Verifying the Accuracy of the Solution of the Wave Equation

January 20, 2010

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Explicit solution of the wave equation for the plane wave

$\mathbf{x} = (x, y, z)$, z is the vertical coordinate looking upwards.
Consider the wave equation in the domain

$$\mathbb{R}_a^3 = \{z < a\}, a = const. \geq 0.$$

Neumann boundary condition

$$\begin{aligned} u_{tt} &= \triangle u, \quad \text{in } \mathbb{R}^3_a \times (0, T), \\ u(\mathbf{x}, 0) &= u_t(\mathbf{x}, 0) = 0, \\ u_z(x, y, a, t) &= f(t). \end{aligned}$$

Obviously that the function u is independent on x, y, because f is independent on x, y. So, $u = u(z, t)$

$$\begin{aligned} w(z, s) &= \int_0^\infty u e^{-st} dt := Lu, \\ \widetilde{f}(s) &= \int_0^\infty f e^{-st} dt. \end{aligned}$$

Then

$$\begin{aligned} w_{zz} - s^2 w &= 0, z \in (-\infty, a), \\ w_z(a,s) &= \widetilde{f}(s). \end{aligned}$$

Solution of the ODE is

$$w = C_1 e^{-sz} + C_2 e^{sz}.$$

But since $\lim_{z \to -\infty} e^{-sz} = \infty$, then $C_1 = 0$. Now we should find C_2. Changing C_2 a little bit, we can write

$$w = C \exp(s(z-a)) = C \exp(-s|z-a|), \text{ for } z < a.$$

Since $w_z(a,s) = \widetilde{f}(s)$, then $C = \widetilde{f}(s)/s$. Hence

$$w = \frac{\widetilde{f}(s)}{s} \exp(-s|z-a|).$$

Rules of the Laplace transform operator

L

$$\frac{\widetilde{f}(s)}{s} = L\left(\int_0^t f(\tau)\,d\tau\right),$$

$$\widetilde{p}(s)\,\widetilde{g}(s) = L\left(\int_0^t p(\tau)\,g(t-\tau)\,d\tau\right).$$

The inverse Laplace transform of $\exp(-s\,|z-a|)$ is $\delta(t-|z-a|) = \delta(t-a+z)$.

By these rules the inverse Laplace transform of the function w is

$$u(z,t) = \int_0^t \left(\int_0^\tau f(r)\, dr \right) \delta(t-\tau-a+z)\, d\tau$$

$$= H(t-a+z) \int_0^{t-a+z} f(r)\, dr,$$

where H is the Heaviside function,

$$H(p) = \begin{cases} 1, & \text{if } p > 0, \\ 0, & \text{if } p < 0. \end{cases}$$

Then

$$u(z,t)=\begin{cases} 0, \text{ if } t<a-z, \\ \int\limits_0^{t-a+z} f(\tau)\,d\tau, \text{ if } t>a-z. \end{cases} \tag{1}$$

For example, consider the case of the truncated sinusoid,

$$f(t)=\begin{cases} \sin(\omega t), \text{ if } t\in\left(0,\frac{2\pi}{\omega}\right), \\ 0, \text{ if } t>\frac{2\pi}{\omega}. \end{cases} \tag{2}$$

For this function f let's calculate

$$\int\limits_0^t f(\tau)\,d\tau=g(t).$$

Case 1. $t \in (0, 2\pi/\omega]$. Then

$$g(t) = \int_0^t f(\tau)\, d\tau = \int_0^t \sin(\omega\tau)\, d\tau = -\frac{1}{\omega}\cos(\omega\tau)\,\Big|_0^t = \frac{1-\cos(\omega t)}{\omega}.$$

Case 2. $t > 2\pi/\omega$. Then

$$\begin{aligned} g(t) &= \int_0^t f(\tau)\, d\tau = \int_0^{2\pi/\omega} \sin(\omega\tau)\, d\tau + \int_{2\pi/\omega}^t 0 \cdot d\tau = \\ &= \int_0^{2\pi/\omega} \sin(\omega\tau)\, d\tau = \frac{1-\cos(2\pi)}{\omega} = 0. \end{aligned}$$

Therefore we obtain

$$g(t) = \int_0^t f(\tau)\, d\tau = \begin{cases} \frac{1-\cos(\omega t)}{\omega}, \text{ if } t \in \left(0, \frac{2\pi}{\omega}\right], \\ 0, \text{ if } t > \frac{2\pi}{\omega}. \end{cases}$$

Hence, it follows from (1) that for the function f in (2) the solution of our initial boundary value problem is
Then

$$u(z,t) = \begin{cases} 0, \text{ if } t < a - z, \\ \frac{(1-\cos\omega(t-a+z))}{\omega}, \text{ if } t \in \left(a-z, a-z+\frac{2\pi}{\omega}\right), \\ 0, \text{ if } t > a - z + \frac{2\pi}{\omega}. \end{cases} \tag{3}$$

By the way

$$u(z,t) \geq 0, \max u(z,t) = \frac{2}{\omega}. \tag{4}$$

Solution of the wave equation in our computations

Consider the domain

$$G = \{(x, y, z) \in [-3.0, 3.0] \times [-2.0, 2.0] \times [-5.0, 5.0]\} .$$

Let ∂G_1 and ∂G_2 be respectively top and bottom sides of the prism G and $\partial G_3 = \partial G \diagdown (\partial G_1 \cup \partial G_2)$ be the rest of the boundary of the domain G. We have numerically solved the following initial boundary value problem for $T = 12$

$$
\begin{aligned}
u_{tt} &= \triangle u, \quad \text{in } G \times (0, T), \\
u(x,0) &= 0, \ u_t(x,0) = 0, \ \text{in } G, \\
\partial_n u\big|_{\partial G_1} &= \sin(\omega t), \ \text{if } t \in \left(0, \frac{2\pi}{\omega}\right), \\
\partial_n u\big|_{\partial G_1} &= -\partial_t u, \ \text{if } t \in \left(\frac{2\pi}{\omega}, T\right), \\
\partial_n u\big|_{\partial G_2} &= -\partial_t u, \ \text{on } \partial G_2 \times (0, T), \\
\partial_n u\big|_{\partial G_3} &= 0, \ \text{on } \partial G_3 \times (0, T).
\end{aligned} \tag{5}
$$

Then the above function $u(z,t)$ satisfies these conditions. Hence, it is solution of the problem (5). But we should take $a = 5$ in (3). Therefore, the solution of the problem (5) is the function (3).

Dirichlet Boundary Condition

Let

$$\begin{aligned} u_{tt} &= \triangle u, \quad \text{in } \mathbb{R}_a^3 \times (0, T), \\ u(\mathbf{x}, 0) &= u_t(\mathbf{x}, 0) = 0, \\ u(x, y, a, t) &= f(t). \end{aligned}$$

Then

$$\begin{aligned} w_{zz} - s^2 w &= 0, z \in (-\infty, a), \\ w(a, s) &= \widetilde{f}(s). \end{aligned}$$

Solution of the ODE is

$$w = C_1 e^{-sz} + C_2 e^{sz}.$$

But since $\lim_{z\to-\infty} e^{-sz} = \infty$, then $C_1 = 0$. Now we should find C_2. Changing C_2 a little bit, we can write

$$w = C \exp(s(z-a)) = C \exp(-s|z-a|), \text{ for } z < a.$$

Hence,

$$w(a, s) = \widetilde{f}(s) = C.$$

Hence,

$$w = \widetilde{f}(s) \exp\left(-s\left|z - a\right|\right).$$

Again calculating the inverse Laplace transform, we obtain

$$u(z, t) = \int_0^t f(\tau)\,\delta(t - \tau - a + z)\,d\tau = f(t - a + z)\,H(t - a + z).$$

Consider now the case

$$f\left(t\right)=\left\{\begin{array}{c}\sin\left(\omega t\right),\ \text{if}\ t\in\left(0,\frac{2\pi}{\omega}\right),\\ 0,\ \text{if}\ t>\frac{2\pi}{\omega}.\end{array}\right. .$$

Then

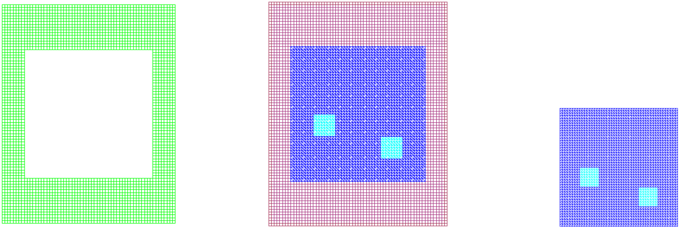
$$u\left(z,t\right)=\left\{\begin{array}{c}0,\ \text{if}\ t\in\left(0,a-z\right).\\ \sin\omega\left(t-a+z\right),\ \text{if}\ t\in\left(a-z,a-z+\frac{2\pi}{\omega}\right),\\ 0,\ \text{if}\ t>a-z+\frac{2\pi}{\omega}.\end{array}\right. \qquad (6)$$

Consider now the problem

$$\begin{aligned} u_{tt} &= \triangle u, \quad \text{in } G \times (0, T), \\ u(x,0) &= 0,\ u_t(x,0) = 0,\ \text{in } G, \\ u\big|_{\partial G_1} &= \sin(\omega t),\ \text{if } t \in \left(0, \frac{2\pi}{\omega}\right), \\ u\big|_{\partial G_1} &= -\partial_t u,\ \text{if } t \in \left(\frac{2\pi}{\omega}, T\right), \\ \partial_n u\big|_{\partial G_2} &= -\partial_t u,\ \text{on } \partial G_2 \times (0, T), \\ \partial_n u\big|_{\partial G_3} &= 0,\ \text{on } \partial G_3 \times (0, T). \end{aligned} \tag{7}$$

Then the function (6) satisfies conditions (7). Hence, this function is the solution of this problem.

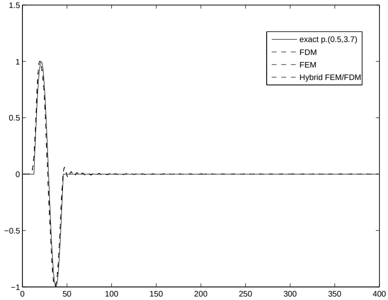
Hybrid method



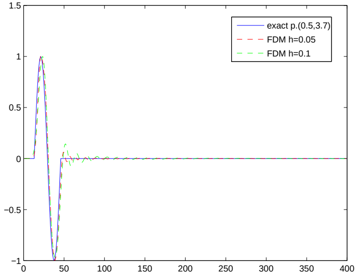
(a) G_{FDM} (b) $G = G_{FEM} \cup G_{FDM}$ (c) $G_{FEM} = \Omega$

The computational domain for the forward problem in our test is $G = [-4.0, 4.0] \times [-5.0, 5.0]$. This domain is split into a finite element domain $G_{FEM} := \Omega = [-3.0, 3.0] \times [-3.0, 3.0]$ and a surrounding domain G_{FDM} with a structured mesh, $G = G_{FEM} \cup G_{FDM}$.

Comparison of the exact and computed solutions on different meshes when $c = 1$



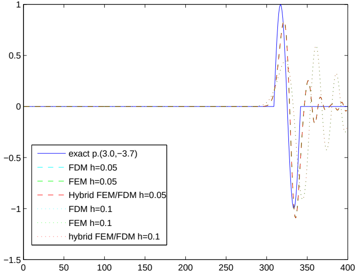
$h = 0.05$

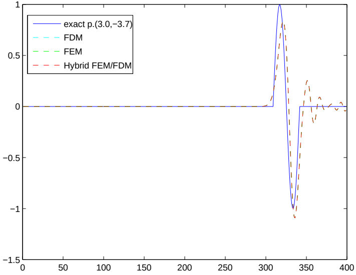


$h = 0.05$ and $h = 0.1$

Exact solution (6) compared with the computed solution of the problem (7) in one point (0.5,3.7), which is located at the top of the domain.

Comparison of the exact and computed solutions on different meshes when $c = 1$





$h = 0.05$ and $h = 0.1$ $\qquad$ $h = 0.05$

Exact solution (6) compared with the computed solution of the problem (7) in one point (3.0,-3.7), which is located at the bottom of the domain.

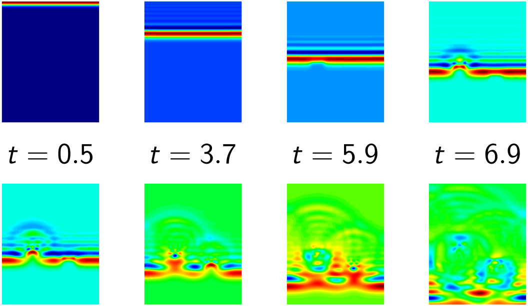
Forward problem

$$
\begin{aligned}
c\left(x\right)u_{tt} - \triangle u &= 0, \quad \text{in } G \times (0, T), \\
u(x,0) &= 0, \; u_t(x,0) = 0, \; \text{in } G, \\
\partial_n u\big|_{\partial G_1} &= f\left(t\right), \; \text{on } \partial G_1 \times (0, t_1], \\
\partial_n u\big|_{\partial G_1} &= -\partial_t u, \; \text{on } \partial G_1 \times (t_1, T), \\
\partial_n u\big|_{\partial G_2} &= -\partial_t u, \; \text{on } \partial G_2 \times (0, T), \\
\partial_n u\big|_{\partial G_3} &= 0, \; \text{on } \partial G_3 \times (0, T),
\end{aligned} \tag{1}
$$

where $f(t)$ is the plane wave defined as

$$
f(t) = \frac{\left(\sin\left(\bar{s}t - \pi/2\right) + 1\right)}{10}, \; 0 \leq t \leq t_1 := \frac{2\pi}{\bar{s}}, \; T = 17.8t_1.
$$

Simulated exact solution



$t = 0.5$	$t = 3.7$	$t = 5.9$	$t = 6.9$
$t = 7.5$	$t = 8.5$	$t = 9.6$	$t = 11.2$

Figure: *Isosurfaces of the simulated exact solution to the forward problem on the mesh with mesh size $h = 0.125$ at different times with a plane wave initialized at the top boundary.*