# Lecture 1. Tichonov scheme for solution ill-posed problems. 

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## 1 Tikhonov theorem (1943)

Let metric space $F$ maps to metric space $U, F \rightarrow U$, and $U_{0} \in U$ such that $F_{0} \in F$. If mapping $F \rightarrow U$ is continuous, one-to-one and $F_{0}$ is compact in $F$, then inverse mapping $U_{0} \rightarrow F_{0}$ is also continuous in the metric of space $F$.

Proof
Let $z \in F, u \in U$ and function $u=\varphi(z)$, function $z=\psi(u)$. Let us take any element $u_{0} \in U_{0}$. We want to show that function $\psi(u)$ is continuous in $u_{0}$. Assume, that this is not true. Then there exists such number $\epsilon_{1}>0$ that for $\forall \delta>0$ will be found such element $\tilde{u} \in U_{0}$ such that $\left\|\tilde{u}-u_{0}\right\|_{U}<\delta$, but $\left\|\tilde{z}-z_{0}\right\|>\epsilon_{1}$.

Let take sequence $\delta_{n}: \lim _{n \rightarrow \infty} \delta_{n}=0$. For any $\delta_{n}$ there $\exists \tilde{u_{n}} \in U_{0}:\left\|\tilde{u_{n}}-u_{0}\right\|_{U}<\delta_{n}$, but $\left\|\tilde{z_{n}}-z_{0}\right\|>\epsilon_{1}$. Clearly, that sequence $\lim _{n \rightarrow \infty} \tilde{u_{n}}=u_{0}$. Because $\tilde{z} \in F_{0}$ and $F_{0}$ is compact, then from $\tilde{z}_{n}$ is possible to take subsequence $\tilde{z}_{n_{k}}$ such that $\left\|\tilde{z}_{n_{k}}\right\|_{F} \rightarrow \tilde{z} \in F$. Note, that $\tilde{z}_{0} \neq z_{0}$ because for any $n_{k} \tilde{z}_{n_{k}}-z_{0} \geq \epsilon_{1}$ and thus, $\left\|\tilde{z}_{0}-z_{0}\right\|_{F} \geq \epsilon_{1}$. Next, to this $\tilde{z}_{n_{k}}$ will correspond sequence of elements $\tilde{u}_{n_{k}}=\varphi\left(\tilde{z}_{n_{k}}\right) \in U_{0}$ such that this sequence $\tilde{u}_{n_{k}} \rightarrow \tilde{u}_{0}=\varphi\left(\tilde{z}_{0}\right)$. This sequence also is subsecuence of sequence $\tilde{u}$. But since sequence $\tilde{u} \rightarrow u_{0}$ and $u_{0}=\varphi\left(z_{0}\right)$, then

$$
\begin{equation*}
\tilde{u_{o}}=\varphi\left(\tilde{z}_{0}\right)=u_{0}=\varphi\left(z_{0}\right) \tag{1}
\end{equation*}
$$

and thus $\varphi\left(\tilde{z}_{0}\right)=\varphi\left(z_{0}\right)$. Since mapping $F \rightarrow U$ is oneto_one, then $\tilde{z}_{0}=z_{0}$.

## 2 The concept of overdetermination

Definition 1.3.1. A coefficient inverse problem is said to be $n$-dimensional if an unknown coefficient depends on $n$ independent variables. In particular, the problem is said to be multidimensional if $n \geq 2$.

Definition 1.3.2. a coefficient inverse problem is said to be non-overdetermined if the number $m$ of independent variables in the boundary data equals the number $n$ of independent variables in the unknown coefficient, i.e., $m=n$. Conversely, such a problem is said to be overdetermined if $m>n$.

There is a conjecture that the uniqueness result may not be established if $m<n$. Although this conjecture remains to be unproven, it can be motivated by considering a system of $m$ simultaneous algebraic equations with $n>m$ unknowns. It is, however, well known that such a system may have an infinite number of solutions. In the mathematics literature, the terms 'the inverse problems with single or multiple measurements' often appear. They are understood as synonyms for the non-overdetermined and overdetermined inverse problems. Also, there is a class of CIPs, in which unknown coefficients should be determined using values of solution of a corresponding PDE given at a set of interior points of a domain $\Omega$. We refer to the book (Engl, 1996) for some regularization-based numerical methods for such problems.

Consider some examples of such inverse problems. Let $u\left(x, t, x_{0}\right)$ be the solution of the Cauchy problem for a hyperbolic equation

$$
\begin{gather*}
u_{t t}=\Delta u+a(x) u, \quad x \in R^{3}, \quad t \in(0, T),  \tag{1.2}\\
\left.u\right|_{t=0}=0,\left.\quad u_{t}\right|_{t t=0}=\delta\left(x-x_{0}\right) \tag{1.3}
\end{gather*}
$$

Assume that the coefficient $a(x) \in C^{k}\left(R^{3}\right)$ for a certain $k \geq 1$ and $a(x)=$ const for $|x|>A$, where $A$ is a certain positive constant. Let $\Omega \subset R^{3}$ be a bounded convex domain, $T$ be a positive constant, $Q_{T}=\Omega \times(0, T)$ be the time cylinder and $S_{T}=\partial \Omega \times(0, T)$ be its lateral surface. Also, let $\Gamma \in C^{1}$ be a part of the boundary $\partial \Omega, \Gamma \subseteq \partial \Omega$ and $\Gamma_{T}=\Gamma \times(0, T)$. We pose the following inverse problem for the equation (1.2).

Inverse Problem 1.3.1. Given the functions

$$
\begin{equation*}
\left.u\right|_{(x, t) \in \Gamma_{T}}=\varphi\left(x, x_{0}, t\right),\left.\quad \frac{\partial u}{\partial n_{x}}\right|_{(x, t) \in \Gamma_{T}}=\psi\left(x, x_{0}, t\right) \tag{1.4}
\end{equation*}
$$

$\forall(x, t) \in \Gamma_{T}, \forall x_{0} \in \Gamma_{0}$, where $n_{x}$ is the outward normal vector on $\Gamma$,

$$
\frac{\partial}{\partial n_{x}}=\sum_{i=1}^{3} \cos \left(n_{x}, x_{i}\right) \cdot \frac{\partial}{\partial x_{i}},
$$

and $\Gamma_{0} \subset R^{3} \backslash \bar{\Omega}$ is either a $k$-dimensional manifold with $k=1,2$ or a single point, or a collection of few points. Determine the coefficient $a(x)$ in $\Omega$ assuming that this coefficient is known outside of $\Omega$.

Taking into account that $\Gamma_{T}$ is a part of the lateral surface $S_{T}$ of the time cylinder $Q_{T}$, we introduce

Definition 1.3.3. A $n$-dimensional inverse problem for a hyperbolic or parabolic equation is called the inverse problem with the lateral data if both the Dirichlet and Neumann data are given on a part $\Gamma_{T} \subseteq S_{T}$ of the surface $S_{T}=\partial \Omega \times(0, T)$ of the time cylinder $Q_{T}=\Omega \times(0, T)$, where $\Omega \subset R^{n}$ is a domain, where an unknown coefficient of this equation is to be determined.

The manifold $\Gamma_{0}$ is a curve if $k=1$, and it is a surface if $k=2$. Since $\Gamma$ is a twodimensional surface and $t \in(0, T)$, the number of the so-called free variables in the data
$\varphi, \psi$ is $k+3 \geq 3$. Since the number of free variables in the unknown coefficient $a(x)$ equals 3 , the Inverse Problem 1.3.1 is non-overdetermined only if $\Gamma_{0}$ is either a point or a collection of few points, i.e., if there is only one source or there are few separated sources.

Let us pose an over-determined inverse problem. Suppose that the equation (1.2) is satisfied in $Q_{T}=\Omega \times(0, T), \partial \Omega \in C^{\infty}$, and both initial conditions (1.3) are zeros. Consider the Dirichlet boundary data

$$
\begin{gather*}
\left.u\right|_{(x, t) \in S_{T}}=F\left(x, x_{0}, t, t_{0}\right),  \tag{1.5}\\
\forall(x, t) \in S_{T}, \quad \forall x_{0} \in \partial \Omega, \quad \forall t_{0} \in(0, T),
\end{gather*}
$$

where the function $F \in C^{\infty}$ is known. If the function $a \in C^{2}(\bar{\Omega})$ is known, then the forward problem (1.2), (1.3), (1.5) has a unique solution $u \in C^{2}\left(\overline{Q_{T}}\right)$. Therefore, the function $G$ is uniquely determined by

$$
\begin{equation*}
G\left(x, x_{0}, t, t_{0}\right)=\left.\frac{\partial u}{\partial n_{z}}\right|_{(x, t) \in \Gamma_{T}}, \tag{1.6}
\end{equation*}
$$

This function depends nonlinearly on the coefficient $a(x)$. The inverse problem of determining the coefficient $a(x)$ from conditions (1.2), (1.3), (1.5), and (1.6) is overdetermined. This is because the data $G$ depends on six variables, whereas the function $a(x)$ depends only on three variables.

Let $\psi(x, t) \in^{\infty}\left(\bar{S}_{T}\right)$ be an arbitrary function. Consider the function $\bar{\psi}(x, t)$ defined by

$$
\begin{equation*}
\bar{\psi}(x, t)=\int_{0}^{T} \int_{\partial \Omega} F\left(x, x_{0}, t, t_{0}\right) \psi\left(x_{0}, t_{0}\right) d S\left(x_{0}\right) d t_{0} \tag{1.7}
\end{equation*}
$$

Here, we integrate with respect to $x_{0} \in \partial \Omega$. Then, the quantity $\bar{\psi}$ can be considered as the Dirichlet data for the equation (1.2) in the time cylinder $Q_{T}$ with the zero initial conditions at $\{t=0\}$. Let $\hat{u}(x, t)$ be the solution of this Dirichlet boundary value problem. Let

$$
\begin{equation*}
\hat{\varphi}(x, t)=\left.\frac{\partial u}{\partial n_{x}}\right|_{\Gamma_{T}}, \forall(x, t) \in \Gamma_{T} \tag{1.8}
\end{equation*}
$$

Then, one can write

$$
\begin{equation*}
\wedge_{a}(\bar{\psi})=\hat{\varphi}, \tag{1.9}
\end{equation*}
$$

where $\wedge_{a}$ is the Dirichlet-to-Neumann map (D-t-N) operator. The Inverse Problem 1.3.1 can be reformulated as a problem of finding the coefficient $a(x)$ from the equation (1.9) for all functions $\hat{\psi}$ generated by (1.7) when the function $\psi$ runs over the set $C^{\infty}\left(\bar{S}_{T}\right)$. If $\Gamma=\partial \Omega$, then the D-t-N map (1.9) is complete. If $\Gamma \subset \partial \Omega, \Gamma \neq \partial \Omega$, then the D-t-N map is incomplete.

Inverse Problem 1.3.2. Given the functions $\hat{\varphi}$ defined by (1.8), (1.9) for all $\bar{\psi}$ defined by (1.7) for all $\psi \in C^{\infty}\left(\bar{S}_{T}\right)$, where $u \in C^{2}\left(\bar{Q}_{T}\right)$ is the solution of the boundary value problem for the equation (1.2) with the zero initial data in (1.3) and the Dirichlet data $\left.u\right|_{S_{T}}=\hat{\psi}(x, t)$. Find the coefficient a (x) from the $D-t-N$ map (1.9).

If the function $F$ satisfies some additional conditions, then given the data (1.9) for all functions $\hat{\psi}$ of the form (1.7), one can obtain (1.9) for all functions $\hat{\psi} \in C\left(\bar{S}_{T}\right)$ as a limit.

One of such conditions can be formulated by the following lemma. Consider the equation

$$
\begin{equation*}
\int_{0}^{T} \int_{\partial \Omega} F\left(x, x_{0}, t, t_{0}\right) g(x, t) d S_{x} d t=0, \forall x_{0} \in \partial \Omega, \quad \forall t_{0} \in(0, T) \tag{1.10}
\end{equation*}
$$

where the function $g \in C\left(\bar{S}_{T}\right)$ is unknown.
Lemma 1.3.1. Suppose that this equation has the only trivial solution. Then the set

$$
\left\{\int_{0}^{T} \int_{\Gamma} F\left(x, x_{0}, t, t_{0}\right) \psi\left(x_{0}, t_{0}\right) d x_{0} d t_{0}\right\} \subset C\left(\bar{S}_{T}\right)
$$

is dense in $C\left(\bar{S}_{T}\right)$ if the functions $\psi$ run over $C^{\infty}\left(\bar{S}_{T}\right)$.
Proof. It is sufficient to prove that if $g \in C\left(\bar{S}_{T}\right)$, then the relation

$$
\int_{0}^{T} \int_{\partial \Omega} g(x, t)\left[\int_{0}^{T} \int_{\partial \Omega} F\left(x, x_{0}, t, t_{0}\right) \psi\left(x_{0}, t_{0}\right) d S_{x_{0}} d t_{0}\right] d S_{x} d t=0, \forall \psi \in C^{\infty}\left(\bar{S}_{T}\right)
$$

implies (1.10).
This lemma allows one to construct some examples of the function $F$. For example, let $\partial \Omega=\Gamma=\left\{x_{3}=0\right\}$. Then one can consider the function

$$
F\left(x, x_{0}, t, t_{0}\right)=\exp \left[-\alpha\left|x-x_{0}\right|^{2}-\alpha\left|t-t_{0}\right|^{2}\right], \quad x, x_{0} \in\left\{x_{3}=0\right\}
$$

where $\alpha=$ const $>0$. In this case, we should also assume that functions $\psi$ are bounded in $\left\{x_{3}=0\right\} \times(0, T)$. One can also choose the delta-function $\delta\left(x, x_{0}, t-t_{0}\right)$.

Now we assume that $\Omega=\{|x|<R\}$ is a ball, and $(\varphi, \theta) \in[0,2 \pi] \times[0, \pi]$ are the angles in the spherical coordinate system. Then the function $F$ can be taken in the form

$$
F=\exp \left[-\alpha\left(\varphi-\varphi_{0}\right)^{2}-\alpha\left(\theta-\theta_{0}\right)^{2}-\alpha\left(t-t_{0}\right)^{2}\right]
$$

where $\alpha \neq 0$ is a complex number. In this case, the relation (1.10) becomes

$$
\begin{gather*}
\int_{0}^{T} \int_{0}^{2 \pi} \int_{0}^{\pi} \exp \left[-\alpha\left(\varphi-\varphi_{0}\right)^{2}-\alpha\left(\theta-\theta_{0}\right)^{2}-\alpha\left(t-t_{0}\right)^{2}\right] g(\theta, \varphi, t t) \sin \theta d \theta d \varphi d t=0, \\
\forall \varphi_{0} \in[0,2 \pi], \quad \forall \varphi_{0} \in[0, \pi], \quad \forall t_{0} \in[0, T] \tag{1.11}
\end{gather*}
$$

To prove that (1.11) implies $g=0$, one should divide (1.11) by $\exp \left[-\alpha\left(\varphi_{0}^{2}+\theta_{0}^{2}+t_{0}^{2}\right)\right]$ and differentiate the result with respect to $\varphi_{0}, \theta_{0}, t_{0}$. This leads to the relation

$$
\begin{aligned}
\int_{0}^{T} \int_{0}^{2 \pi} \int_{0}^{\pi} \varphi^{k_{1}} \theta^{k_{2}} t^{k_{3}} & \left\{\exp \left[-\alpha\left(\varphi \varphi_{0}+\theta \theta_{0}+t t_{0}\right)\right] \exp \left[-\alpha\left(\varphi^{2}+\theta^{2}+t^{2}\right)\right]\right. \\
& \times g(\theta, \varphi, t)\} \sin \theta d \theta d \varphi d t=0,
\end{aligned}
$$

where $k_{1}, k_{2}, k_{3}$ are arbitrary non-negative integers. Hence, all moments of the product of functions including in curly brackets are equal to zero. This implies $g=0$.

Suppose that the conditions of Lemma 1.3.1 are fulfilled. Then the equation (1.9) can be considered for all $\hat{\psi} \in C\left(\bar{S}_{T}\right)$.

Following (Sylvester, 1987), consider a more complicated case of time independent data. Let $u\left(x, x_{0}\right)$ be solution of the elliptic equation

$$
\begin{equation*}
\triangle_{x} u+c(x) u=0, \quad x \in \Omega \tag{1.12}
\end{equation*}
$$

with the Dirichlet boundary data

$$
\begin{equation*}
\left.u\right|_{x \in \partial \Omega}=F\left(x, x_{0}\right), \quad \forall x, x_{0} \in \partial \Omega \tag{1.13}
\end{equation*}
$$

where $\Omega \subset R^{3}$ is a bounded domain with $\partial \Omega \in C^{\infty}$. If $c(x)>0$, then the Dirichlet problem (1.12), (1.13) may have many solutions. Therefore, we assume that $c(x) \leq 0$.

Inverse Problem 1.3.3. Let

$$
\begin{equation*}
G\left(x, x_{0}\right)=\left.\frac{\partial u}{\partial n_{x}}\right|_{x \in T}, \quad \forall x \in \Gamma, \forall x_{0} \in \partial \Omega \tag{1.14}
\end{equation*}
$$

where $u\left(x, x_{0}\right)$ is a solution of the Dirichlet boundary value problem (1.12), (1.13). Suppose that the function $G\left(x, x_{0}\right)$ is given. Find the coefficient $c(x)$ from the data (1.12) - (1.14).

Under conditions formulated in Lemma 1.3.1, one can obtain the D-t-N map from the data (1.12) - (1.14). Specifically, the function $f$ is given by

$$
f=\wedge_{c}(g), \quad \forall g \in C(\partial \Omega),
$$

where $g$ is the Dirichlet data for the equation (1.12). Obviously, the Inverse Problem 1.3.3 is overdetermined if $n \geq 3$, and it is non-overdetermined if $n=2$. A well known application of this problem concerns the inverse conductivity problem for the equation

$$
\begin{equation*}
\nabla(\gamma(x) v)=0, \quad \gamma \in C^{k}(\bar{\Omega}), \quad \gamma \geq \text { const }>0 \tag{1.15}
\end{equation*}
$$

with an appropriate $k \geq 0$. If $k \geq 2$, then replacing variables $u=\gamma^{1 / 2} v$, one can obtain the equation (1.11) with $c(x)=-\gamma^{1 / 2} \triangle(\sqrt{\gamma})$.

## 3 Tikhonov scheme

Let us consider following functional in the space $H^{1}$ :

$$
\begin{equation*}
\Phi(x)=\frac{1}{2}\|F(x)\|_{H_{2}}^{2} \tag{2}
\end{equation*}
$$

where $F: H_{1} \rightarrow H_{2}$ is actually nonlinear operator. Clear, that this functional is positive and is bounded on the whole space $H_{1}$.
let'us assume that there exist (possibly not unique) element $x^{*}$, where functional $\Phi(x)$ attains its global minimum, or

$$
\begin{equation*}
x^{*} \in H_{1}, \Phi\left(x^{*}\right)=\inf _{x \in H_{1}} \Phi(x) . \tag{3}
\end{equation*}
$$

Particularly, when equation

$$
\begin{equation*}
F(x)=0, x \in H_{1}, F: H_{1} \rightarrow H_{2} \tag{4}
\end{equation*}
$$

have non-empty set of solutions, then infimum is exactly on this set,or

$$
\begin{equation*}
\inf _{x \in H_{1}} \Phi(x)=0 \tag{5}
\end{equation*}
$$

In the more common case, when is possible situation that

$$
\begin{equation*}
\inf _{x \in H_{1}} \Phi(x)>0 . \tag{6}
\end{equation*}
$$

element $x^{*}$ defined by condition (3) can be considered as general solution of equation or also quasi-solution of equation (4).
in the non-regular case (for ill-posed problems) usage of gradient-like methods (newton-Kantorovich,Gauss-newton,gardient) is not possible. A.N.Tichonov is introduced commonly used now theory how to get minimizing sequence or find min of (2) in the non-regular case. He did it in $60-$ s without assumption about regularity of the equation $F(x)=0$. Main assumption:

We assume that there exists a set of solutions of this equation which we denote as $X^{*}(F)$. So, we assume that $X^{*}(F) \neq \varnothing$ not empty. Note, that numerical implementation of the Tichonov principle is difficult for any class of operators $F\left(B_{1}, B_{2}\right)$ because this construction works only for operators $F$, when functional (2) is convex. But all other alg. are using main conception of Tichonov theory.

### 3.1 Case of operator $F(x)=0$

Let us consider new functional

$$
\begin{equation*}
\Phi_{\alpha}=\Phi(x)+\frac{\alpha}{2}\left\|x-x_{0}\right\|_{H_{1}}^{2}, x \in H_{1} . \tag{7}
\end{equation*}
$$

This functional is called Tichonov functional. $\alpha>0$ is reg.parameter, $x_{0}$ is some fixed element in $H_{1}$. It is clear that

$$
\begin{equation*}
\inf _{x \in H_{1}} \Phi_{\alpha}(x) \geq 0 \forall \alpha \geq 0 \tag{8}
\end{equation*}
$$

Obviously that

$$
\forall \alpha \geq 0 \forall \epsilon>0 \exists x_{\alpha}^{\epsilon} \in H_{1}: \inf _{x \in H_{1}} \Phi_{\alpha}(x) \leq \Phi_{\alpha}\left(x_{\alpha}^{\epsilon}\right) \leq \inf _{x \in H_{1}} \Phi_{\alpha}(x)+\epsilon .
$$

We will consider that

$$
\begin{equation*}
\epsilon:=\epsilon(\alpha): \lim _{\alpha \rightarrow 0} \frac{\epsilon(\alpha)}{\alpha}=0 . \tag{9}
\end{equation*}
$$

(or $\epsilon=\epsilon(\alpha)$ ).

Let us take any element $x^{*} \in X^{*}(F)$, where we assumed that $X^{*}(F) \neq \oslash$. By definition of $x_{\alpha}^{\epsilon}$ :

$$
\begin{equation*}
\Phi_{\alpha}\left(x_{\alpha}^{\epsilon}\right) \leq \Phi_{\alpha}\left(x^{*}\right)+\epsilon(\alpha) . \tag{10}
\end{equation*}
$$

From this and using also (7) we get

$$
\begin{equation*}
\Phi\left(x_{\alpha}^{\epsilon}\right)+\frac{\alpha}{2}\left\|x_{\alpha}^{\epsilon}-\xi\right\|_{H_{1}}^{2} \leq \frac{\alpha}{2}\left\|x^{*}-\xi\right\|_{H_{1}}^{2}+\epsilon(\alpha) . \tag{11}
\end{equation*}
$$

From (11) follows following inequalities:

$$
\begin{gather*}
\Phi\left(x_{\alpha}^{\epsilon}\right) \leq \frac{\alpha}{2}\left\|x^{*}-\xi\right\|_{H_{1}}^{2}+\epsilon(\alpha)  \tag{12}\\
\left\|x_{\alpha}^{\epsilon}-\xi\right\|_{H_{1}}^{2} \leq\left\|x^{*}-\xi\right\|_{H_{1}}^{2}+\frac{2 \epsilon(\alpha)}{\alpha} . \tag{13}
\end{gather*}
$$

Set of elements $\left\{x_{\alpha}^{\epsilon(\alpha)}\right\}_{\alpha \in\left(0, \alpha_{0}\right]}$ we will call sequence. Using (9) we get

$$
\lim _{\alpha \rightarrow 0} \epsilon(\alpha)=0
$$

Then, from (12) follows that sequence $x_{\alpha}^{\epsilon(\alpha)}$ is minimizing for functional $\Phi$, or

$$
\lim _{\alpha \rightarrow 0} \Phi\left(x_{\alpha}^{\epsilon(\alpha)}\right)=\inf _{x \in H^{1}} \Phi(x)=0 .
$$

## Theorem

Now, if we assume that operator $F$ is weakly continuous, then sequence $\left\{x_{\alpha}^{\epsilon(\alpha)}\right\}$ is strongly convergent to the set $X^{*}(F)$ or

$$
\lim _{\alpha \rightarrow 0} \operatorname{dist}\left(x_{\alpha}^{\epsilon(\alpha)}, X^{*}(F)\right)=0
$$

where by definition for any set $D \subset H_{1}$

$$
\operatorname{dist}(x, D)=\inf _{y \in D}\|x-y\|_{H_{1}}, \quad x \in H_{1}
$$

## Proof.

First, recall some definitions:
Definition 1.
A sequence of points $x_{n}$ converges weakly to $x \in H$ if $\left(x_{n}, y\right) \rightarrow(x, y) \forall y \in H$.
Definition 2.
A sequence of points $x_{n}$ converges strongly $x_{n} \rightarrow x$ if $\left\|x_{n}-x\right\| \rightarrow 0$ as $n \rightarrow \infty$.
Definition 3.
Properties of weakly convergent sequence:

1. Since any closed and bounded set is relatively compact (because the closure is compact), then every bounded sequence $x_{n} \in H$ contains a weakly convergent subsequence.
2. Every weakly convergent sequence is bounded.
3. If $x_{n} \rightharpoonup x$ then $\|x\| \leq \lim _{n \rightarrow \infty} \inf \left\|x_{n}\right\|$
4. If $x_{n} \rightharpoonup x$ and $\left\|x_{n}\right\|=\|x\|$, then $x_{n} \rightarrow x$ strongly.

Because of (13) set $\left\{x_{\alpha}^{\epsilon(\alpha)}\right\}$ is bounded for $\forall \alpha_{0}>0$. Let us take any set of regularization parameters $\alpha_{n}$ such that $\lim _{n \rightarrow \infty} \alpha_{n}=0$. Then corresponding sequence of elements $\left\{x_{\alpha}^{\epsilon\left(\alpha_{n}\right)}\right\}$ is weakly compact because it is bounded. This means that any infinite subset of elements of $\left\{x_{\alpha}^{\epsilon\left(\alpha_{n}\right)}\right\}$ have subsequence which is weakly convergent in $H_{1}$. Thus we can consider that itself $\left\{x_{\alpha}^{\epsilon\left(\alpha_{n}\right)}\right\}$ is weakly convergent in $H_{1}$ to some element $z^{*} \in H_{1}$.

Now using assumption that operator $F$ is weakly convergent, then using (12) we can conclude that $z^{*} \in X^{*}(F)$. Weakly convergent operator means that if sequence $\left\{z_{n}\right\} \rightharpoonup \bar{z} \in$ $H_{1}$, then $\left\{F\left(z_{n}\right)\right\} \rightharpoonup F(\bar{z}) \in H_{2}$.

Using 3 property of weak convergence we conclude that

$$
\left\|F\left(z^{*}\right)\right\|_{H_{2}} \leq \lim _{n \rightarrow \infty} \inf \left\|F\left(x_{\alpha_{n}}^{\epsilon\left(\alpha_{n}\right)}\right)\right\|_{H_{2}}
$$

But using (12) we conclude that

$$
\Phi\left(x_{\alpha_{n}}^{\epsilon\left(\alpha_{n}\right)}\right)=\frac{1}{2}\left\|F\left(x_{\alpha_{n}}^{\epsilon\left(\alpha_{n}\right)}\right)\right\| \leq \frac{\alpha_{n}}{2}\left\|x^{*}-\xi\right\|_{H_{1}}^{2}+\epsilon\left(\alpha_{n}\right) .
$$

and thus we can conclude that

$$
\lim _{n \rightarrow \infty} \Phi\left(x_{\alpha_{n}}^{\epsilon\left(\alpha_{n}\right)}\right)=0 .
$$

Thus, $F\left(z^{*}\right)=0$ and $z^{*} \in X^{*}(F)$. Also

$$
\begin{equation*}
x_{\alpha_{n}}^{\epsilon\left(\alpha_{n}\right)}-\xi \rightarrow z^{*}-\xi, \quad n \rightarrow \infty \tag{14}
\end{equation*}
$$

weakly in $H_{1}$. In reality, this convergence is strong because of 4 property of weakly convergence, or

$$
\lim _{n \rightarrow \infty}\left\|x_{\alpha_{n}}^{\epsilon\left(\alpha_{n}\right)}-z^{*}\right\|_{H_{1}}=0, \quad n \rightarrow \infty
$$

From above equation follows what we want to get

$$
\begin{equation*}
\lim _{\alpha \rightarrow 0} \inf _{x^{*} \in X^{*}(F)}\left\|x_{\alpha_{n}}^{\epsilon\left(\alpha_{n}\right)}-x^{*}\right\|_{H_{1}}=0, \quad n \rightarrow \infty \tag{15}
\end{equation*}
$$

Note, that in assumption of existence of weak continuity of operator $F$ the

$$
\begin{equation*}
\inf _{x \in H_{1}} \Phi_{\alpha}(x) \tag{16}
\end{equation*}
$$

is achieved for $\forall \alpha>0$. Thus, we can assume that $\epsilon(\alpha)=0 \forall \alpha>0$.
To prove that (16) is achieved let us consider any minimizing sequence $x_{n}$ :

$$
\lim _{n \rightarrow \infty} \Phi_{\alpha}\left(x_{n}\right)=d_{\alpha} \equiv \inf _{x \in H_{1}} \Phi_{\alpha}(x)
$$

For any $\omega>0$ there $\exists N(\omega): \forall n \geq N(\omega)$ is valid

$$
\Phi_{\alpha}\left(x_{n}\right) \leq d_{\alpha}+\omega
$$

Then using (7) we get estimates

$$
\begin{aligned}
\frac{1}{2}\left\|F\left(x_{n}\right)\right\|_{H_{2}}^{2} & \leq d_{\alpha}+\omega \\
\left\|x_{n}-\xi\right\|_{H_{1}}^{2} & \leq \frac{2\left(d_{\alpha}+\omega\right)}{\alpha} ; n \geq N(\omega), \omega>0
\end{aligned}
$$

Thus, sequences $\left\{x_{n}\right\}$ and $\left\{F\left(x_{n}\right)\right\}$ are bounded. Thus we can consider them as weakly convergent in $H_{1}$ and $H_{2}$ to elements $\tilde{x}$ and $\tilde{f}$, correspondingly. Using that operator $F$ is weakly continuous, we have $\tilde{f}=F(\tilde{x})$. Using weak convergence, we have

$$
\begin{equation*}
d_{\alpha} \leq \Phi_{\alpha}(\tilde{x})=\frac{1}{2}\|F(\tilde{x})\|_{H_{2}}^{2}+\frac{\alpha}{2}\|\tilde{x}-\xi\|_{H_{1}}^{2} \leq \liminf \left(\frac{1}{2}\left\|F\left(x_{n}\right)\right\|_{H_{2}}^{2}+\frac{\alpha}{2}\left\|x_{n}-\xi\right\|_{H_{1}}^{2}\right) \leq d_{\alpha}+\omega . \tag{17}
\end{equation*}
$$

Since value of $\omega>0$ is any it follows that $d_{\alpha}=\Phi_{\alpha}(\tilde{x})$ and thus infimum is achieved on element $\tilde{x}$.

Remarks.
If set of solutions $X^{*}(F)$ consists of one point $x^{*}$, then by (15) sequence $x_{\alpha_{n}}^{\epsilon\left(\alpha_{n}\right)}$ converges to this point.

If set $X^{*}(F)$ consists of many points, then because of weakly convergence from equation (15) follows: let us define $X^{* *}$ set of all elements in $X^{*}(F)$ closest to $\xi$ :

$$
\begin{equation*}
X^{* *}(F)=\left\{x^{* *} \in X^{*}(F):\left\|x^{* *}-\xi\right\|_{H_{1}}=\inf _{x^{*} \in X^{*}(F)}\left\|x^{*}-\xi\right\|\right\} \tag{18}
\end{equation*}
$$

We see that $X^{* *} \neq \oslash$ and then equation (15) have the form

$$
\begin{equation*}
\lim _{\alpha \rightarrow 0} \inf _{x^{* *} \in X^{* *}(F)}\left\|x_{\alpha_{n}}^{\epsilon\left(\alpha_{n}\right)}-x^{* *}\right\|_{H_{1}}=0, \quad n \rightarrow \infty \tag{19}
\end{equation*}
$$

