# Two-phase free boundary problems and the Friedland-Hayman inequality

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# A two-phase free boundary problem

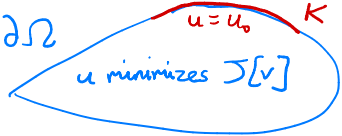
Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, convex domain, with $K \subset \partial\Omega$ closed. Consider the functional

$$J[v] = \int_\Omega |\nabla v|^2 + 1_{\{v>0\}} \, \mathrm{d}x.$$

Here $v \in H^1(\Omega)$, with $v = u_0 \in C^\infty(K)$ on $K$, and $1_{\{v>0\}}$ is the indicator function of the set $\{v > 0\}$.

We assume that $u_0$ takes positive and negative values on $K$ (two-phase).

It is straightforward to establish the existence of the minimizer $u \in H^1(\Omega)$.




**Aim**

*Determine what further regularity the minimizer u has.*

Application to the irrotational flow of two ideal fluids, and other applications in fluid mechanics, electromagnetism, and optimal shape design.

# Properties of the minimizer

**Aim**

*Determine what regularity the minimizer u of* $J[v] = \int_\Omega |\nabla v|^2 + 1_{\{v>0\}}$ *has.*

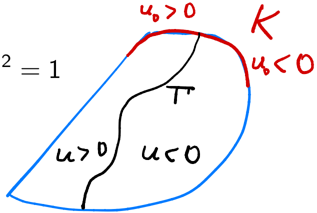
Formally, the Euler-Lagrange equations $J'[u] = 0$ are

1) $u$ is harmonic in the positive phase $\Omega^+ = \{u > 0\}$ and non-negative phase $\Omega^- = \{u \leq 0\}$;
2) $\partial_\nu u = 0$ on the Neumann part of the boundary $\partial\Omega \backslash K$;
3) $u$ satisfies the gradient jump condition

$$|\nabla u^+(x)|^2 - |\nabla u^-(x)|^2 = 1$$

on $\Gamma = \partial\Omega^+ \cap \partial\Omega^-$ (the free boundary).

Cartoon picture of the two-phase minimizer:




But a priori, $u$ is only in $H^1(\Omega)$ and so a major goal of the regularity theory is to show that 3) holds in a suitable sense.

# Properties of the minimizer

Using the fact that $u$ is a minimizer it is (fairly) straightforward to show that it satisfies the following properties:
(Alt-Caffarelli-Friedman '84, Gurevich '99, Raynor '08)

1) $u$ is subharmonic in $\Omega$ and harmonic in the two phases $\Omega^+ = \{u > 0\}$ $\Omega^- = \{u \leq 0\}$
(that is, $\Delta u$ is a positive measure supported on the free boundary)

2) $u$ is Hölder continuous (up to the boundary) for some exponent $\alpha > 0$

3) $\partial_\nu u = 0$ weakly on the Neumann boundary $\partial\Omega \setminus K$

The key idea behind proving these properties is to combine $u$ minimizing the functional with harmonic replacement.

The first major step in the regularity theory is to determine if $u$ is **Lipschitz continuous**.

# Lipschitz continuity of minimizers

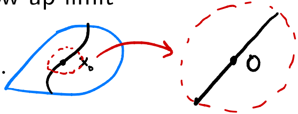
## Theorem (Alt-Caffarelli-Friedman (ACF), '84)

*The minimizer is Lipschitz continuous in the interior of $\Omega$.*

Why is Lipschitz continuity a key step in the regularity theory?

It allows a rescaling of $u$ by dilation and to study the blow-up limit

$$u^0(x) = \lim_{r\to 0} \frac{u(x_0 + rx) - u(x_0)}{r}.$$



This is used by ACF to show that minimizer and free boundary are smooth.

## Question

*Is the minimizer $u$ Lipschitz continuous up to the Neumann boundary?*

$u$ may only be Hölder continuous at the intersection of $K$ with the free boundary (Gurevich '99).

Convexity is a natural (and close to necessary) restriction on $\Omega$ for a positive answer.

# The Alt-Caffarelli-Friedman functional

**Theorem (Alt-Caffarelli-Friedman (ACF), '84)**

*The minimizer is Lipschitz continuous in the interior of $\Omega$.*

To prove this interior Lipschitz regularity they introduced the following functional:

$$\Phi(t) = \left(\frac{1}{t^2}\int_{B_t(x_0)} \frac{|\nabla u^+|^2}{|x - x_0|^{n-2}}\,\mathrm{d}x\right)\left(\frac{1}{t^2}\int_{B_t(x_0)} \frac{|\nabla u^-|^2}{|x - x_0|^{n-2}}\,\mathrm{d}x\right)$$

Here $x_0$ is an interior point on the free boundary and $t > 0$.

**Proposition (Monotonicity of the ACF functional, '84)**

*The functional $\Phi(t)$ is a monotone increasing function of $t$, and so in particular $\Phi(t)$ is uniformly bounded by $\Phi(1)$ for all $0 < t \leq 1$.*

This proposition is the key step in their proof of Lipschitz continuity.

**Remark**

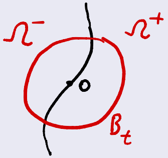
*In the one phase case, Lipschitz continuity can be obtained without using the functional (Alt-Caffarelli '81, Raynor '08).*

# The Alt-Caffarelli-Friedman functional

## Proposition (Alt-Caffarelli-Friedman, '84)

*The functional* $\Phi(t)$

$$\Phi(t) = \left(\frac{1}{t^2}\int_{B_t}\frac{|\nabla u^+|^2}{|x|^{n-2}}\,\mathrm{d}x\right)\left(\frac{1}{t^2}\int_{B_t}\frac{|\nabla u^-|^2}{|x|^{n-2}}\,\mathrm{d}x\right)$$




*is a monotone increasing function of* $t$.

Idea of the proof: By direct calculation,

$$\frac{\Phi'(1)}{\Phi(1)} = \frac{\int_{\partial B_1}|\nabla u^+|^2\,\mathrm{d}\sigma}{\int_{B_1}\frac{|\nabla u^+|^2}{|x|^{n-2}}\,\mathrm{d}x} + \frac{\int_{\partial B_1}|\nabla u^-|^2\,\mathrm{d}\sigma}{\int_{B_1}\frac{|\nabla u^-|^2}{|x|^{n-2}}\,\mathrm{d}x} - 4$$

and also

$$\int_{\partial B_1}|\nabla u^\pm|^2 \geq \int_{\partial B_1}|\partial_r u^\pm|^2 + \lambda^\pm(1)\int_{\partial B_1}|u^\pm|^2,$$

$$\int_{B_1}\frac{|\nabla u^\pm|^2}{|x|^{n-2}} \leq \left(\int_{\partial B_1}(u^\pm)^2\right)^{1/2}\left(\int_{\partial B_1}(\partial_r u^\pm)^2\right)^{1/2} + \frac{n-2}{2}\int_{\partial B_1}(u^\pm)^2.$$

Here $\lambda^+(1)$ is the first Dirichlet eigenvalue of $\{u > 0\} \cap \partial B_1$.

# The Alt-Caffarelli-Friedman functional

Setting

$$z^{\pm} = \int_{\partial B_1} |\partial_r u^{\pm}|^2, \qquad w^{\pm} = \int_{\partial B_1} |u^{\pm}|^2,$$

therefore gives

$$\frac{\Phi'(1)}{\Phi(1)} \geq \frac{z^+ + \lambda^+(1) w^+}{(z^+ w^+)^{1/2} + \frac{n-2}{2} w^+} + \frac{z^- + \lambda^-(1) w^-}{(z^- w^-)^{1/2} + \frac{n-2}{2} w^-} - 4.$$

It then becomes a calculus exercise to minimize the right hand side over $z^{\pm}$, $w^{\pm} \geq 0$,

$$\frac{\Phi'(1)}{\Phi(1)} \geq 2 \left[ -\frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} + \lambda^+(1)} - \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} + \lambda^-(1)} - 2 \right].$$

**Question**

*Is this right hand side positive?*

# The Friedland-Hayman inequality

To answer this, consider the following eigenvalue problem on $\mathbb{S}^{n-1}$.

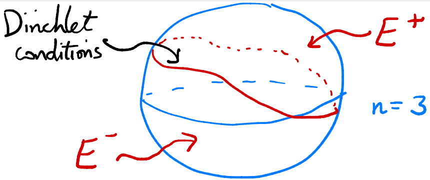
## Definition

*Given disjoint subsets $E^{\pm}$ of $\mathbb{S}^{n-1}$, define $\lambda(E^{\pm})$ to be the first Dirichlet eigenvalue of $E^{\pm}$.*

Call

$$\alpha(E^{\pm}) = -\frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} + \lambda(E^{\pm})}$$

the characteristic exponent of $E^{\pm}$.




## Theorem (Friedland-Hayman '76, Beckner-Kenig-Pipher '88)

*The characteristic exponents $\alpha(E^{\pm})$ satisfy*

$$\alpha(E^+) + \alpha(E^-) \geq 2.$$

*Equality if and only if $E^{\pm}$ are hemispheres.*

# The Alt-Caffarelli-Friedman functional

## Theorem (Friedland-Hayman '76, Beckner-Kenig-Pipher '88)

*The characteristic exponents $\alpha(E^{\pm})$ satisfy*

$$\alpha(E^{+}) + \alpha(E^{-}) \geq 2.$$

*Equality if and only if $E^{\pm}$ are hemispheres.*

The lower bound on $\Phi'(1)/\Phi(1)$ can be written as

$$\frac{\Phi'(1)}{\Phi(1)} \geq 2(\alpha^{+}(1) + \alpha^{-}(1) - 2).$$

So the monotonicity of $\Phi$ follows from the Friedland-Hayman inequality!

Strict monotonicity unless $\{u > 0\} \cap B_t$, $\{u \leq 0\} \cap B_t$ are hemispheres.

## Remark

*The characteristic exponent $\alpha(E^{\pm})$ is the positive homogeneities of the harmonic extensions of the eigenfunctions to the cone generated by $E^{\pm}$.*

# Regularity near the convex boundary

So, the Friedland-Hayman inequality directly gives the monotonicity of $\Phi(t)$ and leads to the interior Lipschitz regularity of the minimizer.

**Question**

*Can we extend the Lipschitz continuity to the convex Neumann boundary?*

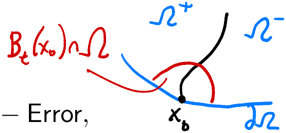
A natural change of functional for $x_0 \in \partial\Omega$ is

$$\Psi(t) = \left( \frac{1}{t^2} \int_{B_t(x_0)\cap\Omega} \frac{|\nabla u^+|^2}{|x - x_0|^{n-2}} \, \mathrm{d}x \right) \left( \frac{1}{t^2} \int_{B_t(x_0)\cap\Omega} \frac{|\nabla u^-|^2}{|x - x_0|^{n-2}} \, \mathrm{d}x \right).$$

Just as in the interior case, Lipschitz regularity reduces to the boundedness of $\Psi(t)$.

Following the calculation in the interior case gives

$$\Psi'(1)/\Psi(1) \geq 2(\alpha^+(1) + \alpha^-(1) - 2) - \text{Error},$$




with the Error term on $\partial\Omega$ measuring the non-conic nature of the boundary.

But the characteristic exponents are now different!

# A variant of the Friedland-Hayman inequality

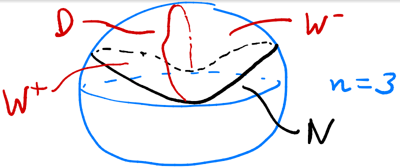
## Definition

*Let $W \subset \mathbb{S}^{n-1}$ be a geodesically convex subset of $\mathbb{S}^{n-1}$. Given disjoint subsets $W^{\pm}$ of $W$, define $\mu(W^{\pm})$ to be the first eigenvalue of $W^{\pm}$ with Neumann boundary conditions on $\partial W^{\pm} \cap \partial W$ and Dirichlet boundary conditions otherwise.*

Again, call

$$\alpha(W^{\pm}) = -\frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} + \mu(W^{\pm})}$$

the characteristic exponent of $W^{\pm}$.




## Theorem (B-Jerison-Raynor '20)

*The characteristic exponents $\alpha(W^{\pm})$ satisfy*

$$\alpha(W^{+}) + \alpha(W^{-}) \geq 2.$$

## Remark (Work in preparation with David Jerison)

*Equality precisely when $W \subset \mathbb{S}^{n-1}$ has antipodal points.*

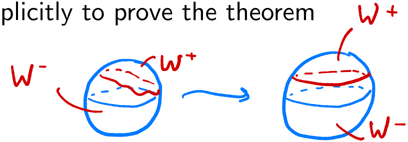
# A variant of the Friedland-Hayman inequality

**Theorem (B-Jerison-Raynor '20)**

*The characteristic exponents* $\alpha(W^{\pm})$ *satisfy*

$$\alpha(W^{+}) + \alpha(W^{-}) \geq 2.$$

On $\mathbb{S}^1$ the eigenvalues can be computed explicitly to prove the theorem (Gemmer-Moon-Raynor '18).

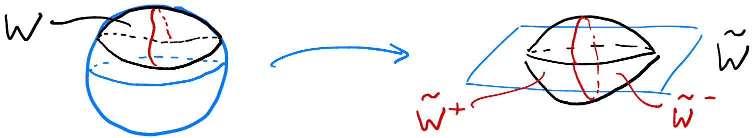



The key steps in the proof of the original Friedland-Hayman inequality:

1) A symmetrization argument to reduce to studying Dirichlet eigenvalues of spherical caps;
2) Obtain a lower bound for spherical caps either by a direct numerical calculation or comparing to Gaussian eigenvalues.

Step 1) breaks down in our Dirichlet-Neumann case.

# A variant of the Friedland-Hayman inequality



Key steps in the proof of the Dirichlet-Neumann version of the inequality:

1) Construct a closed manifold $\tilde{W}$ by gluing two copies of $W$ together along its convex boundary;
2) Can ensure that $\tilde{W}$ is smooth with a Ricci curvature lower bound of $1^-$;
3) The Dirichlet-Neumann eigenvalues $\mu(W^\pm)$ become Dirichlet eigenvalues on the doubled sets $\tilde{W}^\pm$;
4) An application of the Lévy-Gromov isoperimetric inequality bounds $\mu(W^\pm)$ from below in terms of eigenvalues of the sphere (Gromov '99, Bérard-Meyer '82);
5) The original Friedland-Hayman inequality then gives the result.

# Back to the monotonicity of the functional

## Theorem (Gemmer-Moon-Raynor, '18)

*In 2-dimensions, the functional $\Psi(t)$ is monotonically increasing, and the minimizer is Lipschitz continuous up to the Neumann boundary.*

In all dimensions higher than 2, we run into an issue when bounding the functional

$$\Psi(t) = \left( \frac{1}{t^2} \int_{B_t(x_0)\cap\Omega} \frac{|\nabla u^+|^2}{|x - x_0|^{n-2}} \, \mathrm{d}x \right) \left( \frac{1}{t^2} \int_{B_t(x_0)\cap\Omega} \frac{|\nabla u^-|^2}{|x - x_0|^{n-2}} \, \mathrm{d}x \right)$$

for $x_0 \in \partial\Omega$.

In general, the spherical slices $\partial B_t(x_0) \cap \Omega$ will not be geodesically convex, and so our Friedland-Hayman inequality does not directly apply.

However, the limiting spherical slice $V_0 = \lim_{t\to 0} t^{-1} \left(\partial B_t(x_0) \cap \Omega\right)$ is geodesically convex.

## Remark

*When $\Omega$ is a cone with vertex at $x_0$, then this problem vanishes, and $\Psi(t)$ is monotonic.*

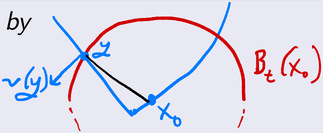
# A Dini condition on the spherical slices

We therefore want to measure the rate at which the slices $V_t = t^{-1}(\partial B_t(x_0) \cap \Omega)$ approach the limiting slice.

## Definition

*Given* $x_0 \in \partial\Omega$, $t \in (0,1)$, *define the function* $M_{x_0}(t)$ *by*

$$M_{x_0}(t) = \sup_{y \in \partial\Omega : |y - x_0| \le t} \nu(y) \cdot (y - x_0).$$




We impose a Dini integrability assumption on the rate that $M_{x_0}(t)$ approaches 0.

## Assumption (Dini condition for $t^{-1}M_{x_0}(t)$)

*There exists a constant* $C_*$ *such that for all* $x_0 \in \partial\Omega$,

$$\int_{0^+}^{1} \frac{M_{x_0}(t)}{t^2}\, \mathrm{d}t < C_*.$$

This is a sufficient condition to bound the functional $\Psi(t)$.

# Lipschitz regularity under the Dini condition

**Assumption (Dini condition for $t^{-1}M_{x_0}(t)$)**

*There exists a constant $C_*$ such that for all $x_0 \in \partial\Omega$,*

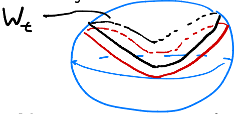
$$\int_{0^+}^{1} \frac{M_{x_0}(t)}{t^2}\,\mathrm{d}t < C_*, \qquad M_{x_0}(t) = \sup_{y\in\partial\Omega:|y-x_0|\leq t} \nu(y)\cdot(y-x_0).$$

**Theorem (B.-Jerison-Raynor '20)**

*Under this assumption on $\partial\Omega$, the functional $\Psi(t)$ is bounded and the minimizer is Lipschitz continuous up to the boundary.*

The quantity $M_{x_0}(t)$ plays a role, as there exists a geodesically convex set $W_t$ with

$\mathrm{Haus}_{\mathbb{S}^{n-1}}\left(W_t, t^{-1}\left(\partial B_t(x_0)\cap\Omega\right)\right) \leq Ct^{-1}M_{x_0}(t).$




Transferring this Hausdorff distance control to Dirichlet-Neumann eigenvalues of the spherical slices, the Dini condition ensures that $\Psi'(t)/\Psi(t)$ is integrable.

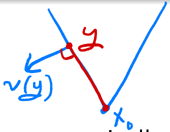
# Discussion of the Dini condition

**Assumption (Dini condition for $t^{-1}M_{x_0}(t)$)**

*There exists a constant $C_*$ such that for all $x_0 \in \partial\Omega$,*

$$\int_{0^+}^{1} \frac{M_{x_0}(t)}{t^2}\,\mathrm{d}t < C_*, \qquad M_{x_0}(t) = \sup_{y \in \partial\Omega : |y - x_0| \leq t} \nu(y) \cdot (y - x_0).$$

Some remarks:




1) If $\Omega$ is a cone with vertex at $x_0$, then $M_{x_0}(t) \equiv 0$;
2) Trivially, $M_{x_0}(t) \leq t$, and so $t^{-2}M_{x_0}(t)$ just fails to be automatically integrable;
3) The Dini condition holds for any $C^{1,\beta}$-domain;
4) We found it very challenging to find an example where the condition fails. In fact, it always holds for 2-dimensional convex domains, but we have a (very) delicate counterexample in 3 and higher dimensions.

# Future directions

**Question**

*Can we establish the Lipschitz continuity result for domains which fail the Dini condition?*

Two possible approaches:

1) Change the definition of monotonicity functional from spherical slices;
2) Interior regularity has been established by Dipierro-Karakhanyan '18 without using the monotonicity formula.

**Question**

*Is the free boundary smooth up to the boundary?*

Just as in the interior case, the Friedland-Hayman inequality (plus case of equality) should allow for a classification of blow-up limits.

Numerical evidence in 2 dimensions (Gemmer-Moon-Raynor) that the free boundary avoids corners and meets the convex boundary at right angles.

Thank you for your attention!