

A globally convergent numerical method for some coefficient inverse problems

Michael V. Klibanov*, Larisa Beilina[△]

* University of North Carolina at Charlotte Charlotte, USA

△ Norwegian Institute of Science and Technology Trondheim,
Norway

Solution of the problem of local minima for a class of coefficient inverse problems

- Developed by L. Beilina and M.V. Klibanov (2007)
- A numerical method X is globally convergent if:
 1. A theorem is proved claiming convergence to a good approximation for the correct solution regardless on a priori availability of a good guess
 2. This theorem is confirmed by numerical experiments for at least one applied problem

$$y' - ay = 0 \rightarrow y(a, t) = Ce^{at}.$$

- Solution of any PDE depends nonlinearly on its coefficients.
- Any coefficient inverse problem is nonlinear.
- Two major challenges in numerical solution of any coefficient inverse problem: NONLINEARITY and III-POSEDNESS.
- Local minima of objective functionals.
- Locally convergent methods: linearizatton, Newton-like and gradient-like methods.

FOUR OTHER NUMERICAL METHODS INDEPENDENT ON THE FIRST GOOD GUESS

1. The method of Novikov, $-\Delta u + q(x)u = Eu$. Numerical experiments: Burov, Morozov and Rumyantseva.
2. The method of Nachman, $\operatorname{div}(\sigma(x, y)\nabla u) = 0$. Numerical results: Isaacson, Newell, Mueller, Siltanen.
3. The BC method of Belishev. General hyperbolic equations.
4. The 2-D Gelfand-Levitan-Krein method: Kabanikhin.
Numerical results in the book of Kabanikhin, Satybaev and Shishlenin

$$u_{tt} = \Delta u + q(x, y)u \text{ in } \mathbb{R}^2$$

- Multiple measurements.
- Non-iterative methods: direct reconstruction.
- Previously a layer stripping with respect to the frequency for

$$\Delta u + k^2 (1 + c(x)) u = 0$$

was developed by Yu Chen (Inverse Problems 1997).

- Convergence theorem was not proven.
- The starting frequency was $k = 0$.
- The case of the unknown coefficient $a(x)$ cannot be handled.
- Linearization instead of Carleman Weight Functions.
- Numerical reconstruction was nice.

A hyperbolic equation

$$c(x) u_{tt} = \Delta u - a(x)u \text{ in } \mathbb{R}^n \times (0, \infty), n = 2, 3,$$

$$u(x, 0) = 0, u_t(x, 0) = \delta(x - x_0).$$

INVERSE PROBLEM. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let one of coefficients $c(x)$ or $a(x)$ be unknown in Ω but it is a given constant outside of Ω . Determine this coefficient in Ω , given the function $f(x, t)$,

$$u(x, t) = f(x, t), x \in \partial\Omega, t \in (0, \infty).$$

Similarly for parabolic equation

$$c(x)\tilde{u}_t = \Delta\tilde{u} - a(x)\tilde{u} \text{ in } \mathbb{R}^n \times (0, \infty),$$

$$\tilde{u}(x, 0) = \delta(x - x_0).$$

APPLICATIONS: Acoustics, electromagnetics, defects in photonic crystals (nano physics!), medical optical imaging

Laplace transform:

$$w(x, s) = \int_0^{\infty} u(x, t) e^{-st} dt = \int_0^{\infty} \tilde{u}(x, t) e^{-s^2 t} dt.$$

$$\Delta w - [s^2 c(x) + a(x)] w = -\delta(x - x_0),$$
$$\forall s > s_0 = \text{const} > 0.$$

$$\lim_{|x| \rightarrow \infty} w(x, s) = 0, \forall s > s_0 = \text{const} > 0.$$

$$w(x, s) > 0, \forall s > s_0.$$

THE TRANSFORMATION PROCEDURE

First, we eliminate the unknown coefficient from the equation:

$$v = \ln w.$$

$$\Delta v + |\nabla v|^2 = s^2 c(x) + a(x) \text{ in } \Omega,$$

- Let, for example $c(x) = ?$ For simplicity let $a(x) = 0$. It follows from Romanov that

$$D_x^\alpha D_s^\beta (v) = D_x^\alpha D_s^\beta \left[-\frac{s l(x, x_0)}{g(x, x_0)} \left(1 + O\left(\frac{1}{s}\right) \right) \right], s \rightarrow \infty.$$

- Introduce a new function

$$\tilde{v} = \frac{v}{s^2}.$$

Then

$$\tilde{v}(x, s) = O\left(\frac{1}{s}\right), s \rightarrow \infty.$$

- Eliminate the unknown coefficient $c(x)$ via the differentiation:
 $\partial_s c(x) \equiv 0$

$$q(x, s) = \partial_s \tilde{v}(x, s),$$

$$\tilde{v}(x, s) = - \int_s^\infty q(x, \tau) d\tau \approx - \int_s^{\bar{s}} q(x, \tau) d\tau + V(x, \bar{s}).$$

- $V(x, \bar{s})$ is the tail function, $V(x, \bar{s}) \approx 0$. But still we iterate with respect to the tail.

- This truncation is similar to the truncation of high frequencies.
- Obtain Dirichlet boundary value problem for the nonlinear equation

$$\Delta q - 2s^2 \nabla q \cdot \int_s^{\bar{s}} \nabla q(x, \tau) d\tau + 2s \left[\int_s^{\bar{s}} \nabla q(x, \tau) d\tau \right]^2 \quad (1)$$

$$+ 2s^2 \nabla q \nabla V - 2s \nabla V \cdot \int_s^{\bar{s}} \nabla q(x, \tau) d\tau + 2s (\nabla V)^2 = 0,$$

$$q(x, s) = \psi(x, s), \quad \forall (x, s) \in \partial\Omega \times [\underline{s}, \bar{s}]. \quad (2)$$

- Backwards calculations

$$c(x) = \Delta \tilde{v} + \underline{s}^2 (\nabla \tilde{v})^2,$$

How To Solve the Problem (1)-(2)?

- Layer stripping with respect to the pseudo frequency s .
- On each step Dirichlet boundary value problem is solved for an elliptic equation.

$$\underline{s} = s_N < s_{N-1} < \dots < s_1 < s_0 = \bar{s}, s_{i-1} - s_i = h$$

$$q(x, s) = q_n(x) \text{ for } s \in (s_n, s_{n-1}] .$$

$$\int_s^{\bar{s}} \nabla q(x, \tau) d\tau = (s_{n-1} - s) \nabla q_n(x) + h \sum_{j=1}^{n-1} \nabla q_j(x), s \in (s_n, s_{n-1}] .$$

- Dirichlet boundary condition:

$$q_n(x) = \bar{\psi}_n(x), x \in \partial\Omega,$$

$$\bar{\psi}_n(x) = \frac{1}{h} \int_{s_n}^{s_{n-1}} \psi(x, s) ds.$$

Hence,

$$\begin{aligned}
 \tilde{L}_n(q_n) &:= \Delta q_n - 2(s^2 - 2s(s_{n-1} - s)) \left(h \sum_{j=1}^{n-1} \nabla q_j(x) \right) \cdot \nabla q_n \\
 &\quad + 2(s^2 - 2s(s_{n-1} - s)) \nabla q_n \cdot \nabla V(x, \bar{s}) - \varepsilon q_n \\
 &= 2(s_{n-1} - s) [s^2 - s(s_{n-1} - s)] (\nabla q_n)^2 - 2sh^2 \left(\sum_{j=1}^{n-1} \nabla q_j(x) \right)^2 \\
 &\quad + 4s \nabla V(x, \bar{s}) \cdot \left(h \sum_{j=1}^{n-1} \nabla q_j(x) \right) - 2s [\nabla V(x, \bar{s})]^2, s \in (s_{n-1}, s_n].
 \end{aligned}$$

The Carleman Weight Function

Introduce the s -dependent Carleman Weight Function $\mathcal{C}_{n\mu}(s)$ by

$$\mathcal{C}_{n\mu}(s) = \exp [\mu(s - s_{n-1})], s \in (s_n, s_{n-1}],$$

where $\mu \gg 1$ is a parameter.

- Multiply equation by $\mathcal{C}_{n\mu}(s)$ and integrate with respect to $s \in [s_n, s_{n-1}]$.

$$\begin{aligned}
 L_n(q_n) &:= \Delta q_n - A_{1n}(\mu, h) \left(h \sum_{i=1}^{n-1} \nabla q_i(x) \right) \cdot \nabla q_n - \varepsilon q_n \\
 &= 2 \frac{I_{1n}(\mu, h)}{I_0(\mu, h)} (\nabla q_n)^2 - A_{2n}(\mu, h) h^2 \left(\sum_{i=1}^{n-1} \nabla q_i(x) \right)^2 \\
 &\quad + 2A_{1n}(\mu, h) \nabla V(x, \bar{s}) \cdot \left(h \sum_{i=1}^{n-1} \nabla q_i(x) \right) \\
 &\quad - A_{2n}(\mu, h) \nabla q_n \cdot \nabla V(x, \bar{s}) - A_{2n}(\mu, h) [\nabla V(x, \bar{s})]^2,
 \end{aligned}$$

where

$$I_0(\mu, h) = \int_{s_{n-1}}^{s_n} \mathcal{C}_{n\mu}(s) ds = \frac{1 - e^{-\mu h}}{\mu},$$

$$I_{1n}(\mu, h) = \int_{s_{n-1}}^{s_n} (s_{n-1} - s) \left[s^2 - s(s_{n-1} - s) \right] \mathcal{C}_{n\mu}(s) ds,$$

$$A_{1n}(\mu, h) = \frac{2}{I_0(\mu, h)} \int_{s_{n-1}}^{s_n} \left(s^2 - 2s(s_{n-1} - s) \right) \mathcal{C}_{n\mu}(s) ds,$$

$$A_{2n}(\mu, h) = \frac{2}{I_0(\mu, h)} \int_{s_{n-1}}^{s_n} s \mathcal{C}_{n\mu}(s) ds.$$

- Important observation:

$$\frac{|I_{1n}(\mu, h)|}{I_0(\mu, h)} \leq \frac{C}{\mu} \ll 1, \text{ for } \mu h > 1.$$

- Iterative solution for every q_n

$$\Delta q_{nk}^i - A_{1n} \left(h \sum_{j=1}^{n-1} \nabla q_j \right) \cdot \nabla q_{nk}^i - \varepsilon q_{nk}^i + A_{1n} \nabla q_{nk}^i \cdot \nabla V_n^i =$$

$$2 \frac{I_{1n}(\mu, h)}{I_0(\mu \cdot h)} \left(\nabla q_{n(k-1)}^i \right)^2 - A_{2n} h^2 \left(\sum_{j=1}^{n-1} \nabla q_j(x) \right)^2$$

$$+ 2A_{2n} \nabla V_n^i \cdot \left(h \sum_{j=1}^{n-1} \nabla q_j(x) \right) - A_{2n} \left(\nabla V_n^i \right)^2, k \geq 1,$$

$$q_{nk}^i(x) = \bar{\psi}_n(x), x \in \partial\Omega.$$

- Hence, we obtain the function

$$q_n^i = \lim_{k \rightarrow \infty} q_{nk}^i, \text{ in } C^{2+\alpha}(\overline{\Omega}).$$

CONVERGENCE THEOREM

For any function $c(x) \in C^\alpha(\overline{\Omega})$, $c(x) \geq d = \text{const} > 0$ consider the solution $w_c(x, \bar{s}) \in C^{2+\alpha}(\overline{\Omega})$ of the boundary value problem

$$\Delta w_c - \bar{s}^2 c(x) w_c = 0, x \in \Omega,$$

$$w_c |_{\partial\Omega} = \varphi(x, \bar{s}).$$

Denote the corresponding tail function as $V_c = \ln w_c(x, \bar{s}) / \bar{s}^2$. Suppose that the cut-off pseudo frequency \bar{s} is so large that for any such function $c(x)$ satisfying the inequality $\|c - c^*\|_{C^\alpha(\overline{\Omega})} \leq M^*$ the following estimate holds

$$\|\nabla V_c - \nabla V^*\|_{C^\alpha(\overline{\Omega})} \leq \xi,$$

where $\xi \in (0, 1)$ is sufficiently small number.

Let $V_1^0(x, \bar{s}) \in C^{2+\alpha}$ be the initial tail function, and let

$$\left\| \nabla V_1^0 - \nabla V^* \right\|_{C^\alpha(\bar{\Omega})} \leq \xi.$$

Let $\bar{N} < N$ be the total number of functions q_n calculated by the above algorithm. Suppose that the number $\bar{N} = \bar{N}(h)$ is connected with the step size h via $\bar{N}h = \beta$, where the constant $\beta > 0$ is independent on h . Let β be so small that

$$\beta \leq \min \left(\frac{1}{2M^*}, \frac{3}{56KM^*} \right).$$

Then there exists a sufficiently small number

$\eta_0 = \eta_0(K(M^*, \Omega), M^*) \in (0, 1)$ *and a sufficiently large number*
 $\mu = \mu(K(M^*, \Omega), M^*, \eta) > 1$ *such that for all* $\eta \in (0, \eta_0)$, *for every integer* $n \in [1, \bar{N}]$ *and for every integer* $i \in [1, m_n]$ *the sequence*
 $\{q_{nk}^i\}_{k=1}^{\infty}$ *converges in the* $C^{2+\alpha}(\bar{\Omega})$ *norm. In addition, the following estimates hold*

$$\|q_n - q_n^*\|_{C^{2+\alpha}(\bar{\Omega})} \leq 2K \cdot M^* \left(\frac{1}{\sqrt{\mu}} + 3\eta \right),$$

$$\|q_n\|_{C^{2+\alpha}(\bar{\Omega})} \leq 2M^*,$$

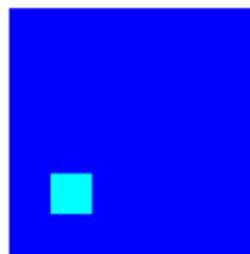
$$\|c_{\bar{N}} - c^*\|_{C^\alpha(\bar{\Omega})} \leq 18K(M^*)^2 \left(1 + \underline{s}^2 \right) \left(\frac{1}{\sqrt{\mu}} + 3\eta \right).$$

REMARK. Our starting tail function is $V = 0$.

Numerical examples

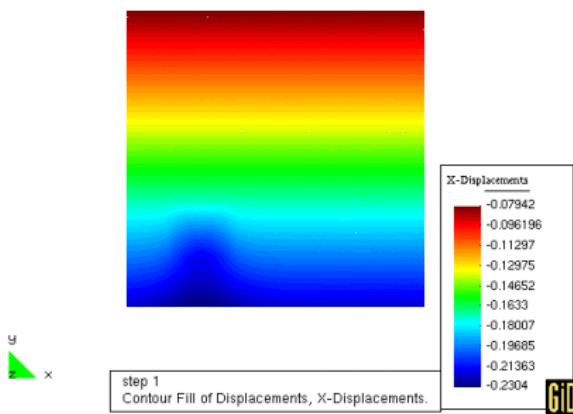
- **Example 1**

Our goal is to reconstruct one square using globally convergent method. Exact $c(x) = 3.4$

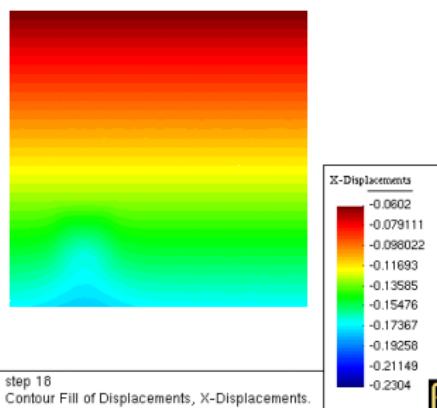


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Tail function for the exact solution

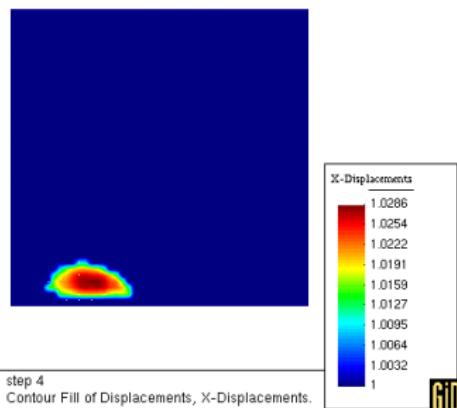
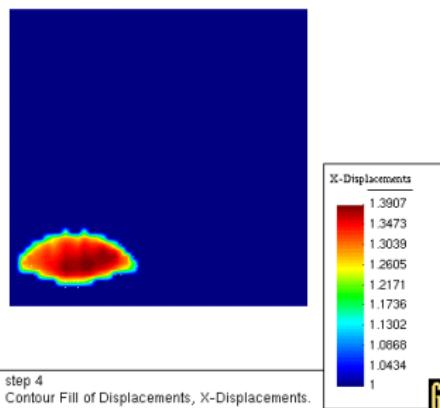


a) $s = 6.55$

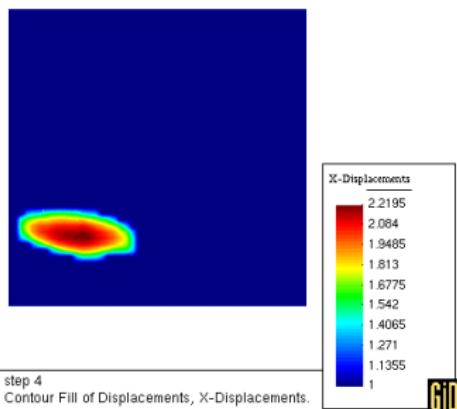
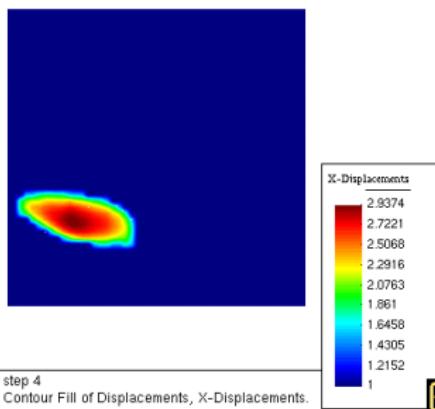


b) $s = 7.45$

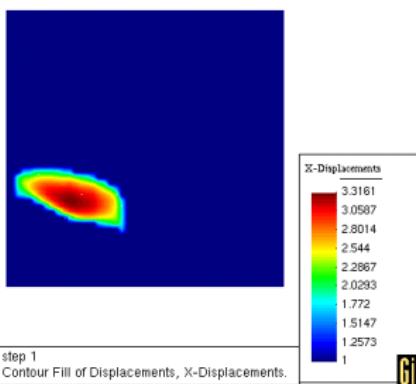
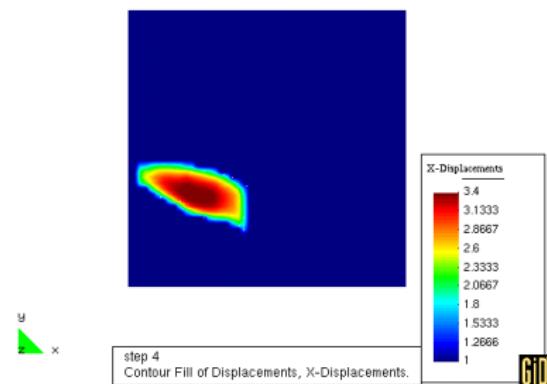
Reconstruction of the coefficient $c(x)$ with noise 5% with computed $q_{2,4}$ and $q_{4,4}$

a) $q_{2,4}$ b) $q_{4,4}$

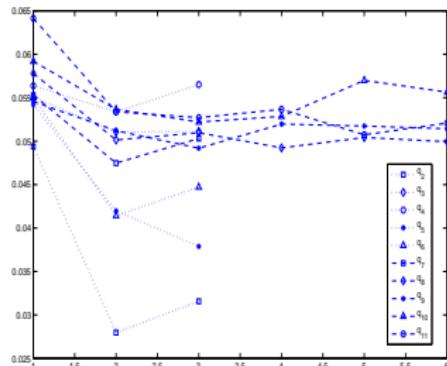
Reconstruction of the coefficient $c(x)$ with noise 5% with computed $q_{7,4}$ and $q_{9,4}$

a) $q_{7,4}$ b) $q_{9,4}$

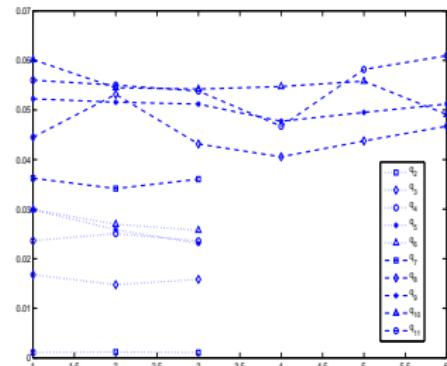
Reconstruction of the coefficient $c(x)$ with noise 5% with computed $q_{11,1}$ and $q_{11,4}$.

a) $q_{11,1}$ b) $q_{11,4}$

Stopping criterion in reconstruction of a single square



a) $\sigma = 5\%$



b) $\sigma = 5\%$

Figure: Computed relative L_2 -norms: a) $\frac{||\nabla V_{n,k} - \nabla V_{n,k-1}||}{||\nabla V_{n,k}||}$; b) $\frac{||c_{n,k} - c_{n,k-1}||}{||c_{n,k}||}$

Stopping criterion

We compute relative L_2 -norms of gradients of tails

$$\frac{\|\nabla V_{n,k} - \nabla V_{n,k-1}\|}{\|\nabla V_{n,k}\|}$$
(1)

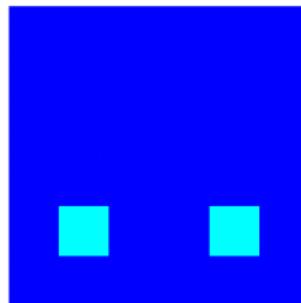
and relative L_2 -norms of the target coefficient

$$\frac{\|c_{n,k} - c_{n,k-1}\|}{\|c_{n,k}\|}.$$
(2)

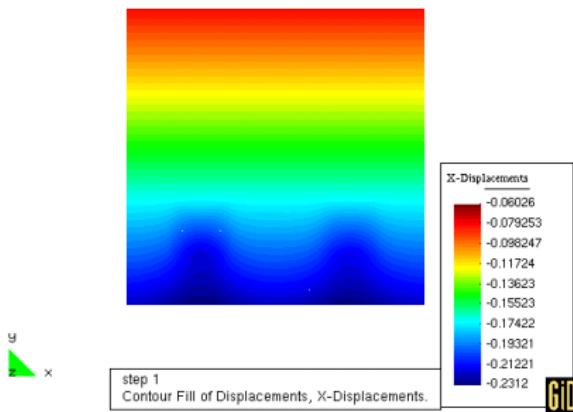
We use these norms as the stopping rule for computation in our iterative algorithm. We stop our iterative algorithm for computing of the new function q_n when both relative norms (1) and (2) are stabilized.

- **Example 2**

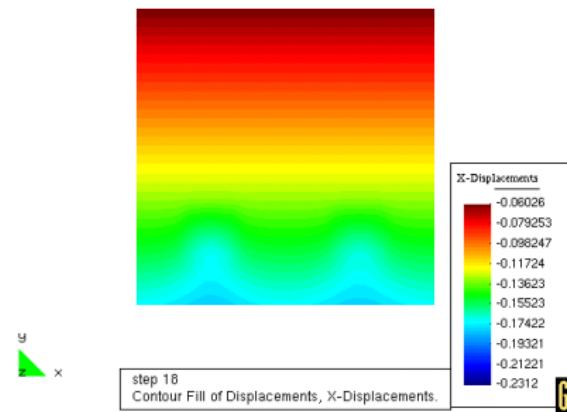
Our goal is to reconstruct 2 squares using globally convergent method. Exact $c(x)=3.4$ in both squares.



Tail function for the exact solution

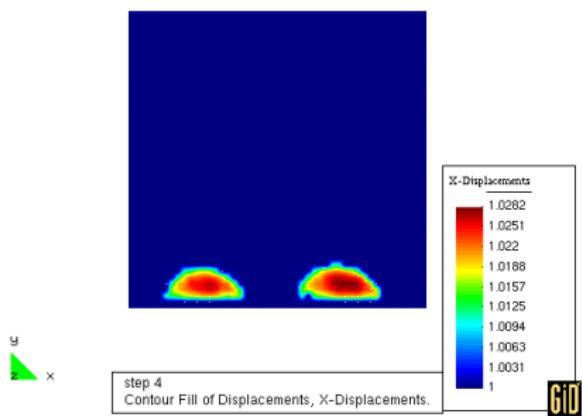


a) $s = 6.55$

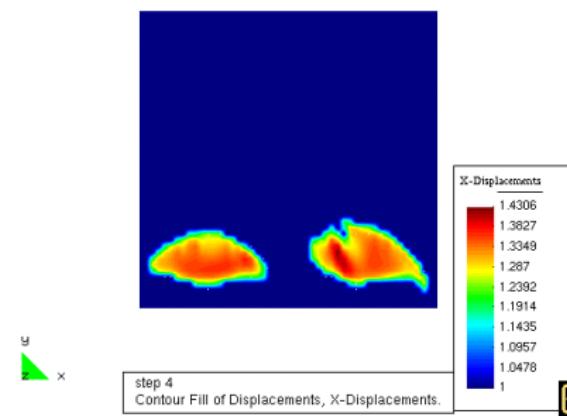


b) $s = 7.45$

Reconstruction of the coefficient $c(x)$ with noise 5% with computed $q_{2,4}$ and $q_{4,4}$

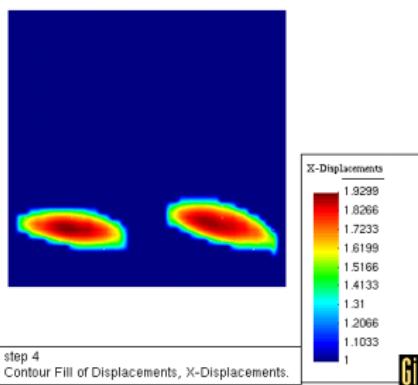
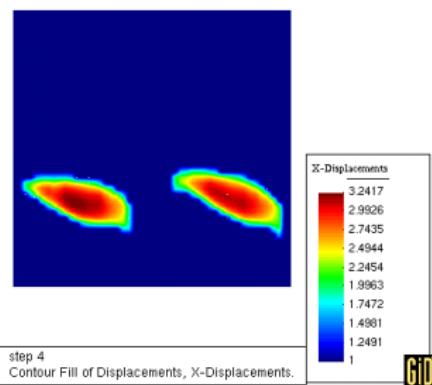


a) $q_{2,4}$

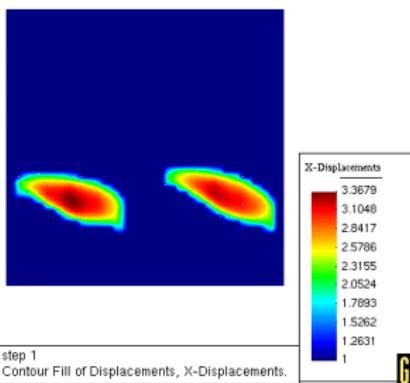
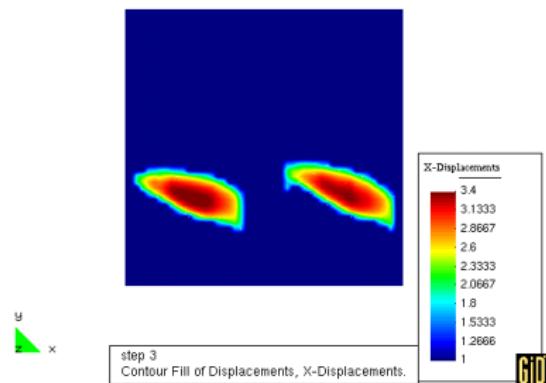


b) $q_{4,4}$

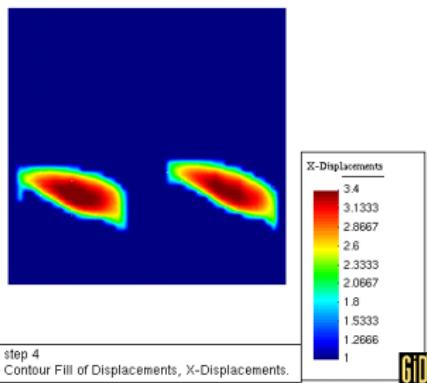
Reconstruction of the coefficient $c(x)$ with noise 5% with computed $q_{6,4}$ and $q_{10,4}$

a) $q_{6,4}$ b) $q_{10,4}$

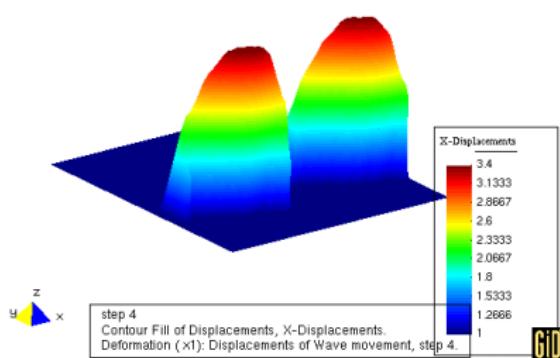
Reconstruction of the coefficient $c(x)$ with noise 5% with computed $q_{11,1}$ and $q_{11,3}$

a) $q_{11,1}$ b) $q_{11,3}$

Reconstruction of the coefficient $c(x)$ with noise 5% with computed $q_{11,4}$.

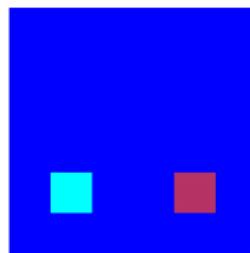


a) $q_{11,4}$

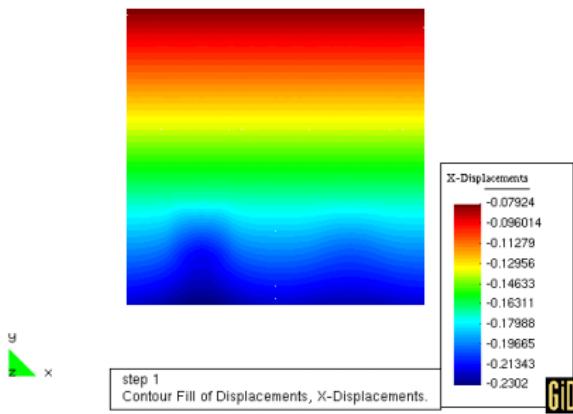


b) $q_{11,4}$

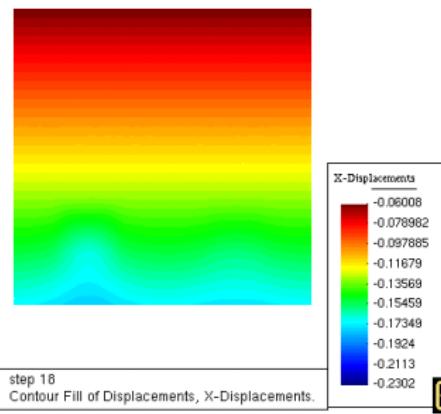
Example 3: Reconstruction of the coefficient $c(x)$ with noise 5%, when exact $c(x) = 3.4$ in the left square, and $c(x) = 2.0$ in the right square



Tail function for the exact solution

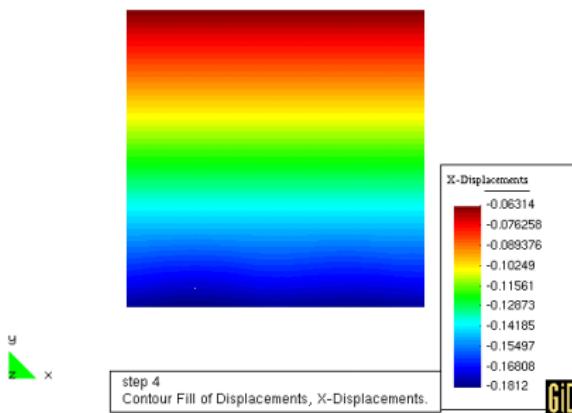


a) $s = 6.55$

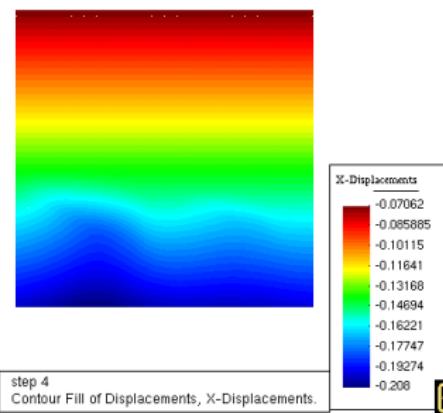


b) $s = 7.45$

Computed tail function

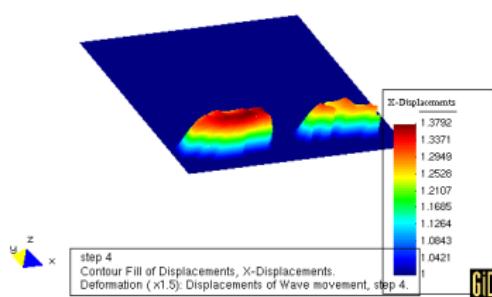


a) tail in $q_{4,4}$

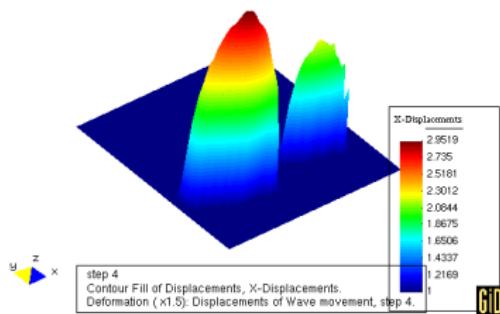


b) tail in $q_{11,4}$

Reconstruction of the coefficient $c(x)$ with noise 5% with computed $q_{4,4}$ and $q_{8,4}$

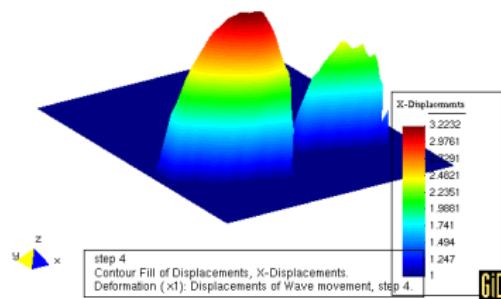
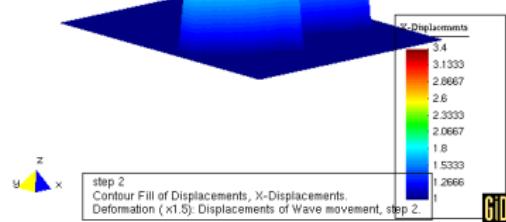


a) $q_{4,4}$

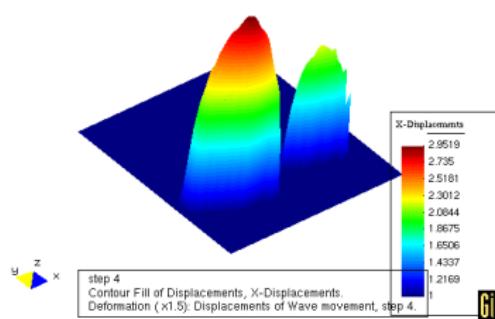
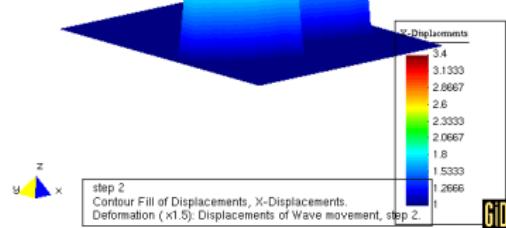


b) $q_{8,4}$

Reconstruction of the coefficient $c(x)$ with noise 5% with computed $q_{10,4}$ and $q_{11,3}$

a) $q_{10,4}$ b) $q_{11,3}$

Reconstruction of the coefficient $c(x)$ with noise 5% with computed $q_{8,4}$ and $q_{11,3}$

a) $q_{8,4}$ b) $q_{11,3}$

Locally convergent reconstruction by quasi-Newton method

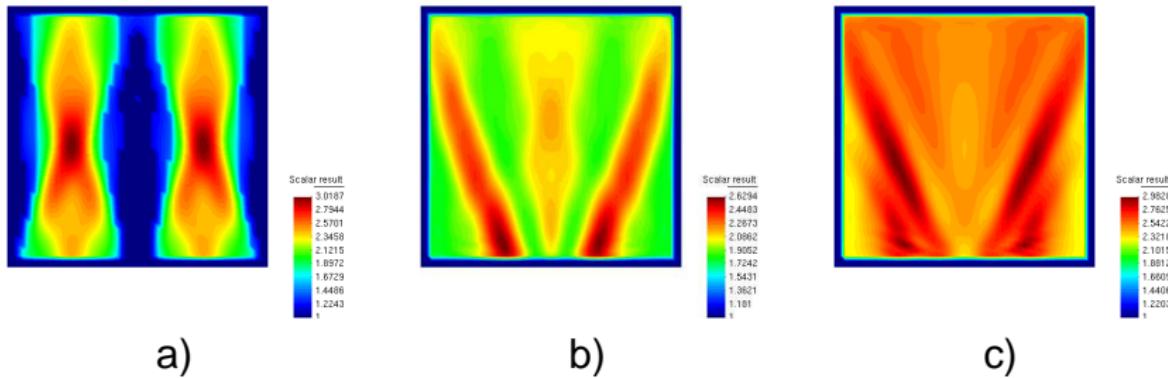


Figure: Spatial distribution of c_h in Example 2 (reconstruction of the two small squares) with initial guess: a) $c=1$, b) $c=1.5$, c) $c=2$.

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