Lecture 1. Introduction to well- and ill-posed problems.

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1 Mathematical background

Definition 1.2.1. Let $\Omega \subseteq R^n$ be a domain, and $u(x), x = (x_1, \ldots, x_n) \in R^n$ is a m times continuously differentiable function defined on Ω. Denote

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}}$$

the partial derivative of order $|\alpha| = \alpha_1 + \ldots + \alpha_n$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, and α_i are non-negative integer numbers.

Definition 1.2.2. Denote $C^m\left(\bar{\Omega}\right)$ a set of functions u defined on the closure $\bar{\Omega}$ of the open set Ω, such that the function u is m times continuously differentiable, and its norm is given by

$$\|u\|_{C^m\left(\bar{\Omega}\right)} =_{|\alpha| \le m^{x \in \bar{\Omega}}} \sup |D^\alpha u(x)| < \infty.$$

Definition 1.2.3. A set *supp* $u = \overline{\{x : u(x) \neq 0\} \cap \Omega}$, where $u(x)$ is a continuous function defined on an open set Ω, is called a support of this function.

Definition 1.2.4. Denote $C_0^m(\Omega)$ a set of all m times continuously differentiable functions whose support is compact in $\Omega \subset R^n$. By analogy, denote $C_0^\infty(\Omega)$ a set of infinitely times differentiable functions.

Definition 1.2.5. Denote $W_{k,p}(\Omega), (1 \le p < \infty, k \ge 0)$ a set of all functions belonging to the space $L_p(\Omega)$ together with their generalized derivatives $D^\alpha u$ of order $|\alpha| \le k$. This set is called the Sobolev space. In this space, the norm is given by

$$\|u\|_{W^{k,p}(\Omega)} = (\left(\int_\Omega \left(_{|\alpha| \le k} |D^\alpha u|^2\right)^{p/2}\right) dx)^{1/p}$$

Definition 1.2.6. In the specific case $p = 2$, we denote a Hilbert space $H^k(\Omega) = W^{k,2}(\Omega)$ with the inner product

$$(u, v)_{H^k(\Omega)} =_{|\alpha| \le k} (D^\alpha u, D^\alpha v) =_{|\alpha| \le k} \int_\Omega D^\alpha u D^\alpha v dx.$$

Definition 1.2.7. Denote $C^{\alpha}\left(\bar{\Omega}\right), 0<\alpha<1$ a set of all continuous functions u defined on $\bar{\Omega}$, such that

$$\|u\|_{C^{\alpha}} = \|u\|_{C^{0}} + \sup_{x,y\in\Omega,\ x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} < \infty,$$

where $\|u\|_{C^{0}} = \sup_{x\in\Omega}|u(x)|$.

Definition 1.2.8. Denote $C^{k+\alpha}\left(\bar{\Omega}\right)$, where k is a positive integer and $\alpha\in(0,1)$, a Hölder space with the norm

$$\|u\|_{C^{k+\alpha}} = \|u\|_{C^{k}} + \max_{|\alpha|=k}|D^{\alpha}u|_{C^{0,\alpha}} < \infty.$$

If $\alpha=0$, these are Banach spaces. Note that if the domain Ω is unbounded, then we have to assume the boundedness of the functions and their derivatives of order $k+\alpha$. In this case, the set $C^{k+\alpha}\left(\bar{\Omega}\right)$ is not a Banach space.

Definition 1.2.9. A pair (X,τ), where τ is a topology in X, i.e., it is a family of subsets of the set X, such that: (1) an empty set and the whole set X belong to τ, (2) the intersection of any finite number of set on τ belongs to τ, is said to be a topological space. For convenience, the topological space is denoted as X.

Definition 1.2.10. A topological space X is called a Hausdorff space if for any elements $x\in X, y\in X$, there exist their non-intersecting vicinities.

2 Ill-Posedness

Let U, F be the topological spaces, and A be an operator acting from U in F. Consider an operator equation

$$A(u) = f,\ u \in U,\ f \in F. \tag{1.1}$$

Definition 1.2.1. (Hadamard, 1932) The problem (1.1) is said to be well-posed in the sense of Hadamard if the following conditions are fulfilled.

1. For any $f \in F$ there exists an element $u \in U$, such that $A(u) = f$, i.e., the range $R(A)$ of the operator A coincides with the whole space F.

2. A solution u of the equation (1.1) is uniquely determined by the element f. In other words, there exists the inverse A^{-1} of the operator A.

3. The solution u depends continuously on the element F. In other words, the operator A^{-1} is continuous.

If at least one of these conditions is not fulfilled, the problem is said to be ill-posed in the sense of Hadamard.

Clearly, the problem (1.1) is well-posed in the sense of Hadamard if and only if there exists the continuous inverse A^{-1} of the operator A defined on the whole space F.

A typical example of an ill-posed problem is given by the operator equation (1.1) whose operator is linear and compact. In this case, the inverse A^{-1} cannot be defined on the whole space F. Furthermore, it is not continuous even on the set AU. In general, the inverse A^{-1} of the operator A generated by an applied problem cannot be defined on the whole space F. In other words, the third Hadamard's condition is extremely strong.

3 Examples of ill-posed problems

3.1 Example 1. Hadamard example - Caushy problem for the Laplace equation

Hadamard presented an exmaple o an ill-posed problem for PDE which in his opinion did not correspond to any real physical formulation.

The solution is unique.

But it have not continuous dependence on its data.

Let

$$u = u(x,y), \triangle u = 0, y > 0, u(x,0) = 0, \frac{\partial}{\partial y}u(x,0) = \alpha \sin(nx), x \in [0,\pi], \tag{1}$$

The Caushy problem (1) has the solution

$$u(x,y)=\frac{\alpha}{n}\sinh(ny)\sin(nx). \tag{2}$$

For any pair of functional spaces C^k, L_p, H_p^l, W_p^l and any $\epsilon>0, c>0, y>0$ it is possible to choose α and n such that

$$||\alpha sin(nx)||<\epsilon, \tag{3}$$

but

$$||\frac{\alpha}{n}\sinh(ny)\sin(nx)||>c \tag{4}$$

since

$$\lim_{n\to 0}||\frac{\alpha}{n}\sinh(ny)\sin(nx)||=\infty. \tag{5}$$

This problem does not depends continuously on its data and hence is not well-posed.

3.2 Example 2. The differentiation problem.

Suppose the function $f(x)$ is given with a noise. In other words, the given function is

$$f_\delta\left(x\right)=f\left(x\right)+\delta f(x), x\in[0,1]$$

$$\|\delta f\|_{C[0,1]}\leq\delta,$$

where δ is small and this number is the level of noise. Then the problem of calculating $f_\delta'\left(x\right)$ is ill-posed. Indeed, let for example

$$\delta f(x)=\frac{\sin nx}{n}.$$

Then

$$\|\delta f\|_{C[0,1]}\leq\frac{1}{n}$$

and is small for large n. However, $f_\delta'\left(x\right)-f'\left(x\right)=\cos nx$ and is certainly not small.

The differentiation problem can be reduced to an equivalent problem of solving the integral equation of the first kind, assuming that $f\left(0\right)=0$, i.e. $f\left(0\right)$ is known,

$$f_\delta\left(x\right)=\int\limits_0^x f_\delta'\left(y\right)dy. \tag{1}$$

Hence the problem is to find the solution $f_\delta'\left(y\right)$ of equation (1), assuming that $f_\delta\left(x\right)$ is known.

By the way, there exists a simple method of regularization of the differentiation problem. Indeed,

$$f_\delta'\left(x\right)\approx\frac{f\left(x+h\right)-f\left(x\right)}{h}+\frac{\delta f(x+h)-\delta f(x)}{h}.$$

We want to make small the following

$$\left| f'_\delta (x) - \frac{f(x+h) - f(x)}{h} \right| \leq \frac{2\delta}{h}.$$

Hence, we should take $h = h(\delta)$ such that

$$\lim_{\delta \to 0} \frac{2\delta}{h(\delta)} = 0.$$

We can take then $h(\delta) = \delta^\mu$, where $\mu \in (0,1)$. Hence, $h(\delta)$ is the regularization parameter here. Basically it says that the mesh step size cannot be too small.

3.3 Example 3. Solution of the integral equation of the first kind.

First, some well known facts. Let $G, \Omega \subset \mathbb{R}^n$ be two bounded domains. Let $K(x,y) \in C\left(\overline{G} \times \overline{\Omega}\right), x \in G, y \in \Omega$ be a function. Consider the integral operator $K : C\left(\overline{\Omega}\right) \to C\left(\overline{G}\right)$ defined as

$$(Kf)(x) = \int_\Omega K(x,y) f(y)\, dy, x \in G.$$

Then it is known from the Functional Analysis (somewhere where Fredholm's theory is studied) that K is a compact operator. Consider two equations

$$f(x) + \int_\Omega K(x,y) f(y)\, dy = g(x), \text{ in the case } \Omega = G, f = ? \tag{2}$$

(2) is called integral equation of the second kind. This equation is a standard thing in the theory of elliptic and parabolic PDEs. The Fredholm theory works for (2). It basically says that if solution of (2) is unique, then the existence theorem holds. Furthermore, solution of (2) is stable, i.e.

$$\|f\|_{C(\overline{\Omega})} \leq C \|g\|_{C(\overline{\Omega})},$$

again if uniqueness theorem can be proven for (2).

Now we consider another type of integral equations, in which it is **not** necessary that $\Omega = G$.

$$\int_\Omega K(x,y) f(y)\, dy = g(x), x \in G, f = ? \tag{3}$$

This is the so-called integral equation of the first kind. Tikhonov has started his theory on the basis of (3).

I now show that the problem (3) is ill-posed. Let $\Omega = (0,1), G = (a,b)$. Consider instead of f the function

$$f_n(x) = f(x) + \sin nx. \tag{4}$$

Then

$$\int_{\Omega} K(x,y)\, f_n(y)\, dy = g_n(x),$$

where $g_n(x) = g(x) + p_n(x)$, where

$$p_n(x) = \int_{\Omega} K(x,y) \sin nydy.$$

By the Lebesque lemma $\lim_{n\to\infty} \|p_n\|_{C[a,b]} = 0$. However, it is clear from (4) that $\|f_n(x) - f(x)\|_{C[0,1]} = \|\sin nx\|_{C[0,1]}$ is not small for large n.

3.4 Example 4.

Coefficient Inverse Problems are ill-posed. The proof of this statement follows from regularity estimates for solutions of PDEs and is therefore space consuming.

Forward ,or operator A is compact:

A(x)=y , where A is compact operator (linear or nonlinear).

It is known that:

a. Domain of values of compact operator does not have inner points: And thus it is impossible to prove existence theorem.

b. Because of a) compact operator does't have continuous inverse operator.

Indeed, the following result shows that even for a linear operator, the solvability of the equation (1.1) in the whole space and the continuity of the inverse A^{-1} are closely connected.

Theorem 1.2.1. (Ivanov, 1978) *Let A be a linear injective continuous operator with the domain $D(A) = U$ and with the range $R(A) \subseteq F$, where U, F are Banach spaces. Then the inverse A^{-1} of the operator A is bounded if and only if $\overline{R(A)} = R(A)$.*

Thus if the inverse A^{-1} is unbounded, then the range $R(A)$ of the operator A is not closed in F, i.e., the operator equation is not solvable in the whole space $\dot{F}$.

Numerous problems of linear algebra, applicable analysis, calculus of variations, control theory, signal processing, imaging, etc. are proven to be ill-posed in the sense of Hadamard. As such, they should be excluded from consideration within the framework of classical mathematics. Meanwhile, they are of particular interest to scientists and practitioners. To eliminate such a discrepancy, the concept of conditional correctness was introduced by Tikhonov.

4 Conditional correctness

Definition 1.2.2. (Tikhonov, 1943) The problem (1.1) is said to be well-posed in the sense of Tikhonov (or conditionally well-posed) if the following conditions are fulfilled.

1. It is known a *priori* that there exists a solution of this problem and it belongs to a certain set $M \subset U$, i.e., $f \in N = AM$.

2. This solution is unique on the set M, i.e., the inverse A^{-1} is defined on this set.

3. The inverse A^{-1} of the operator A is continuous on N. The set M is called a correctness set.

The concepts of correctness by Hadamard and Tikhonov are fundamentally different. Unlike Hadamard's definition, the solvability of the operator equation (1.1) in the whole space U is not required by Tikhonov's definition. Furthermore, the continuity condition for A^{-1} is required only on the correctness set M. Therefore, the problems satisfying all conditions indicated in Definition 1.2.2 are also called the conditionally well-posed problems.

According to Tikhonov, any compact set of U can be taken as a correctness set M. In connection with this, Tikhonov pointed to the following general topological theorem that establishes the stability result.

Theorem 1.2.2. (Dunford, 1962) *Let U be a topological space, F be a topological Hausdorff space, and $M \subset U$ be a compact set. If a continuous one-to-one operator A maps the set M onto a set $N \subset F$, then the inverse A^{-1} is continuous on N in a relative topology.*

Due to this theorem, the third condition in Definition 1.2.2 follows from the second one. This fact is widely exploited in the theory of coefficient inverse problems (see, e.g., (Lavrentiev, 1986) and (Isakov, 1998) for establishing the stability results followed the uniqueness theorem. Furthermore, the continuity condition in Theorem 1.2.1 can be weakened.

Definition 1.2.3. An operator A is closed if its graph $\{(u, Au) : u \in D(A)\}$ is closed in the topological product $U \times F$.

Theorem 1.2.3. (Ivanov, 1962) *Let U, F be topological Hausdorff spaces satisfying the first axiom of countability, and A be a close one-to-one operator acting from U to F. let $D(A)$ be the domain of the operator A, $M = D(A) \cap K, N = AM \subset F$. Then the set N is closed on F, and the inverse A^{-1} is continuous on N in a relative topology.*

In practice, the right-hand side f of the equation (1.1) is usually determined from measurements. Specifically, instead of the element $f \in N$, a certain approximation $\bar{f}$ of f is given. If F is a metric space, then the deviation estimate is given by $\rho_F\left(f, \bar{f}\right) \leq \delta, \delta > 0$. Since there are no effective criteria of belonging the element $\bar{f}$ to the set $N = AM$, the element $A^{-1}\bar{f}$ may not exist. Because of this, both theorems 1.2.1 and 1.2.2 can not, in general, be used for constructing the efficient algorithms for the numerical solution of the problem 1.1.

The theory of regularization (see, e.g., (Tikhonov, 1977), (Ivanov, 1978), (Bakushinsky, 1994)) allows for constructing such algorithms. The core of this theory is the concept of regularizability in the sense of Tikhonov. Consider a certain operator $\mathcal{F}$ (e.g., $\mathcal{F} = A^{-1}$) acting from a metric space F into a metric space U. Assume that this operator is defined on a subset $D \subseteq F$, and it is not continuous. We wish to compute the approximate values of $\mathcal{F}$ given the pair $\left(\bar{f}, \delta\right)$.

Definition 1.2.4. An operator $\mathcal{F}$ is called regularizable on D if there exists a parametric operator R_δ defined on $F \times \{0 < \delta \leq \delta_0\}$, such that

$$\lim_{\delta \to 0} \sup \left\{\rho_U \left(R_\delta \left(\bar{f}\right), \mathcal{F}(f) : \rho_F \left(\bar{f}, f\right) \leq \delta\right)\right\} = 0, \quad \forall f \in D.$$

In this case, the operator R_δ is called the regularizing algorithm for computing the values of $\mathcal{F}$, and the element $R_\delta \left(\bar{f}\right)$ is called the approximate solution of this problem.

There exist both the regularizable and non-regularizable operators (see (Bakushibsky, 1994) for details). As an example, consider a linear one-to-one operator A acting from V_a^b into $\bar{L}_2(a, b)$, where V_a^b is a space of all functions with bounded variations on the interval (a, b). Clearly, the operator A^{-1} cannot be regularizable on $AV_a^b \subseteq L_2$, because the space V_a^b is not separable, whereas the space L_2 is separable. In the theory of regularization, there is a general criterion of regularizability in linear normed spaces.

Theorem 1.2.4. (Vinokurov, 1971) *An operator $\mathcal{F}$ acting from a linear normed space F into a separable linear normed space U is regularizable on $D \subseteq F$ if and only if it is a pointwise limit (on D) of a sequence of continuous operators $\mathcal{F}_\alpha$ defined on F.*

In particular, all operators are regularizable if they are the pointwise limits of continuous operators. This criterion establishes the close connection between the problem of constructing the regularizing algorithms and the problem of constructing a family of approximating continuous operators for $\mathcal{F}$.

Definition 1.2.5. A family $\{\mathcal{F}_\alpha\}$ of continuous operators $\mathcal{F}_\alpha$ on F, where α runs over an ordered number set, such that

$$\lim_{\alpha \to \infty} \rho_U \left(\mathcal{F}_\alpha(f), \mathcal{F}(f)\right) = 0, \forall f \in D,$$

is called the family of approximating operators.

Once such a family is constructed, based on Theorem 1.2.3, one can make a transition from $\mathcal{F}_\alpha$ to R_δ specifying the function $\alpha(\delta)$, so that the family $\mathcal{F}_{\alpha(\delta)}\left(\bar{f}\right) = R_\delta \left(\bar{f}\right)$ satisfies the definition 1.2.4.

Theorem 1.2.5. (Baksushinsky, 1994) *Suppose the operator F is approximated on D by operators F_α satisfying the Lipschitz condition, i.e., for all $f_1, f_2 \in F$*

$$\rho_U \left(\mathcal{F}_\alpha(f_1), \mathcal{F}_\alpha(f_2)\right) \leq C_\alpha \rho_F (f_1, f_2).$$

Then one can choose the function $\alpha = \alpha(\delta)$, so that the operator $F_{\alpha(\delta)}$ generates the regularizing algorithm for F on D.

Thus, within the framework of Tikhonov's regularization, knowledge of the pair $\left(\delta, \bar{f}\right)$ and choosing the function $\alpha(\delta)$ are necessary for constructing any regularizing algorithm.

There is another way for constructing the regularizing algorithms. This way utilizes the concept of quasisolution. Now assume that the operator $\mathcal{F} = A^{-1}$, where A is the continuous one-to-one operator in the equation (1.1). For brevity, assume that U and F are Banach spaces, and M is a compact set in U.

Definition 1.2.6. (Ivanov, 1962) Any element $\bar{u} \in M$ of the set

$$Arg \min \{\|A(u) - f\| : u \in M\}$$

is called a quasisolution of the equation 1.1.

By virtue of continuity of the operator A, the functional $\|A(u) - f\|$ is continuous. Hence, the quasisolution exists for any $f \in F$. It coincides with the classical solution if $f \in AM$. Consider the operator $R_\delta\left(\bar{f}\right) = \bar{u}$, where $\bar{u} \in M$ is a quasisolution, i.e., a minimizer of the functional $\left\{\left\|A(u) - \bar{f}\right\| : u \in M\right\}$.

Theorem 1.2.6. *The operator* $R_\delta\left(\bar{f}\right)$ *generates the regularizing algorithm for* A^{-1} *on the set* AM.

Clearly, if the compacts set M can be established a priori, the method of quasisolutions is advantageous, because it does not require knowledge of δ, i.e., the operator R_δ does not depend explicitly on this parameter. Furthermore, one can obtain the error estimate of quasisolutions. Indeed, since $\|A\bar{u} - Au_0\| \leq \left\|A\bar{u} - \bar{f}\right\| + \left\|\bar{f} - f_0\right\| \leq 2\delta$, where $Au_0 = f_0$, then $\sup\{\|\bar{u} - u_0\| : z_0 \in M\} \leq \omega\left(2\delta, A^{-1}, N\right)$, where $N = AM$.

5 Examples of inverse problems

Let us consider some examples of inverse problems.

Example 1.

Let $q(x)$ be continuous function for all $x \in R$ and $u(x,t)$ is the Cauchy problem solution

$$\begin{gathered} u_x - u_y + q(x)u = 0, \quad (x,y) \in R^2, \\ u(x,0) = \varphi(x), \quad x \in R. \end{gathered} \tag{6}$$

The problem (6) is well-posed for known functions q, φ.

It is necessary to demand the continuous differentiability of $\varphi(x)$ for existing classical solution (i.e. solution and its partial derivatives are continuous in R^2). Suppose that $\varphi(x)$ is continuously differentiable. Let us consider the problem of finding function $q(x)$ by knowing information about solution of (6):

$$u(0,y) = \varphi(y), \quad y \in R. \tag{7}$$

Indeed the solution of (6) is known:

$$u(x,y) = \varphi(x+y)exp\left(\int_{x+y}^{x} q(\xi)d\xi\right), \quad (x,y) \in R^2. \tag{8}$$

It follows from (7) that

$$\psi(y) = \varphi(y)exp\left(\int_{y}^{0} q(\xi)d\xi\right), y \in R. \tag{9}$$

Therefore the solution to the inverse problem exists if and only if

1. $\psi(y)$ is continuously differentiable for $y \in R$.
2. If

$$\frac{\psi(y)}{\varphi(y)} > 0, \quad y \in R; \quad \psi(0) = \varphi(0). \tag{10}$$

Thus the solution of inverse problem is given by formula

$$q(x) = -\frac{\partial}{\partial x}\left(ln\frac{\psi(y)}{\varphi(y)}\right), \quad x \in R. \tag{11}$$

Example 2.

It is necessary to find initial state of bounded heated bar, if the boundary value problem solution

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < \pi, t > 0, \\ u(0,t) &= u(,t) = 0, \quad t > 0, \\ u(x,0) &= \varphi(x), \quad 0 \leq x \leq \pi, \end{aligned} \tag{12}$$

is known in fixed moment $t = T$:

$$u(x,T) = \psi(x), 0 \leq x \leq \pi. \tag{13}$$

Using Fourier method the solution of (12) has the form

$$u(x,t) = \sum_{n=1}^{\infty} e^{-n^2 t}\varphi_n \sin nx. \tag{14}$$

Here φ_n are Fourier coefficients of function $\varphi(x)$:

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n \sin nx. \tag{15}$$

Let $t = T$ in (14) we obtain

$$\psi(x) = \sum_{n=1}^{\infty} e^{-n^2 T}\varphi_n \sin nx, x \in [0,\pi]. \tag{16}$$

Therefore

$$\psi(x) = \psi_n e^{-n^2 T}, n = 1, 2, ..., \tag{17}$$

where ψ_n are Fourier coefficients of function $\psi(x)$. The coefficients $\varphi_n, n = 1, 2, ...$ uniquely define function $\varphi(x) \in L_2$. Notice, that in this case the boundary condition (14) holds in the following sense

$$\lim_{t \to +0} \int_0^{\pi} [u(x,t) - \varphi(x)]^2 dx = 0. \tag{18}$$

From (18) follows that the solution of inverse problem (12) exists if and only if, when

$$\sum_{n=1}^{\infty} \psi_n^2 e^{2n^2 T} < \infty \tag{19}$$

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