(x,t,c), \\ u_1 &:= u_1(x,t,c,b) = \left(u^{c+b} - u^c - \widetilde{u}\right)(x,t). \end{aligned}$$

Hence, $u_1 \in H^2(Q_T)$.

We now figure out the equation for the function u_1. By (19) and (27)

$$\begin{aligned}\Delta u_1 &= (c+b)\, u_{tt}^{c+b} - c u_{tt}^{c} - c\widetilde{u}_{tt} - b u_{tt}^{c} \\ &= (c+b)\, u_{tt}^{c+b} - (c+b)\, u_{tt}^{c} - c\widetilde{u}_{tt} \\ &= (c+b)\left(u^{c+b} - u^{c} - \widetilde{u}\right)_{tt} + b\widetilde{u}_{tt} = (c+b)\, u_{1tt} + b\widetilde{u}_{tt}.\end{aligned}$$

Hence, the function u_1 is the solution of the following intial boundary value problem

$$(c+b)\, u_{1tt} = \Delta u_1 - b\widetilde{u}_{tt}; \quad u_1(x,0) = u_{1t}(x,0) = 0, \quad \partial_n u_1 \mid_{S_T} = 0. \tag{24}$$

Hence, (17), (23) and (24) imply that

$$\left\| u_1 \right\|_{H^1(Q_T)} \leq C \left\| b \right\|_Z^2 . \tag{25}$$

Hence,

$$\lim_{\|b\|_Z \to 0} \left(\frac{\|u_1\|_{H^1(Q_T)}}{\|b\|_Z} \right) =$$
$$\lim_{\|b\|_Z \to 0} \left(\frac{\|u(x,t,c+b) - u(x,t,c) - \widetilde{u}(x,t,c,b)\|_{H^1(Q_T)}}{\|b\|_Z} \right) = 0. \tag{26}$$

Note that we set $A_i : \widetilde{Y} \to H^1(Q_T), i = 1, 2$ instead of $A_i : \widetilde{Y} \to H^2(Q_T)$ only for the sake of the estimate (25). Since the function $\widetilde{u}(x,t,c,b)$ depends linearly on b, then (25) and (26) imply that the function $\widetilde{u}$ is indeed the Frechét derivative of the operator A_1 at the point c.

- Hence, we now can consider the function $\widetilde{u}(x,t,c,b)$ for any $b \in Z'$.
- The proof for the operator A_2 is similar.
- it follows from (18) and the embedding theorem that functions $u, \lambda \in C^1\left(\overline{Q}_T\right)$, which implies the statement about the operator A_3. $\square$

How to derive equation (21) ?

If we replace h with b and t with τ in the definition of the Frechet derivative, we have

$$(c + \tau b)\, u_{tt}^{c+\tau b} = \Delta u^{c+\tau b}. \tag{6}$$

Consider

$$\left(\frac{d}{d\tau} u^{c+\tau b} \right) |_{\tau=0} = \widetilde{u}\,(x,t)\,. \tag{7}$$

To find the left hand side of (7), just differentiate (6) *formally* with respect to τ. Let $U^{c+\tau b} = \partial_\tau u^{c+\tau b}$. By (7) $U^{c+\tau b}\, |_{\tau=0} = \widetilde{u}\,(x,t)\,.$ Hence, by (6)

$$b u_{tt}^{c+\tau b} + (c + \tau b)\, U_{tt}^{c+\tau b} = U^{c+\tau b}. \tag{8}$$

Now we set in (8) $\tau := 0$ and obtain equation (21):

$$c\widetilde{u}_{tt} = \Delta \widetilde{u} - c u_{tt}^{c}, \tag{9}$$

The Frechét derivative of the Tikhonov functional

Assume that conditions of Theorem 3.1 hold. We define the Tikhonov functional $E : \widetilde{Y} \to \mathbb{R}$ as

$$E(c) = \frac{1}{2} \int_{S_T} (u\mid_{S_T} - \widetilde{g}(x,t))^2 z_\varepsilon(t)\, dxdt + \frac{1}{2}\alpha \int_{\Omega} (c - c_{glob})^2 \; dx, \forall c \in \widetilde{Y},$$

where $\alpha \in (0,1)$ is the regularization parameter and $c_{glob} \in \widetilde{Y}$ is the approximation for the exact solution c^* obtained on the globally convergent stage, see the end of section 2. We use the domain Ω rather than Ω_1 in the second integral term because of (20).

Consider the associated Lagrange functional $L(c)$,

$$L(c) = E(c) + \int\limits_{Q_T} \left(-c(x)\, u_t \lambda_t + \nabla u \nabla \lambda\right) dxdt - \int\limits_{S_T} p\lambda dxdt,$$
$$u := u(x,t,c) \in H^4(Q_T), \lambda := \lambda(x,t,c) \in H^4(Q_T), \tag{27}$$

where functions $u(x,t,c)$ and $\lambda(x,t,c)$ are solutions of initial boundary value problems (13), (14). The reason why we consider $L(c)$ is that we want to simplify the calculation of the Frechét derivative of $E(c)$. By (13), (14) and (16) the integral term in the first line of (29) equals zero. Hence,

$$L(c) = E(c) \text{ implying that } L'(c) = E'(c), \forall c \in \widetilde{Y},$$

where $L'(c)$ and $E'(c)$ are Frechét derivatives of functionals $L(c)$ and $E(c)$ respectively.

To obtain the explicit expression for $L'(c)$, we need to vary in (29) the function c via considering $c+b \in \widetilde{Y}$ for $b \in Z'$ and then to single out the term, which is linear with respect to b. When varying c, we also need to consider respective variations of functions u and λ in (29), since these functions depend on c as solutions of state and adjoint problems (13) and (14). By Theorem 3.1, linear, with respect to c, parts of variations of u and λ are functions $\widetilde{u}(x,t,c,b)$, $\widetilde{\lambda}(x,t,c,b)$.

Theorem 3.2. *Assume that conditions of Theorem 3.1 hold. Then for every function* $c \in \widetilde{Y}$

$$E'(c) = L'(c) = \alpha (c - c_{glob}) - \int_0^T u_t \lambda_t dt, \tag{28}$$

$$E'(c) \in C\left(\overline{\Omega}\right).$$

Proof.

We have

$$L(c) = E(c) + \int\limits_{Q_T} \left(-c(x)\, u_t \lambda_t + \nabla u \nabla \lambda\right) dxdt - \int\limits_{S_T} p\lambda dxdt,$$
$$u := u(x,t,c) \in H^4(Q_T), \lambda := \lambda(x,t,c) \in H^4(Q_T). \tag{29}$$

But here functions $u = u^c = u(x,t,c)$ and $\lambda = \lambda(x,t,c)$ depend on c, because they are solutions of (19) and (20).

Now, consider in (29) $L(c+b) - L(c) = E(c+b) - E(c)$, we get

$$L(c+b) - L(c) = E(c+b) - E(c)$$

$$+\int_{Q_T} \left[-(c+b)(x)(u_t\lambda_t)(x,t,c+b) + (\nabla u \nabla \lambda)(x,t,c+b)\right] dxdt$$

$$-\int_{S_T} p\lambda(x,t,c+b)\,dxdt$$

$$-\int_{Q_T} \left[-c(x)(u_t\lambda_t)(x,t,c) + (\nabla u \nabla \lambda)(x,t,c)\right] dxdt + \int_{S_T} p\lambda(x,t,c)\,dxdt.$$

Then single out the part which is linear with respect to b and thus obtain $L'(c)(b)$:

$$
\begin{aligned}
L'(c)(b) = E'(c)(b) = & \int_{\Omega} \left[\alpha (c - c_{glob}) - \int_0^T u_t \lambda_t dt \right] b(x)\, dx \\
& + \int_{Q_T} \left(-c u_t \widetilde{\lambda}_t + \nabla u \nabla \widetilde{\lambda} \right) dxdt - \int_{S_T} p \widetilde{\lambda} dxdt \\
& + \int_{Q_T} \left(-c \lambda_t \widetilde{u}_t + \nabla \lambda \nabla \widetilde{u} \right) dxdt - \int_{S_T} \left(\widetilde{g} - u \mid_{S_T} \right) z_{\varepsilon_1}(t) \widetilde{u} dxdt, \\
& \forall c \in \widetilde{Y}, \forall b \in Z',
\end{aligned}
\tag{30}
$$

- $\widetilde{u}$ and $\widetilde{\lambda}$ are solutions of problems (21) and (22) respectively.
- Since $\widetilde{u}(x,0) = \widetilde{\lambda}(x,T) = 0$, then (13), (14) and (16) imply that second and third lines in (30) equal zero, which proves the first line of (28).
- The validity of the second line of (28) follows from the statement of Theorem 3.1 about the operator A_3. $\square$

A Posteriori Error Estimates in The Adaptivity

- We work only with piecewise linear finite elements, because they are used in our computations.
- Consider a finite element mesh with the maximal grid step size h. Let the function $f \in Z$ and let f^I be its standard interpolant on this mesh. From standard interpolation estimates follows that

$$\left\| f - f^I \right\|_{C(\overline{\Omega}_1)} \leq K \left\| \nabla f \right\|_{L_\infty(\Omega_1)} h, \tag{31}$$

where the positive constant $K = K(\Omega_1)$ depends only on the domain Ω_1.

- We introduce the space of finite elements C_h with the norm $\|\cdot\|_{C_h} := \|\cdot\|_{C(\overline{\Omega})}$.
- Since $\dim C_h < \infty$, then all norms in this space are equivalent. Also, $C_h \subset Z$ as a set. Hence, if the function $\widetilde{c}(x)$ is defined in Ω_1 and is such that

$$\widetilde{c}\,(x) \in C_h \text{ for } x \in \Omega_1;\ \widetilde{c}\,(x) \in (1-\omega, d+\omega) \text{ in } \Omega; \\ \widetilde{c}\,(x) = 1 \text{ in } \Omega_1 \diagdown \Omega, \text{ then } \widetilde{c}\,(x) \in \widetilde{Y}. \tag{32}$$

In Theorem 4.1 we assume that state and adjoint problems are solved exactly for the case when the coefficient belongs to C_h.

Theorem 4.1. *Assume that conditions of Theorem 3.1 hold. Suppose that there exists a minimizer* $c_\alpha \in \widetilde{Y}$ *of the functional* $E(c)$ *on the set* V_r *as well as a minimizer* $c_{\alpha h} \in \widetilde{Y}$ *of* $E(c)$ *on the set* $V_r \cap C_h$ *(also, see (37)). Assume also that state and adjoint problems (13) and (14) are solved exactly for both coefficients* c_α *and* $c_{\alpha h}$*.*

Then the following approximate error estimate for the above defined Tikhonov functional is valid

$$\left|E\left(c_{\alpha}\right)-E\left(c_{h\alpha}\right)\right| \leq \left(A\left(\Omega\right)K\left\|E'\left(c_{\alpha h}\right)\right\|_{C\left(\overline{\Omega}\right)}\right)\left\|\nabla c_{\alpha}\right\|_{L_{\infty}(\Omega)}h,$$

where $A\left(\Omega\right)$ is the volume of the domain Ω, K is the interpolation constant from (31) and by (28)

$$E'\left(c_{\alpha h}\right)=\alpha\left(c_{\alpha h}-c_{glob}\right)-\int_{0}^{T}\left(u_t\lambda_t\right)\left(x,t,c_{\alpha h}\right)dt, \qquad (33)$$

where functions $u\left(x,t,c_{\alpha h}\right)\in H^{4}\left(Q_T\right)$ and $\lambda\left(x,t,c_{\alpha h}\right)\in H^{4}\left(Q_T\right)$ are solutions of problems (13) and (14) respectively with $c:=c_{\alpha h}$.

Proof. Since the function $c_{h\alpha}$ is a minimizer of $E(c)$ on $V_r \cap C_h$, then

$$E'\left(c_{\alpha h}\right)(b) = 0,\ \forall b \in C_h. \tag{34}$$

Now we use the Galerkin orthogonality . We have splitting ,

$$c_\alpha - c_{h\alpha} = \left(c_\alpha - c_\alpha^I\right) + \left(c_\alpha^I - c_{h\alpha}\right).$$

Since, $c_\alpha^I - c_{h\alpha} \in C_h$, then by (34) $E'\left(c_{\alpha h}\right)\left(c_\alpha^I - c_{h\alpha}\right) = 0$. Hence,

$$E'\left(c_{\alpha h}\right)\left(c_\alpha - c_{h\alpha}\right) = E'\left(c_{\alpha h}\right)\left(c_\alpha - c_\alpha^I\right). \tag{35}$$

By Theorem 3.2 the function $E'(c_{\alpha h}) \in C\left(\overline{\Omega}\right)$. Using (31), (**??**), (33) and (35), we obtain the following approximate error estimate

$$
\begin{aligned}
\|E(c_\alpha) - E(c_{h\alpha})\| &\leq \|E'(c_{\alpha h})(c_\alpha - c_\alpha^I)\| \\
&\leq \|c_\alpha - c_\alpha^I\|_{C(\overline{\Omega})} \int\limits_\Omega \|\alpha(c_{\alpha h} - c_{glob}) - \int\limits_0^T (u_t\lambda_t)(x,t,c_{\alpha h})dt\|dx \\
&\leq (A(\Omega)K\|E'(c_{\alpha h})\|_{C(\overline{\Omega})})\|\nabla c_\alpha\|_{L_\infty(\Omega)} h. \ \square
\end{aligned}
\tag{36}
$$

While it was assumed in Theorem 4.1 that state and adjoint problems (13) and (14) are solved exactly, in the computational practice they are solved approximately with a small error. Hence, it is desirable to express a posteriori error estimate through these approximate solutions. This is done in Theorem 4.2.

Theorem 4.2. *Assume that conditions of Theorem 3.1 hold. Suppose that there exists a minimizer $c_\alpha \in \widetilde{Y}$ of the functional $E(c)$ on the set V_r as well as a minimizer $c_{\alpha h} \in \widetilde{Y}$ of $E(c)$ on the set $V_r \cap C_h$. Suppose that state and adjoint problems (13) and (14) are solved exactly for $c := c_\alpha$ and that they are solved computationally with an error for $c := c_{\alpha h}$. Let functions $u_h := u_h(x, t, c_{\alpha h}) \in H^1(Q_T), \lambda_h := \lambda_h(x, t, c_{\alpha h}) \in H^1(Q_T)$ be those approximate solutions. Suppose that functions $u_{ht}, \lambda_{ht} \in L_\infty(Q_T)$.*

Let functions $u := u\left(x, t, c_{\alpha_h}\right), \lambda := \lambda\left(x, t, c_{\alpha h}\right) \in H^4\left(Q_T\right)$ (see (18)) be exact solutions of problems (13) and (14) with $c := c_{\alpha h}$. Assume that

$$\|u - u_h\|_{H^1(Q_T)} + \|\lambda - \lambda_h\|_{H^1(Q_T)} \leq \zeta, \tag{37}$$

where $\zeta \in (0,1)$ is a small number.
Then the following approximate a posteriori error estimate is valid

$$|E\left(c_\alpha\right) - E\left(c_{h\alpha}\right)| \leq K\left(A\left(\Omega\right)\|D\left(c_{\alpha h}\right)\|_{L_\infty(\Omega)} + C\zeta\right)\|\nabla c_\alpha\|_{L_\infty(\Omega)} h, \tag{38}$$

where the positive constant C was introduced in section 3 and

$$D\left(c_{\alpha h}\right) := \alpha\left(c_{\alpha h} - c_{glob}\right) - \int_0^T \left(u_{ht}\lambda_{ht}\right)\left(x, t, c_{\alpha h}\right) dt. \tag{39}$$

Proof. Since functions $u_{ht}, \lambda_{ht} \in L_\infty(Q_T)$ and functions $c_{\alpha h}, c_{glob} \in \widetilde{Y}$, then by (39) the function $D(c_{\alpha h}) \in L_\infty(\Omega)$. Next, using (41) and (45), we obtain the following approximate error estimate

$$
\begin{aligned}
|E(c_\alpha) - E(c_{h\alpha})| &\leq |E'(c_{\alpha h})(c_\alpha - c_\alpha^I)| \\
&\leq |D(c_{\alpha h})(c_\alpha - c_\alpha^I)| + \|[E'(c_{\alpha h}) - D(c_{\alpha h})](c_\alpha - c_\alpha^I)| \\
&= \|J_1\| + \|J_2\|
\end{aligned}
\tag{40}
$$

It follows from (31), (39) and (40) that

$$|J_1| \leq \left(A(\Omega) K \|D(c_{\alpha h})\|_{L_\infty(\Omega)} \right) \|\nabla c_\alpha\|_{L_\infty(\Omega)} h. \qquad (41)$$

We now estimate $|J_2|$ in (40). It follows from (31), (33), (39) and (40) that

$$|J_2| \leq K \|\nabla c_\alpha\|_{L_\infty(\Omega)} h \int_\Omega \left| \int_0^T [(u_t \lambda_t) - (u_{ht} \lambda_{ht})] (x, t, c_{\alpha h}) \, dt \right| dx$$

We have $(u_t\lambda_t) - (u_{ht}\lambda_{ht}) = (u_t - u_{ht})\,\lambda_t + u_{ht}\,(\lambda_t - \lambda_{ht})$. Next, by (24) and (37)

$$\|u_h\|_{H^1(Q_T)} \leq \|u_h - u\|_{H^1(Q_T)} + \|u\|_{H^1(Q_T)} \leq \zeta + B\,\|F\|_{H^5(Q_T)}\,.$$

Hence, since $\zeta \in (0,1)$, we obtain, using the Cauchy-Schwarz inequality that

$$\int_{\Omega} \left| \int_0^T \left[(u_t \lambda_t) - (u_{ht} \lambda_{ht}) \right] (x, t, c_{\alpha h}) \, dt \right| dx$$
$$\leq \| u - u_h \|_{H^1(Q_T)} \| \lambda \|_{H^1(Q_T)} + \| u_h \|_{H^1(Q_T)} \| \lambda - \lambda_h \|_{H^1(Q_T)} \leq C\zeta. \tag{42}$$

Hence,

$$|J_2| \leq CK \| \nabla c_\alpha \|_{L_\infty(\Omega)} \zeta h.$$

Combining this with (40) and (41), we obtain (38). □

Mesh Refinement Recommendation.

Assume that conditions of Theorem 4.2 hold and that the function $D(c_{\alpha h})(x) \in C(\overline{\Omega})$. It follows from this theorem and Remark 4.1 that the mesh should be refined in such a subdomain of the domain Ω where values of the function $|D(c_{\alpha h})(x)|$ are close to the number

$$\max_{\overline{\Omega}} |D(c_{\alpha h})(x)| = \max_{\overline{\Omega}} \left| \alpha (c_{\alpha h} - c_{glob})(x) - \int_0^T (u_{ht} \lambda_{ht})(x, t, c_{\alpha h})\, dt \right| .$$

Remarks on adaptivity

- We solve approximately the following equation with respect to the function $c_{\alpha_h}(x)$,

$$\alpha\left(c_{\alpha h}-c_{glob}\right)(x)-\int_0^T\left(u_{ht}\lambda_{ht}\right)\left(x,t,c_{\alpha h}\right)dt=0. \qquad (43)$$

- For each new mesh we first linearly interpolate the function $c_{glob}(x)$ on it.
- On each mesh we iteratively update approximations c_h^n of the function $c_{\alpha h}$ via the quasi-Newton method with the classic BFGS update formula with the limited storage [Nocedal].

- Denote

$$g^n(x) = \alpha(c_h^n - c_{glob})(x) - \int_0^T (u_{ht}\lambda_{ht})(x, t, c_h^n)\, dt,$$

where functions $u_h(x, t, c_h^n)$, $\lambda_h(x, t, c_h^n)$ are computed via solving state and adjoint problems with $c := c_h^n$.

Tne Adaptive algorithm

- Step 0. Choose an initial mesh K_h in Ω_1 and an initial time partition J_0 of the time interval $(0, T)$. Start with the initial approximation $c_h^0 = c_{glob}$ and compute the sequence of c_h^n via the following steps:
- Step 1. Compute solutions $u_h(x, t, c_h^n)$ and $\lambda_h(x, t, c_h^n)$ of state and adjoint problems of (13) and (14) on K_h and J_k.
- Step 2. Update the coefficient $c := c_h^{n+1}$ on K_h.
- Step 3. Stop computing c_h^n if either $||g^n||_{L_2(\Omega_1)} \leq \theta_1$ or norms $||g^n||_{L_2(\Omega_1)}$ are stabilized. Otherwise set $n := n + 1$ and go to step 1. Here θ_1 is the tolerance in quasi-Newton updates. In our computations we took $\theta_1 = 10^{-5}$.

Step 4. Compute the function $B_h(x)$,

$$B_h(x) = \left| \alpha \left(c_{\alpha h} - c_{glob} \right)(x) - \int_0^T \left(u_{ht} \lambda_{ht} \right)(x, t, c_{\alpha h})\, dt \right| .$$

Next, refine the mesh at all points where

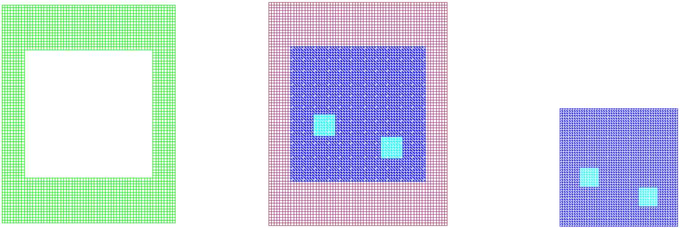
$$B_h(x) \geq \beta_1 \max_{\overline{\Omega}} B_h(x) . \tag{44}$$

Here the tolerance number $\beta_1 \in (0, 1)$ is chosen by the user.

Step 5. Construct a new mesh K_h in Ω_1 and a new time partition J_k of the time interval $(0, T)$. On J_k the new time step τ should be chosen in such a way that the CFL condition is satisfied. Interpolate the initial approximation c_{glob} from the previous mesh to the new mesh. Next, return to step 1 and perform all above steps on the new mesh.

Step 6. Stop mesh refinements if norms defined in step 3 either increase or stabilize, compared with the previous mesh, see Table 1 in section 6 for details.

Hybrid method



(a) G_{FDM} (b) $G = G_{FEM} \cup G_{FDM}$ (c) $G_{FEM} = \Omega$

The computational domain for the forward problem in our test is $G = [-4.0, 4.0] \times [-5.0, 5.0]$. This domain is split into a finite element domain $G_{FEM} := \Omega = [-3.0, 3.0] \times [-3.0, 3.0]$ and a surrounding domain G_{FDM} with a structured mesh, $G = G_{FEM} \cup G_{FDM}$.

Forward problem

$$\begin{aligned} c(x)\, u_{tt} - \triangle u &= 0, \quad \text{in } G \times (0,T), \\ u(x,0) = 0, \ u_t(x,0) &= 0, \ \text{in } G, \\ \partial_n u \big|_{\partial G_1} &= f(t), \ \text{on } \partial G_1 \times (0, t_1], \\ \partial_n u \big|_{\partial G_1} &= -\partial_t u, \ \text{on } \partial G_1 \times (t_1, T), \\ \partial_n u \big|_{\partial G_2} &= -\partial_t u, \ \text{on } \partial G_2 \times (0, T), \\ \partial_n u \big|_{\partial G_3} &= 0, \ \text{on } \partial G_3 \times (0, T), \end{aligned} \tag{45}$$

where $f(t)$ is the plane wave defined as

$$f(t) = \frac{(\sin(\bar{s}t - \pi/2) + 1)}{10}, \ 0 \leq t \leq t_1 := \frac{2\pi}{\bar{s}}, \ T = 17.8 t_1.$$

Simulated exact solution

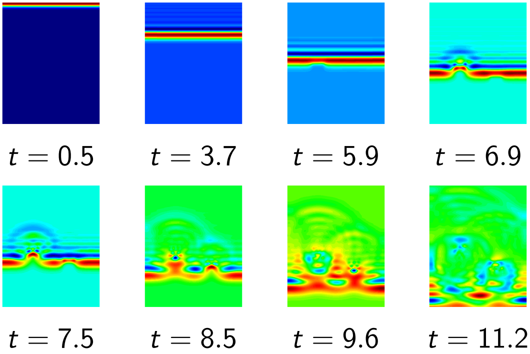
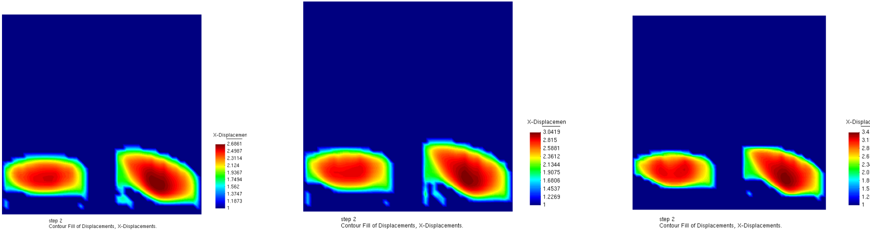


Figure: *Isosurfaces of the simulated exact solution to the forward problem (45) at different times with a plane wave initialized at the top boundary.*

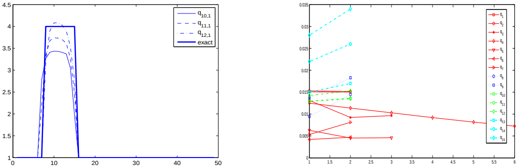
Results of reconstruction in the globally convergent method



a) $c_{9,2}$ b) $c_{10,2}$ c) $c_{11,2}$

Figure: *Spatial distributions of some functions $c_{n,k}$. The function $c_{11,2}$ is taken as the final result. The maximal value of $c_{11,2}(x) = 3.8$ within each imaged inclusion.*

Convergence results in globally convergent method



a) b)

Figure: *a) The one-dimensional cross-sections of the image of the function $c_{n,k}$ computed for corresponding functions $q_{n,1}$ along the vertical line passing through the middle of the right small square; b) Computed L_2 -norms of the $F_{n,k} = ||q_{n,k}\,|_{\partial\Omega} - \overline{\psi}_n||_{L_2(-3,3)}$.*

Adaptivity technique

On all refined meshes we have used a cut-off parameter C_{cut} for the reconstructed coefficient $c_{\alpha h}$. So that we re-define $c_{\alpha h}$ as

$$c_{\alpha h}(x) := \begin{cases} c_{\alpha h}(x), & \text{if } |c_{\alpha h}(x) - c_{glob}(x)| \geq C_{cut} \\ c_{glob}(x), & \text{elsewhere.} \end{cases}$$

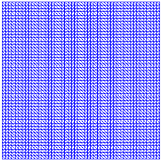
We choose $C_{cut} = 0$ for $m < 3$ and $C_{cut} = 0.3$ for $m \geq 3$, where m is the number of iterations in the quasi-Newton method on each mesh. Hence, the cut-off parameter ensures that we do not go too far from our good first guess for the solution $c_{glob}(x)$.

Convergence results in adaptive method

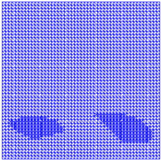
n	4608 elements	5340 elements	6356 elements	10058 elements	14586 elements
1	0.0992683	0.097325	0.0961796	0.0866793	0.0880115
2	0.0988798	0.097322	0.096723	0.0868341	0.0880866
3	0.0959911	0.096723			0.0876543
4		0.096658			

Table: *Norms* $||u\mid_{\Gamma_T} - g||_{L_2(\Gamma_T)}$ *on adaptively refined meshes. Here* $\Gamma_T = \Gamma \times (0, T)$ *and n is the number of updates in the quasi-Newton method. These norms generally decrease as meshes are refined. Then they slightly increase on the* 4^{th} *refinement. Thus, using this table, we conclude that on the four times refined mesh we get the final solution of our inverse problem.*

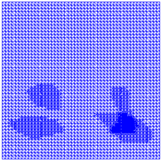
Results of reconstruction using an adaptive algorithm



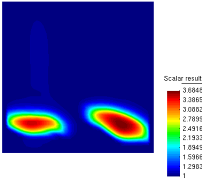
a) 4608 elements



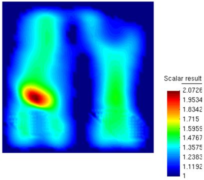
b) 5340 elements



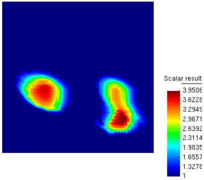
c) 6356 elements



d) 4608 elements

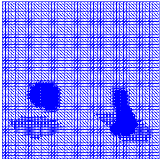


e) 5340 elements

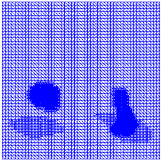


f) 6356 elements

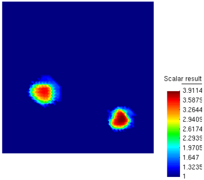
Results of reconstruction using an adaptive algorithm



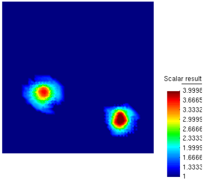
g) 10058 elements



h) 14586 elements



i) 10058 elements



j) 14586 elements

Adaptively refined meshes a)-c),g),h) and corresponding images d)-f), i),j) on the second stage of our two-stage numerical procedure. In a) the same mesh was used as one on the globally convergent stage. Comparison of d) with Fig. 2-c) (for $c_{11,2} = c_{glob}$) shows that the image was not improved compared with the globally convergent stage when the same mesh was used. However, the image was improved due to further mesh refinements. Fig. j) displays the final image obtained after four mesh refinements. Locations of both inclusions as well as 4:1 inclusions/background contrasts in them are imaged accurately.