NTNU

Innovation and Creativity

A posteriori error estimation for an inverse scattering problem

Larisa Beilina

Institutt for matematiske fag

Acoustic wave propagation

The scalar wave equation modeling acoustic wave propagation in a bounded domain $\Omega \subset \mathbf{R}^d, \; d = 2, 3$, with boundary Γ, takes the following form:

$$
\begin{aligned}
\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \triangle p = f, &\qquad \text{in } \; \Omega \times (0,T), \\
p(\cdot,0) = 0, \;\; \frac{\partial p}{\partial t}(\cdot,0) = 0, &\;\; \text{in } \; \Omega, \\
p\big|_{\Gamma} = 0, &\;\; \text{on } \; \Gamma \times (0,T),
\end{aligned} \tag{1}
$$

where $p(x,t) \in \mathbf{R}$ is the pressure satisfying homogeneous boundary and initial conditions, $c(x)$ is the wave speed depending on $x \in \Omega$, t is the time variable and T is a final time, and $f(x,t)$ is a given source function.

Inverse acoustic scattering

Our goal is to find the function $c(x)$ which minimizes the quantity

$$E(p,c) = \frac{1}{2}\int_0^T \int_\Omega (p-\tilde{p})^2 \delta_{obs} dxdt + \frac{1}{2}\gamma \int_\Omega (c-c_0)^2 \; dx, \tag{2}$$

where $\tilde{p}$ is observed data at x_{obs}, p satisfies (1) and thus depends on c, $\delta_{obs} = \sum \delta(x_{obs})$ is a sum of multiples of delta-functions $\delta(x_{obs})$ corresponding to the observation points, and γ is a regularization parameter (small).

To approach this minimization problem, we introduce the Lagrangian

$$L(u) = E(p,c) - ((\frac{1}{c^2} Dp, D\varphi)) + ((\nabla p, \nabla \varphi)) - ((f, \varphi)),$$

where $u = (p, \varphi, c)$, and search for a stationary point with respect to u satisfying $\forall \bar{u}$

$$L'(u; \bar{u}) = 0, \tag{3}$$

where $L'(u; \cdot)$ is the Jacobian of L at u, and we assume that $\varphi(\cdot, T) = \bar{\varphi}(\cdot, T) = 0$ and $p(\cdot, 0) = \bar{p}(\cdot, 0) = 0$, together with homogeneous Dirichlet boundary conditions.

The equation (3) expresses that in $\Omega \times (0, T)$

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \triangle p &= f, & (4)\\ \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \triangle \varphi &= -(p - \tilde{p})\delta_{obs}, & (5)\\ \gamma(c - c_0) + \frac{2}{c^3}\int_0^T \frac{\partial p}{\partial t}\frac{\partial \varphi}{\partial t}\, dt &= 0, & (6) \end{aligned}$$

together with homogeneous boundary and initial conditions.

Finite element discretization.

To formulate the finite element method for (3) we introduce the finite element spaces V_h, W_h^p and W_h^{φ} defined by :

$$\begin{aligned}
V_h &:= \{v \in L_2(\Omega) : v \in P_0(K), \forall K \in K_h\}, \\
W^p &:= \{p \in H^1(\Omega \times J) : p(\cdot, 0) = 0, p|_\Gamma = 0\}, \\
W^{\varphi} &:= \{\varphi \in H^1(\Omega \times J) : \varphi(\cdot, T) = 0, \varphi|_\Gamma = 0\}, \\
W_h^p &:= \{v \in W^p : v|_{K \times J} \in P_1(K) \times P_1(J), \forall K \in K_h, \forall J \in J_k\}, \\
W_h^{\varphi} &:= \{v \in W^{\varphi} : v|_{K \times J} \in P_1(K) \times P_1(J), \forall K \in K_h, \forall J \in J_k\},
\end{aligned}$$

where $P_1(K)$ and $P_1(J)$ are the set of linear functions on K and J, respectively.

The finite element method now reads: Find $c_h \in V_h, \varphi_h \in W_h^{\varphi}, p_h \in W_h^p$, such that

$$L'(\varphi_h, p_h, c_h)(\bar{\varphi}, \bar{p}, \bar{c}) = 0 \;\; \forall \bar{c} \in V_h, \bar{\varphi} \in W_h^{\lambda}, \bar{p} \in W_h^p. \tag{7}$$

A posteriori error estimation for the Lagrangian

We obtain an a posteriori error estimate for error in the Lagrangian by noting that

$$
\begin{aligned}
L(u) - L(u_h) &= \int_0^1 \frac{d}{d\epsilon} L(\epsilon u + (1-\epsilon)u_h) d\epsilon \\
&= \int_0^1 L'(\epsilon u + (1-\epsilon)u_h; u - u_h) d\epsilon \\
&= L'(u_h; u - u_h) + R,
\end{aligned}
$$

where R is a second order remainder term.

Using now the Galerkin orthogonality

$$L'(u_h; \bar{u}) = 0 \;\; \forall \bar{u} \in U_h$$

with the splitting

$$u - u_h = (u - u_h^I) + (u_h^I - u_h),$$

where $u_h^I \in U_h$ denotes an interpolant of u, and neglecting the term R, we get the following error representation:

$$L(u) - L(u_h) \approx L'(u_h; u - u_h^I), \tag{8}$$

involving the residual $L'(u_h; \cdot)$ with $u - u_h^I$ appearing as a weight.

Derivation of an a posteriori error estimate for the Lagrangian

Using the Galerkin orthogonality (3) and the splitting
$\varphi-\varphi_h=(\varphi-\varphi_h^I)+(\varphi_h^I-\varphi_h),\;\; p-p_h=(p-p_h^I)+(p_h^I-p_h),\;\; c-c_h=(c-c_h^I)+(c_h^I-c_h)$,
where $(\varphi_h^I,p_h^I,c_h^I)$ denotes an interpolant of $(\varphi,p,c)\in W_h^{\varphi}\times W_h^{p}\times V_h$, and neglecting the term R, we get:

$$e \approx L'(\varphi_h,p_h,c_h)(\varphi-\varphi_h^I,p-p_h^I,c-c_h^I)=(I_1+I_2+I_3), \tag{9}$$

where

$$
\begin{aligned}
I_1 &= \int_0^T \int_\Omega (-\frac{1}{c_h^2} \frac{\partial(\varphi - \varphi_h^I)}{\partial t} \frac{\partial p_h}{\partial t} + \nabla(\varphi - \varphi_h^I)\nabla p_h \\
&\quad -f(\varphi - \varphi_h^I)) \; dxdt, \\
I_2 &= \int_0^T \int_\Omega (p_h - \tilde{p})(p - p_h^I)\, \delta_{obs} \; dxdt \\
&+ \int_0^T \int_\Omega \left(-\frac{1}{c_h^2} \frac{\partial \varphi_h}{\partial t} \frac{\partial(p - p_h^I)}{\partial t} + \nabla\varphi_h \nabla(p - p_h^I)\right) \; dxdt, \\
I_3 &= \frac{2}{c_h^3} \int_0^T \int_\Omega \frac{\partial \varphi_h(x,t)}{\partial t} \frac{\partial p_h(x,t)}{\partial t} (c - c_h^I) \; dxdt \\
&+ \gamma \int_\Omega (c_h - c_0)(c - c_h^I) \; dx.
\end{aligned}
$$

To estimate I_1 we integrate by parts in the first and second terms to get:

$$
\begin{aligned}
|I_1| &= \Big| \int_0^T \int_\Omega \left(\frac{1}{c_h^2} \frac{\partial^2 p_h}{\partial t^2} (\varphi - \varphi_h^I) - \triangle p_h (\varphi - \varphi_h^I) - f(\varphi - \varphi_h^I) \right) \; dxdt \\
&+ \sum_K \int_0^T \int_{\partial K} \frac{\partial p_h}{\partial n_K} (\varphi - \varphi_h^I) \, dsdt \\
&- \sum_k \int_\Omega \frac{1}{c_h^2} \left[\frac{\partial p_h}{\partial t} (t_k) \right] (\varphi - \varphi_h^I)(t_k) \, dx \Big|,
\end{aligned}
\tag{10}
$$

where the terms $\frac{\partial p_h}{\partial n_K}$ and $\left[\frac{\partial p_h}{\partial t}\right]$ appear during the integration by parts and denote the derivative of p_h in the outward normal direction n_K of the boundary ∂K of element K, and the jump of the derivative of p_h in time, respectively.

In the second term of the (10) we sum over the element boundaries, and each internal side $S \in S_h$ occurs twice. Denoting by $\partial_s p_h$ the derivative of a function p_h in one of the normal directions of each side S, we can write

$$\sum_K \int_{\partial K} \frac{\partial p_h}{\partial n_K} (\varphi - \varphi_h^I) \ ds = \sum_S \int_S \left[\partial_s p_h \right] (\varphi - \varphi_h^I) \ ds, \tag{11}$$

where $\left[\partial_s p_h \right]$ is jump in the derivative $\partial_s p_h$ computed from the two triangles sharing S.

We distribute each jump equally to the two sharing triangles and return to a sum over elements edges ∂K :

$$\sum_S \int_S [\partial_s p_h] \cdot (\varphi - \varphi_h^I) \ ds = \sum_K \frac{1}{2} h_K^{-1} \int_{\partial K} \left[\partial_s p_h\right] (\varphi - \varphi_h^I) \ h_K \ ds. \tag{12}$$

We formally set $dx = h_K ds$ and replace the integrals over the element boundaries ∂K by integrals over the elements K, to get:

$$\left| \sum_K \frac{1}{2} h_K^{-1} \int_{\partial K} \left[\partial_s p_h\right] (\varphi - \varphi_h^I) \ h_K \ ds \right| \leq C \max_{S \subset \partial K} h_K^{-1} \int_\Omega \left| \left[\partial_s p_h\right] \right| \left| (\varphi - \varphi_h^I) \right| \ dx, \tag{13}$$

where $\left[\partial_s p_h\right]\big|_K = \max_{S \subset \partial K} \left[\partial_s p_h\right]\big|_S$.

In a similar way we can estimate the third term in (10):

$$
\begin{aligned}
& \left| \sum_k \int_\Omega \frac{1}{c_h^2} \left[\frac{\partial p_h}{\partial t}(t_k) \right] (\varphi - \varphi_h^I)(t_k)\, dx \right| \leq \\
& \sum_k \int_\Omega \frac{1}{c_h^2} \tau^{-1} \cdot \left| \left[\frac{\partial p_h}{\partial t}(t_k) \right] \right| \cdot \left| (\varphi - \varphi_h^I)(t_k) \right| \, \tau dx \\
\leq \quad & C \sum_k \int_{J_k} \int_\Omega \frac{1}{c_h^2} \tau^{-1} \cdot \left| [\partial p_{h t_k}] \right| \cdot \left| (\varphi - \varphi_h^I) \right| \, dxdt \\
= \quad & C \int_0^T \int_\Omega \frac{1}{c_h^2} \tau^{-1} \cdot \left| [\partial p_{h t}] \right| \cdot \left| (\varphi - \varphi_h^I) \right| \, dxdt,
\end{aligned}
$$

where

$$
[\partial p_{h t_k}] = \max_k \left(\left[\frac{\partial p_h}{\partial t}(t_k) \right], \; \left[\frac{\partial p_h}{\partial t}(t_{k+1}) \right] \right), \tag{14}
$$

$$
[\partial p_{h t}] = [\partial p_{h t_k}] \text{ on } J_k. \tag{15}
$$

Substituting both above expressions for the second and third terms in (10), we get:

$$
\begin{aligned}
|I_1| \quad \leq \quad & C \int_0^T \int_\Omega \left| \frac{1}{c_h^2} \frac{\partial^2 p_h}{\partial t^2} - \triangle p_h - f \right| \cdot \left(\tau^2 \left| \frac{\partial^2 \varphi}{\partial t^2} \right| + h^2 |D_x^2 \varphi| \right) \; dxdt \\
+ \quad & C \int_0^T \int_\Omega \max_{S \subset \partial K} h_k^{-1} \cdot \left| [\partial_s p_h] \right| \cdot \left(\tau^2 \left| \frac{\partial^2 \varphi}{\partial t^2} \right| + h^2 |D_x^2 \varphi| \right) \; dxdt \\
+ \quad & \frac{C}{c_h^2} \int_0^T \int_\Omega \tau^{-1} \cdot \left| [\partial p_{ht}] \right| \cdot \left(\tau^2 \left| \frac{\partial^2 \varphi}{\partial t^2} \right| + h^2 |D_x^2 \varphi| \right) \; dxdt,
\end{aligned}
$$

where we used standard interpolation estimates for $\varphi - \varphi_h^I$, and C denotes interpolation constants. Next, the terms $\frac{\partial^2 p_h}{\partial t^2}$ and $\triangle p_h$ disappears in the first integral in (16) (p_h is continuous piecewise linear function). We estimate $\frac{\partial^2 \varphi}{\partial t^2} \approx \frac{\left[\frac{\partial \varphi_h}{\partial t}\right]}{\tau}$ and $D_x^2 \varphi \approx \frac{\left[\frac{\partial \varphi_h}{\partial n}\right]}{h}$ to get:

$$
\begin{aligned}
\left|I_1\right| \leq \quad & C \int_0^T \int_\Omega |f| \cdot \left(\tau^2 \left| \frac{\left[\frac{\partial \varphi_h}{\partial t}\right]}{\tau} \right| + h^2 \left| \frac{\left[\frac{\partial \varphi_h}{\partial n}\right]}{h} \right| \right) \; dxdt \\
+ \quad & C \int_0^T \int_\Omega \max_{S \subset \partial K} h_k^{-1} \left| \left[\partial_s p_h\right] \right| \cdot \left(\tau^2 \left| \frac{\left[\frac{\partial \varphi_h}{\partial t}\right]}{\tau} \right| + h^2 \left| \frac{\left[\frac{\partial \varphi_h}{\partial n}\right]}{h} \right| \right) \; dxdt \\
+ \quad & \frac{C}{c_h^2} \int_0^T \int_\Omega \tau^{-1} \left| \left[\partial p_{ht}\right] \right| \cdot \left(\tau^2 \left| \frac{\left[\frac{\partial \varphi_h}{\partial t}\right]}{\tau} \right| + h^2 \left| \frac{\left[\frac{\partial \varphi_h}{\partial n}\right]}{h} \right| \right) \; dxdt.
\end{aligned}
\tag{16}
$$

We estimate I_2 similarly to I_1. To estimate I_3 we use a standard approximation estimate of the form $c - c_h^I \approx hD_x c$ to get:

$$
\begin{aligned}
|I_3| \leq & \frac{2}{c_h^3} \int_0^T \int_\Omega \left| \frac{\partial \varphi_h(x,t)}{\partial t} \cdot \frac{\partial p_h(x,t)}{\partial t} \right| \cdot h \cdot \left| D_x c \right| \; dxdt \\
& + \gamma \int_\Omega \left| c_h - c_0 \right| \cdot h \cdot \left| D_x c \right| \; dx \\
& \leq C \, \frac{2}{c_h^3} \int_0^T \int_\Omega \left| \frac{\partial \varphi_h(x,t)}{\partial t} \cdot \frac{\partial p_h(x,t)}{\partial t} \right| \cdot h \cdot \left| \frac{[c_h]}{h} \right| \; dxdt \\
& + \gamma \int_\Omega \left| c_h - c_0 \right| h \cdot \left| \frac{[c_h]}{h} \right| \; dx \\
& \leq C \, \frac{2}{c_h^3} \int_0^T \int_\Omega \left| \frac{\partial \varphi_h(x,t)}{\partial t} \cdot \frac{\partial p_h(x,t)}{\partial t} \right| \cdot \left| [c_h] \right| \; dxdt \\
& + \gamma \int_\Omega \left| c_h - c_0 \right| \cdot \left| [c_h] \right| \; dx.
\end{aligned}
$$

Error representation

Defining the residuals

$$
\begin{aligned}
R_{p_1} &= |f|, \;\; R_{p_2} = \frac{1}{2} \max_{S \subset \partial K} h_k^{-1} \left| \left[\partial_s p_h \right] \right|, \;\; R_{p_3} = \frac{1}{2\, c_h^2} \tau^{-1} \left| \left[\partial p_{ht} \right] \right|, \\
R_{\varphi_1} &= |p_h - \tilde{p}|, R_{\varphi_2} = \frac{1}{2} \max_{S \subset \partial K} h_k^{-1} \left| \left[\partial_s \varphi_h \right] \right|, \;\; R_{\varphi_3} = \frac{1}{2\, c_h^2} \tau^{-1} \left| \left[\partial \varphi_{ht} \right] \right|, \\
R_{c_1} &= \frac{2}{{c_h}^3} \left| \frac{\partial \varphi_h}{\partial t} \right| \cdot \left| \frac{\partial p_h}{\partial t} \right|, \;\; R_{c_2} = |c_h - c_0|,
\end{aligned}
$$

and interpolation errors in the form

$$
\begin{aligned}
\sigma_\varphi &= C\tau \left| \left[\frac{\partial \varphi_h}{\partial t} \right] \right| + Ch \left| \left[\frac{\partial \varphi_h}{\partial n} \right] \right|, \;\; \sigma_p = C\tau \left| \left[\frac{\partial p_h}{\partial t} \right] \right| + Ch \left| \left[\frac{\partial p_h}{\partial n} \right] \right|, \\
\sigma_c &= C \left| [c_h] \right|,
\end{aligned}
$$

we obtain the following a posteriori estimate

$$
\begin{aligned}
|e| \quad \leq \quad & \int_0^T \int_\Omega R_{p_1}\sigma_\varphi \; dxdt + \int_0^T \int_\Omega R_{p_2}\sigma_\varphi \; dxdt + \int_0^T \int_\Omega R_{p_3}\sigma_\varphi \; dxdt + \\
+ \quad & \int_0^T \int_\Omega R_{\varphi_1}\sigma_p \; dxdt + \int_0^T \int_\Omega R_{\varphi_2}\sigma_p \; dxdt + \int_0^T \int_\Omega R_{\varphi_3}\sigma_p \; dxdt \\
+ \quad & \int_0^T \int_\Omega R_{c_1}\sigma_c \; dxdt - \int_\Omega R_{c_2}\sigma_c \; dx
\end{aligned}
$$

An a posteriori error estimate for parameter identification

Now we present a more general a posteriori error estimate, which may be used to estimate the error in the parameter identification. This estimate involves the solution $\tilde{u}$ of the problem:

$$-L''(u_h; \bar{u}, \tilde{u}) = (\psi, \bar{u}) \ \ \forall \bar{u}, \tag{17}$$

where ψ acts as given data, and $L''(u; \cdot, \cdot)$ is the Hessian of the Lagrangian at u, which expresses the sensitivity of the Jacobian $L'(u; \cdot)$ with respect to changes in u. Assuming this problem can be solved, we obtain choosing here $\bar{u} = u - u_h$ and using the fact that $L''(u; \bar{u}, \tilde{u})$ is symmetric in $\bar{u}$ and $\tilde{u}$, the following error representation:

$$\begin{aligned} ((\psi, u - u_h)) &= -L''(u_h; u - u_h, \tilde{u}) \\ &= -L'(u; \tilde{u}) + L'(u_h; \tilde{u}) + R \\ &= L'(u_h; \tilde{u}) + R = L'(u_h; \tilde{u} - \tilde{u}^I) + R, \end{aligned}$$

where $\tilde{u}^I$ is an interpolant of $\tilde{u}$ and again R is a second order remainder.

Neglecting R we obtain the following analog of (8)

$$((\psi, u - u_h)) \approx L'(u_h; \tilde{u} - \tilde{u}^I),$$

with $\tilde{u}$ replacing u in the second argument.
The concrete form of this estimate is the same as that given above for the Lagrangian with only u replaced by $\tilde{u}$ in the weights. Compare with a posteriori error for Lagrangian:

$$\begin{aligned} e = L(u) - L(u_h) &= \int_0^1 \frac{d}{d\epsilon} L(\epsilon u + (1-\epsilon) u_h) d\epsilon \\ &= \int_0^1 L'(\epsilon u + (1-\epsilon) u_h; u - u_h) d\epsilon \\ &= L'(u_h; u - u_h) + R. \end{aligned}$$

We will consider now scalar wave equation in the form

$$
\begin{aligned}
\alpha \frac{\partial^2 p}{\partial t^2} - \triangle p = f, \qquad \text{in } \ \Omega \times (0, T), \\
p(\cdot, 0) = 0, \ \ \frac{\partial p}{\partial t}(\cdot, 0) = 0, \ \text{ in } \ \Omega, \\
p\big|_{\Gamma} = 0, \ \text{ on } \ \Gamma \times (0, T),
\end{aligned}
\tag{18}
$$

where we define $\alpha = \frac{1}{c^2}$.

The Hessian for the acoustic wave equation

In the acoustic case the second derivative L'' takes the form

$$
\begin{aligned}
L''(u;\bar{u},\tilde{u}) = &-((\alpha D\tilde{p}, D\bar{\varphi})) + ((\nabla\tilde{p},\nabla\bar{\varphi})) \\
&+ ((\bar{p},\tilde{p}))_{\delta_{obs}} - ((\bar{\alpha}D\tilde{p}, D\varphi)) \\
&- ((\alpha D\bar{p}, D\tilde{\varphi})) + ((\nabla\bar{p},\nabla\tilde{\varphi})) - ((\bar{\alpha}Dp, D\tilde{\varphi})) \\
&- ((\tilde{\alpha}Dp, D\bar{\varphi})) - ((\tilde{\alpha}D\bar{p}, D\varphi)) + \gamma(\bar{\alpha},\tilde{\alpha}),
\end{aligned}
$$

and the Hessian problem takes the following strong form:

$$
\begin{aligned}
\alpha D^2\tilde{\varphi} - \Delta\tilde{\varphi} + \tilde{p}_{\delta_{obs}} + D^2\varphi\tilde{\alpha} &= \psi_1, \\
\alpha D^2\tilde{p} - \Delta\tilde{p} + D^2 p\tilde{\alpha} &= \psi_2, \\
\int_0^T D^2\varphi\tilde{p}\, dt + \int_0^T \tilde{\varphi}D^2 p\, dt + \gamma\tilde{\alpha} &= \psi_3,
\end{aligned}
\tag{19}
$$

together with initial and boundary conditions.

Algorithm

To get error estimator, we solve iteratively system (19). The iterative algorithm is:

1. Find (p, α, φ), using quasi-Newton method, where p is solution of the state problem, φ is solution of the adjoint problem and α are discrete values of the identification parameter found from the optimality condition.

Algorithm

To get error estimator, we solve iteratively system (19). The iterative algorithm is:

1. Find (p, α, φ), using quasi-Newton method, where p is solution of the state problem, φ is solution of the adjoint problem and α are discrete values of the identification parameter found from the optimality condition.
2. Choose guess for $\psi = (\psi_1, \psi_2, \psi_3)$, for example, $\psi = (0, 0, 1)$.

Algorithm

To get error estimator, we solve iteratively system (19). The iterative algorithm is:

1. Find (p, α, φ), using quasi-Newton method, where p is solution of the state problem, φ is solution of the adjoint problem and α are discrete values of the identification parameter found from the optimality condition.
2. Choose guess for $\psi = (\psi_1, \psi_2, \psi_3)$, for example, $\psi = (0, 0, 1)$.
3. From third equation of system (19) eliminate $\tilde{\alpha}$ using equation

$$\tilde{\alpha}^{new} = \tilde{\alpha}^{old} + \rho(\psi_3 - \int_0^T \tilde{\varphi} D^2 p \, dt - \gamma \tilde{\alpha}^{old}) \tag{20}$$

with already computed (p, α, φ).

Algorithm

To get error estimator, we solve iteratively system (19). The iterative algorithm is:

1. Find (p, α, φ), using quasi-Newton method, where p is solution of the state problem, φ is solution of the adjoint problem and α are discrete values of the identification parameter found from the optimality condition.
2. Choose guess for $\psi = (\psi_1, \psi_2, \psi_3)$, for example, $\psi = (0, 0, 1)$.
3. From third equation of system (19) eliminate $\tilde{\alpha}$ using equation
$$\tilde{\alpha}^{new} = \tilde{\alpha}^{old} + \rho(\psi_3 - \int_0^T \tilde{\varphi} D^2 p \, dt - \gamma \tilde{\alpha}^{old}) \tag{20}$$
with already computed (p, α, φ).
4. From second equation eliminate $\tilde{p}$ solving scalar wave equation
$$\alpha D^2 \tilde{p} - \Delta \tilde{p} = \psi_2 - D^2 p \tilde{\alpha} \tag{21}$$

Algorithm

To get error estimator, we solve iteratively system (19). The iterative algorithm is:

1. Find (p, α, φ), using quasi-Newton method, where p is solution of the state problem, φ is solution of the adjoint problem and α are discrete values of the identification parameter found from the optimality condition.
2. Choose guess for $\psi = (\psi_1, \psi_2, \psi_3)$, for example, $\psi = (0, 0, 1)$.
3. From third equation of system (19) eliminate $\tilde{\alpha}$ using equation

$$\tilde{\alpha}^{new} = \tilde{\alpha}^{old} + \rho(\psi_3 - \int_0^T \tilde{\varphi} D^2 p \, dt - \gamma \tilde{\alpha}^{old}) \tag{20}$$

with already computed (p, α, φ).

4. From second equation eliminate $\tilde{p}$ solving scalar wave equation

$$\alpha D^2 \tilde{p} - \Delta \tilde{p} = \psi_2 - D^2 p \tilde{\alpha} \tag{21}$$

5. From first equation eliminate $\tilde{\varphi}$ solving scalar wave equation

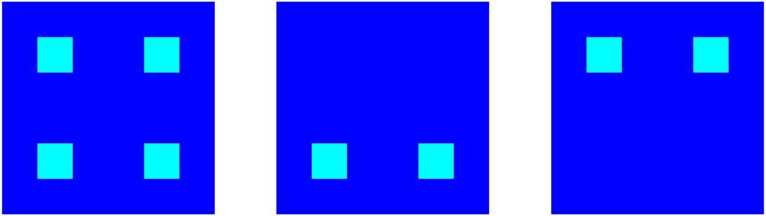
$$\alpha D^2 \tilde{\varphi} - \Delta \tilde{\varphi} = \psi_1 - \tilde{p}_{\delta_{obs}} \tag{22}$$

Algorithm

To get error estimator, we solve iteratively system (19). The iterative algorithm is:

1. Find (p, α, φ), using quasi-Newton method, where p is solution of the state problem, φ is solution of the adjoint problem and α are discrete values of the identification parameter found from the optimality condition.
2. Choose guess for $\psi = (\psi_1, \psi_2, \psi_3)$, for example, $\psi = (0, 0, 1)$.
3. From third equation of system (19) eliminate $\tilde{\alpha}$ using equation
$$\tilde{\alpha}^{new} = \tilde{\alpha}^{old} + \rho(\psi_3 - \int_0^T \tilde{\varphi} D^2 p\, dt - \gamma \tilde{\alpha}^{old}) \tag{20}$$
with already computed (p, α, φ).
4. From second equation eliminate $\tilde{p}$ solving scalar wave equation
$$\alpha D^2 \tilde{p} - \Delta \tilde{p} = \psi_2 - D^2 p \tilde{\alpha} \tag{21}$$
5. From first equation eliminate $\tilde{\varphi}$ solving scalar wave equation
$$\alpha D^2 \tilde{\varphi} - \Delta \tilde{\varphi} = \psi_1 - \tilde{p}_{\delta_{obs}} \tag{22}$$
6. Repeat steps $2 - 4$ until desired convergence is achieved.

A two-dimensional photonic crystal.



We show the square lattice of a crystal from above. Material is a square lattice of columns with wave speed $c(x)$. The material is homogeneous in the z direction and periodic along x and y with lattice constant a.

Applications of photonic crystals

1. Reflecting dielectric, which reflects light (to control light propagation in the microwave regime with wavelengths from 1 mm to 10 cm);

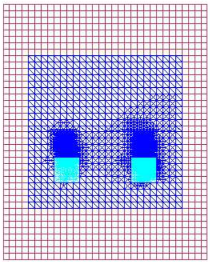
Applications of photonic crystals

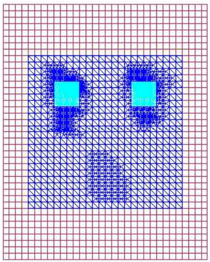
1. Reflecting dielectric, which reflects light (to control light propagation in the microwave regime with wavelengths from 1 mm to 10 cm);
2. Resonant cavity, which traps light (placement of the defects to serve as a resonant cavity which are crucial components of laser systems);

Applications of photonic crystals

1. Reflecting dielectric, which reflects light (to control light propagation in the microwave regime with wavelengths from 1 mm to 10 cm);
2. Resonant cavity, which traps light (placement of the defects to serve as a resonant cavity which are crucial components of laser systems);
3. Waveguide, which transports the light (using line defects to guide light from one location to another): optoelectronic circuit, fi ber-optic network.

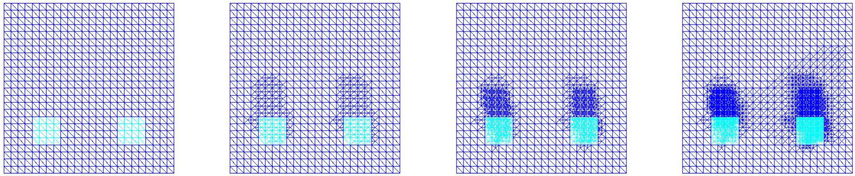
Hybrid FEM/FDM method





Hybrid mesh is a combination of an structured mesh, where FDM is applied, and an unstructured mesh, where we use FEM, with a thin overlapping of structured elements.

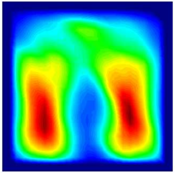
Adaptively refined computational meshes for reconstruction of the lower columns in square lattice.



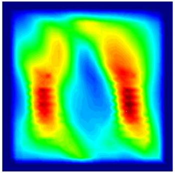
625 nodes 1152 elements	809 nodes 1520 elements	1263 nodes 2428 elements	2225 nodes 4352 elements

Solution to the forward problem

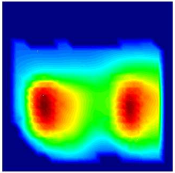
Reconstruction of the lower columns in square lattice.



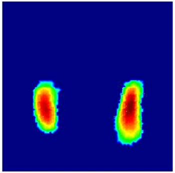
8 Q.N. it.
625 nodes



4 Q.N. it.
809 nodes



5 Q.N. it.
1263 nodes



6 Q.N.it.
2225 nodes

We show reconstructed parameter $c(x)$, indicating domains with a given parameter value: red color corresponds to the maximum parameter value (c=4) on the corresponding meshes, and blue color - to the minimum (c=1).

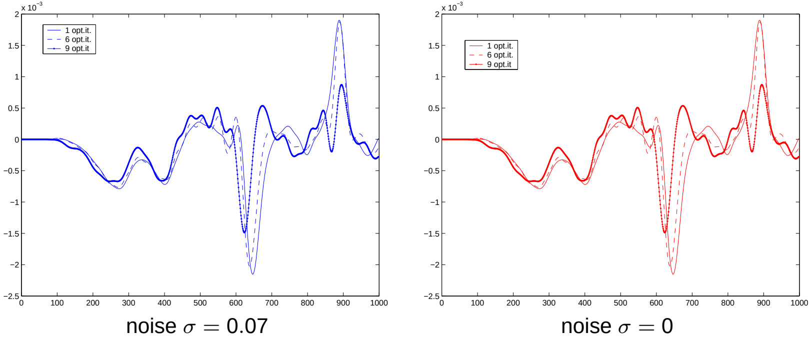
opt.it.	625 nodes	809 nodes	1263 nodes	2225 nodes
1	0.0118349	0.0108764	0.0108764	0.010476
2	0.0095824	0.00987447	0.00965067	0.00954041
3	0.00822312	0.00709372	0.00558728	0.00769998
4	0.00748565	0.00318215	0.00273809	0.00313069
5	0.00619674	0.00291434		
6	0.00528474			
7	0.00471419			
8	0.00354939			
9				
10				

Table: $||p - p_{obs}||$ on adaptively refi ned meshes in reconstruction of the lower columns in square lattice. Number of stored corrections in quasi-Newton method is $m = 15$. Computations was performed with noise level $\sigma = 0$ and regularization parameter $\epsilon = 0.1$.

σ, ϵ	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}
0	0.00630036	0.00630536	0.00475773	0.0046071	0.00313069
0.01	0.00650122	0.00642409	0.00489691	0.00425432	0.00317147
0.03	0.00671315	0.00644934	0.00572624	0.00427946	0.00317955
0.05	0.0068622	0.00661597	0.00639352	0.00428971	0.00318703
0.07	0.00731985	0.00598225	0.00631647	0.00462458	0.00312281
0.1	0.00672832	0.00618862	0.00673036	0.00467998	0.00331152
0.2	0.00702925	0.00696454	0.00640261	0.00448304	0.0037926

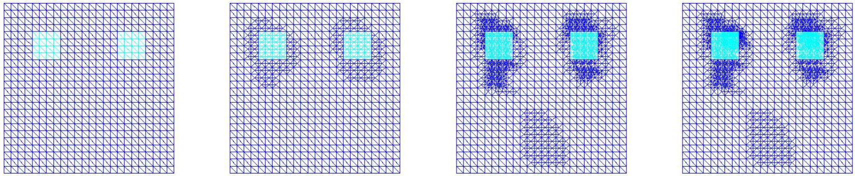
Table: $||p - p_{obs}||$ for the best reconstraction of the lower columns in square lattice. We present results for different noise levels σ and regularization parameters ϵ.

L_2-norms in space of adjoint problem solution



L_2-norms in space of adjoint problem solution λ_h in reconstruction of the lower columns in square lattice. on different optimization iterations. Here the x-axis denotes time steps on $[0, 25.0]$.

Adaptively refined computational meshes for reconstruction of the upper columns in square lattice.



625 nodes
1152 elements

844 nodes
1590 elements

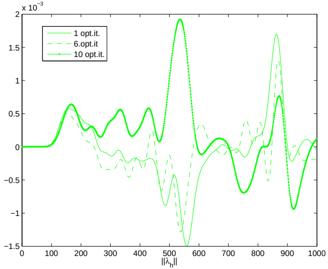
1592 nodes
3086 elements

1945 nodes
3792 elements

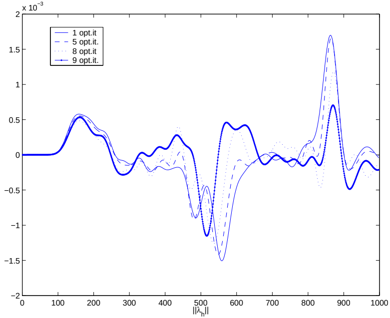
σ, ϵ	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}
0	0.00548847	0.00549544	0.00549544	0.00512397	0.00340977
0.01	0.00547518	0.00549755	0.00489691	0.0055677	0.00345097
0.03	0.00545709	0.00550747	0.00572624	0.0055182	0.0040041
0.05	0.00548414	0.00548424	0.00639352	0.0055076	0.00357293
0.07	0.00544183	0.00544645	0.00631647	0.00552189	0.00353966
0.1	0.00543398	0.00548045	0.00673036	0.00552947	0.00430008
0.2	0.00561054	0.00561999	0.00640261	0.00566159	0.00386997

Table: $||p - p_{obs}||$ for the best reconstraction of the upper columns in square lattice. We present results for different noise levels σ and regularization parameters ϵ.

L_2-norms in space of adjoint problem solution



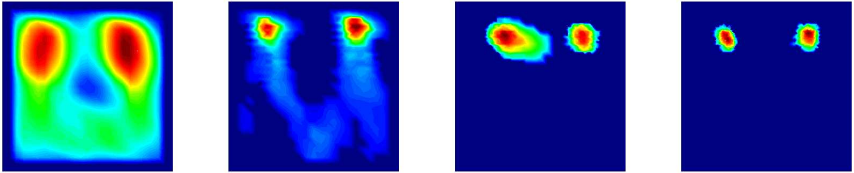
noise $\sigma = 0.07$



noise $\sigma = 0$

L_2-norms in space of adjoint problem solution λ_h in reconstruction of the upper columns in square lattice. on different optimization iterations. Here the x-axis denotes time steps on $[0, 25.0]$.

Reconstruction of the upper columns in square lattice.



9 Q.N. it. 625 nodes	10 Q.N. it. 844 nodes	10 Q.N. it. 1592 nodes	10 Q.N.it. 1945 nodes

Reconstructed parameter $c(x)$ computed with wave frequency $\omega = 25$ on different adaptively refined meshes after different number of quasi-Newton iterations. We show parameter $c(x)$, indicating domains with a given parameter value: red color corresponds to the maximum parameter value (c=4) on the corresponding meshes, and blue color - to the minimum (c=1).