Spectral structure and arithmetic progressions

Thomas Bloom

University of Cambridge

Caltech/UCLA Analysis Seminar

November 2020

What's additive combinatorics all about?

Finding WEAK additive structures under WEAK hypotheses.

- (In any abelian group - usually either $\mathbb{Z}/N\mathbb{Z}$ or $\mathbb{F}_p^n$.)
- The weakest kind of structure you could come up with, that involves some addition, is a three-term arithmetic progression $x, x+d, x+2d$ (hereafter a 3AP).
- So perhaps one of the most natural questions in additive combinatorics is:

What conditions on a set are enough to guarantee that it contains a 3AP?

- Obviously there are infinite sets that contain no 3APs (e.g. $\{1, 2, 4, 8, 16, \ldots\}$).
- But these are very sparse – the reason they don't contain any progressions is that the gaps between successive members keeps increasing.
- After some experimentation, it seems that any set without 3APs must be 'sparse'.

- This kind of question was first considered by Erdős and Turán in 1936. They proved a couple of elementary estimates, and conjectured the following.

Conjecture (Erdős-Turán 1936)

If $A \subset \mathbb{N}$ is such that A contains no 3APs then

$$\lim_{N\to\infty} \frac{|A \cap \{1, \ldots, N\}|}{N} \to 0.$$

- In fact, in later years, Erdős conjectured something even stronger.

Conjecture (Erdős)

If $A \subset \mathbb{N}$ is such that A contains no 3APs then $\sum_{a\in A} \frac{1}{a}$ converges.

- It took some time before the Erdős-Turán conjecture was proved.

Theorem (Roth 1953)

If $A \subset \mathbb{N}$ is such that A contains no 3AP then

$$\lim_{N\to\infty} \frac{|A \cap \{1, \ldots, N\}|}{N} \to 0.$$

- It took even longer for the stronger Erdős version to be proved.

Theorem (Bloom-Sisask 2020)

If $A \subset \mathbb{N}$ is such that A contains no 3APs then $\sum_{a\in A} \frac{1}{a}$ converges.

- As a consequence, we know that the primes have infinitely many three-term arithmetic progressions (already known since the 1930s).
- In fact, any dense subset of the primes has infinitely many three-term arithmetic progressions (already proved by Green in 2005, then extended to arbitrary length progressions by Green-Tao in 2006).
- These proofs use a lot of number theoretic machinery. Now we know that it is true not because of any special properties of the primes - just that there are lots of them.

- In fact, Erdős conjectured that both of these conjectures should be true if we replace 3AP by kAP for any k. The analogue of the first conjecture was proved in 1975 by Szemerédi:

Theorem (Szemerédi 1975)

For any $k \geq 3$, if $A \subset \mathbb{N}$ is such that A contains no kAPs then

$$\lim_{N \to \infty} \frac{|A \cap \{1, \ldots, N\}|}{N} \to 0.$$

- The second, harder, conjecture, is still wide open for the general case. It is has the largest bounty of any surviving Erdős conjecture ($3000):

Conjecture (Erdős)

For any $k \geq 3$, if $A \subset \mathbb{N}$ is such that A contains no kAPs then $\sum_{a \in A} \frac{1}{a}$ converges.

- The qualitative fact $r(N) = o(N)$ was proved by Roth (1953), using an adaptation of the circle method.
- Since then several very different proofs of this - using combinatorics, ergodic theory, harmonic analysis, or various combinations of these.
- For the quantitative question "how quickly does $r(N)/N$ decay?" combinatorics and ergodic theory do terribly, not even able to match Roth's original bound of $r(N) \ll N/\log\log N$.

Roth 1953	$\frac{N}{\log\log N}$
Szemerédi 1986	$\exp(-O(\log\log N)^{1/2}))N$
Heath-Brown 1987	$\frac{N}{(\log N)^c}$ for some tiny $c > 0$
Szemerédi 1990	$\frac{N}{(\log N)^{1/4-o(1)}}$
Bourgain 1999	$\frac{N}{(\log N)^{1/2-o(1)}}$
Bourgain 2008	$\frac{N}{(\log N)^{2/3-o(1)}}$

Sanders 2012	$\frac{N}{(\log N)^{3/4-o(1)}}$
Sanders 2011	$\frac{(\log\log N)^6}{\log N}N$
Bloom 2014	$\frac{(\log\log N)^4}{\log N}N$
Bloom-Sisask 2019	$\frac{(\log\log N)^7}{\log N}N$
Schoen 2020	$\frac{(\log\log N)^{3+o(1)}}{\log N}N$

- Our actual main result is the following bound.

Theorem (Bloom-Sisask 2020)

There exists a constant $c > 0$ such that

$$r(N) \ll \frac{N}{(\log N)^{1+c}}.$$

- In particular, this is $o(N/\log N)$ (and so if A has no non-trivial 3APs then $\sum \frac{1}{a}$ converges).
- The value of the constant c is in principle effectively computable, but very very tiny. Something like

$$c = 2^{-2^{2^{1000}}}$$

should work.

- This particular type of bound might look surprising if you're new to the area. But it came as no surprise to additive combinatorialists.
- Our result uses many of the ideas from previous work by Bateman and Katz on the 'cap set problem': which asks for the maximal size of a subset of $\mathbb{F}_3^n$ that contains no 3APs.

Theorem (Bateman-Katz 2010)

There exists a constant $c > 0$ such that

$$r(\mathbb{F}_3^n) \ll \frac{3^n}{n^{1+c}}.$$

- The use of $\mathbb{F}_3^n$ as an easier 'model setting' for understanding the integers is well-known, and the result of Bateman and Katz, especially as it used Fourier-analytic techniques, raised the hope that a similar result could be proved for the integers.

- Our proof does use the ideas of Bateman and Katz, translated to the integers, and our result would not have been possible without this breakthrough.
- There are significant difficulties in performing this translation, and several other new ideas were necessary.

- Actually, for $\mathbb{F}_3^n$ a completely different method has since done much better than Bateman and Katz with a much simpler proof!
- Ellenberg and Gijswijt have shown, using a new polynomial method introduced by Croot, Lev, and Pach, that

$$r(\mathbb{F}_3^n) \leq 2.756^n.$$

- There does not seem to be any way to adapt these polynomial methods to the integers, however. (Unlike the previous Fourier analytic techniques.)

- The bound of $r(N) \ll N/(\log N)^{1+c}$ for some tiny $c > 0$ is very unlikely to be sharp!
- The best lower bound, due to Behrend (1946) (with slight refinements by Elkin and Green-Wolf), is

$$r(N) \gg \frac{N}{C^{\sqrt{\log N}}}$$

for some constant $C > 1$.
- We believe that the lower bound is closer to the truth.

Conjecture

There exists $c > 0$ and $C > 1$ such that

$$r(N) \ll \frac{N}{C^{(\log N)^c}}.$$

- Now let's talk about the proof...

- Let's approach the problem of estimating $r(N)$, as Roth did, analytically, using the Fourier transform.
- We first ask a different question: not "how big can A be if there are no non-trivial solutions to $x + y = 2z$" but instead "can we get a lower bound for the number of solutions to $x + y = 2z$ knowing only $A \subset \{1, \ldots, N\}$ and the size of A?"
- The second leads directly to an answer to the first, since if there are only trivial solutions, then there are exactly $|A|$ of them – comparing this to the lower bound and rearranging gives an upper bound on $|A| / N$.

From few 3APs to the large spectrum

- We work with some $A \subset G = \mathbb{Z}/N\mathbb{Z}$ of size $|A| = \alpha N$.
- We will use the compact normalisation for G and the discrete normalisation for its dual group, so that, for example, if $\gamma \in \widehat{G}$ is a character then
$$\widehat{1_A}(\gamma) = \frac{1}{N} \sum_{a \in A} \gamma(a).$$
- We want to count the number of 3APS in A. A 3AP is a solution to $x + y = 2z$, so
$$\frac{\#\text{ 3APs in } A}{N^2} = \langle 1_A * 1_A, 1_{2\cdot A} \rangle.$$

From few 3APs to the large spectrum

- Applying Parseval's identity, this gives a Fourier expression for the number of 3APs,

$$\frac{\#\text{ 3APs in } A}{N^2} = \langle 1_A * 1_A, 1_{2\cdot A}\rangle = \sum_{\gamma} \widehat{1_A}(\gamma)^2 \widehat{1_A}(-2\gamma).$$

- Since all we know about A is its size, the only contribution we can control is the trivial character, where $\widehat{1_A}(0) = \alpha$, and so

$$\frac{\#\text{ 3APs in } A}{N^2} = \alpha^3 + O\left(\sum_{\gamma\neq 0} \lvert\widehat{1_A}(\gamma)\rvert^3\right).$$

From few 3APs to the large spectrum

- So either we have $\gg \alpha^3 N^2$ many 3APs (which for $|A| \gg N^{1/2}$ is $\geq |A|$, and hence there must be some non-trivial 3APs), or the error term here must dominate the main term, so

$$\sum_{\gamma \neq 0} |\widehat{1_A}(\gamma)|^3 \gg \alpha^3.$$

- This could happen for a number of reasons - a small number of very large Fourier coefficients, or many small Fourier coefficients.For simplicity, we focus on the latter case, and suppose that if

$$\Delta = \{\gamma \neq 0 : |\widehat{1_A}(\gamma)| \gg \alpha^2\}$$

  then $|\Delta| \gg \alpha^{-3}$.

From few 3APs to the large spectrum

- This set Δ we call the large spectrum of A.
- Most of the progress in understanding three-term arithmetic progressions has come about by new insights into what sets like Δ can look like.
- (Note that by Parseval's identity

$$\alpha^4 \left|\Delta\right| \ll \sum_{\gamma} \left|\widehat{1_A}(\gamma)\right|^2 = \alpha,$$

  and so $\left|\Delta\right| \ll \alpha^{-3}$, so the lower bound from before is near-optimal.)
- In particular, it's useful to find subsets of Δ with small 'dimension', i.e. contained in the linear span of a small number of elements. Let's see why.

From large spectrum to density increment

- Suppose that we can find some $H \subset \Delta$ of size $\gg \alpha^{-1}$ and dimension $\ll 1$.
- Since $H \subset \Delta$,

$$\sum_{H} \left| \widehat{1_A}(\gamma) \right|^2 \gg \alpha^4 \left| H \right| \gg \alpha^3.$$

- Adding in the trivial character, where $\left| \widehat{1_A} \right| = \alpha$, we get that

$$\sum_{H \cup \{0\}} \left| \widehat{1_A}(\gamma) \right|^2 \geq (1 + \Omega(\alpha))\alpha^2.$$

From large spectrum to density increment

- If $G' \leq G$ is the set of approximate annihilators of H, then $|\widehat{1_{G'}}| \approx \frac{1}{N}|G'|$ on H, and also $|G'| \gg |G|$ (if H has dimension $O(1)$).
- So

$$\|1_A * 1_{G'}\|_2^2 = \sum \left|\widehat{1_{G'}}(\gamma)\right|^2 \left|\widehat{1_A}(\gamma)\right|^2$$

$$\geq |G'|^2 \sum_{H \cup \{0\}} \left|\widehat{1_A}(\gamma)\right|^2 \geq (1+\Omega(\alpha))\alpha^2 |G'|^2 / N^2.$$

- Since $\|1_A * 1_{G'}\|_1 = \alpha |G'| / N$, we get that $\|1_A * 1_{G'}\|_\infty \geq (1+\Omega(\alpha))\alpha |G'| / N$, or, unpacking the notation, there is some translate A' of A such that

$$\frac{|A' \cap G'|}{|G'|} \geq (1+\Omega(\alpha))\alpha.$$

Finishing the argument

- So we've basically shown that either:
  1. we have $\gg \alpha^3 N^2$ many 3APs in A, or
  2. there is some (approximate) subgroup $G' \subset G$ of size $|G'| \gg |G|$ and A', which is a translate of A (so has the same number of 3APs) such that
$$\frac{|A' \cap G'|}{|G'|} \geq (1 + \Omega(\alpha))\alpha.$$
- We then repeat the argument, starting with $A' \cap G' \subset G'$, and so on.
- The second 'density increment' step can't happen more than $O(\alpha^{-1})$ many times, since the density can never exceed 1.

Finishing the argument

- So in $O(\alpha^{-1})$ many steps we exit with the first case, and since our group at that point has size $\gg c^{\alpha^{-1}}N$ for some $c > 0$, the number of 3APs in some subset of a translate of A (and hence in A itself) is at least
$$\gg \alpha^3 c^{\alpha^{-1}} N^2 \geq \exp(-O(\alpha^{-1}))N^2.$$
- This implies, for example, that if A has no non-trivial 3APs to begin with then
$$\alpha \ll \frac{1}{\log N}.$$
- How do we get past $\log N$?

More refined structure

- We began with Δ having size α^{-3}, and a subset $H \subset \Delta$ having size α^{-1} and dimension $O(1)$.
- These parameters are all, in some sense, optimal, so we can't hope to do better directly (e.g. by finding a larger set with dimension $O(1)$).
- Instead, following the breakthrough strategy of Bateman and Katz, we instead obtain more refined information about the relationship between H and Δ.
- So that, roughly speaking, not only is $H \subset \Delta$, but also Δ is 'almost invariant under shifts by H', so that

$$\Delta + H - H \approx \Delta.$$

- How we do so is a large part of the proof (building on the earlier work of Bateman and Katz), which we will omit from this talk.

More refined structure

- So in other words, we now have Δ with size $\approx \alpha^{-3}$, and $H \subset \Delta$ with size $\approx \alpha^{-1}$ and dimension $O(1)$, but ALSO with the stronger property that $\Delta + H - H \approx \Delta$.
- What can we hope to do with this? We can take inspiration from a result of Bourgain, that approximately says that, if

$$\Delta' = \{\gamma \neq 0 : |\widehat{1_A}(\gamma)| \gg \alpha^{3/2}\},$$

then

$$\Delta' - \Delta' \subset \Delta.$$

Spectral boosting

- We prove a partial converse to this result, which in particular (approximately) implies that not only is $H \subset \Delta$, but in fact $H \subset \Delta'$.
- We call this 'spectral boosting', since we have used the combinatorial information $\Delta + H - H \approx \Delta$ to 'boost' the trivial fact that $|\widehat{1_A}| \gg \alpha^2$ on H to the much stronger fact that $|\widehat{1_A}| \gg \alpha^{3/2}$ on H.
- (Then running the previous density increment argument with this instead, gives a density increment of $(1 + \Omega(1))\alpha$, rather than the previous $(1 + \Omega(\alpha))\alpha$. This means that the whole argument only needs to be iterated $O(1)$ many times before we exit, which is much stronger!)

How does spectral boosting work?

- The proof of spectral boosting is a careful application of the Cauchy-Schwarz inequality, coupled with 'almost-periodicity', a physical space random sampling technique.
- The latter allows us to assume, for example, that $\langle 1_\Delta * 1_{-\Delta}, \lvert\widehat{1_A}\rvert^2\rangle \ll \alpha^2\lvert\Delta\rvert$.
- You can then use this coupled with the Cauchy-Schwarz inequality with the fact that

$$\langle 1_H * 1_{-H} * 1_\Delta, \lvert\widehat{1_A}\rvert^2\rangle \gg \alpha^4 \lvert\Delta\rvert \lvert H\rvert^2 ,$$

(which is because $\Delta + H - H \approx \Delta$), to deduce that

$$\lVert 1_H * \lvert\widehat{1_A}\rvert^2\rVert_\infty \gg \alpha^3 \lvert H\rvert ,$$

or in other words that (on some translate of H we have $\lvert\widehat{1_A}\rvert \gg \alpha^{3/2}$ as required.

Summary

1. Convert the additive problem into spectral information, e.g. showing that there are at least $\gg \alpha^{-3}$ many Fourier coefficients of 1_A of size $\gg \alpha^2$ (classical);
2. Show that the set Δ of such Fourier coefficients has rich combinatorial structure, e.g. there is a moderately sized H of small dimension such that $\Delta + H - H \approx \Delta$ (Bateman-Katz);
3. Exploit this structure to 'boost' the initial spectral information, e.g. find $\gg \alpha^{-1}$ many Fourier coefficients of size $\gg \alpha^{3/2}$ contained in a set of very low dimension (Bloom-Sisask).