Invariant Gibbs measures for the three-dimensional wave equation with a Hartree nonlinearity

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“Baby” equation

Linear and nonlinear oscillator

$$\text{(Osc)} \qquad \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + u + \lambda \cdot u^3 = 0.$$

$\lambda = 0$: Harmonic oscillator.
$\lambda > 0$: Duffing oscillator.

We define the Hamiltonian $H\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$H(q,p) = \frac{p^2}{2} + \frac{q^2}{2} + \lambda \frac{q^4}{4}.$$

Using the variables $q = u$ and $p = \frac{\mathrm{d}u}{\mathrm{d}t}$, we obtain that

$$\text{(Osc)} \quad \Longleftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} p \\ -q - \lambda q^3 \end{pmatrix} = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix}.$$

The Gibbs measure

(1) Conservation of energy: For any initial data $q(0), p(0) \in \mathbb{R}$,

$$H(q(t), p(t)) = H(q(0), p(0)) \qquad \text{for all } t \in \mathbb{R}.$$

(2) Liouville's theorem: Since $(q, p) \mapsto (\partial_p H, -\partial_q H)$ is divergence-free, the solution map

$$(q(0), p(0)) \in \mathbb{R}^2 \to (q(t), p(t)) \in \mathbb{R}^2$$

preserves the Lebesgue measure $\mathrm{d}q\,\mathrm{d}p$.

Theorem (Invariance). The Gibbs measure

$$\mathrm{d}\mu = \mathcal{Z}^{-1} \exp(-H(q, p))\, \mathrm{d}q\mathrm{d}p$$

is invariant under the Hamiltonian flow.

Example: Harmonic Oscillator → Gaussian measure

$$H(q,p) = \frac{p^2}{2} + \frac{q^2}{2}.$$

Gibbs measure:

$$\mathrm{d}\mu = \mathcal{Z}^{-1} \exp\Big(-\frac{1}{2}q^2 - \frac{1}{2}p^2\Big)\mathrm{d}q\mathrm{d}p.$$

→ Gaussian measure.

Hamiltonian flow:

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} q(0) \\ p(0) \end{pmatrix}.$$

→ Rotation in phase-space.

Then: *Theorem (Invariance)* → (Standard) Gaussian measure is rotation invariant.

Example: Duffing oscillator

$$H(q,p) = \frac{p^2}{2} + \frac{q^2}{2} + \lambda\frac{q^4}{4}.$$

Gibbs measure:

$$\begin{aligned} \mathrm{d}\mu &= \mathcal{Z}^{-1} \exp\Big(-\frac{1}{2}q^2 - \frac{\lambda}{4}q^4 - \frac{1}{2}p^2\Big)\mathrm{d}q\mathrm{d}p \\ &= \mathcal{Z}^{-1} \underbrace{\exp\Big(-\frac{\lambda}{4}q^4\Big)}_{\leqslant 1} \exp\Big(-\frac{1}{2}q^2 - \frac{1}{2}p^2\Big)\mathrm{d}q\mathrm{d}p \end{aligned}$$

is absolutely continuous with respect to the Gaussian measure.

Hamiltonian ODE:

$$\frac{\mathrm{d}q}{\mathrm{d}t} = p, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -q - \lambda q^3.$$

→ Complicated (Jacobi elliptic functions).

Gibbs measure: Implications

Most Hamiltonian ODEs cannot be solved explicitly and individual solutions are difficult to analyze. However,

Theorem (Invariance) $\Rightarrow$ Information on typical solutions.

For example, one can use:

- Poincaré recurrence theorem.
- Furstenberg multiple recurrence theorem.

Nonlinear wave equations (main result)

Nonlinear wave equations

$$
\begin{array}{lll}
\text{(NLW)} & -\partial_t^2 u - u + \Delta_x u = u^p & (t,x) \in \mathbb{R} \times \mathbb{T}^d \\
\text{(HNLW)} & -\partial_t^2 u - u + \Delta_x u = (V * u^2) u & (t,x) \in \mathbb{R} \times \mathbb{T}^d
\end{array}
$$

In the Hartree-nonlinearity, $V\colon \mathbb{T}^d \to \mathbb{R}$ is an interaction potential.

Hamiltonian structure:

$$H(u, \partial_t u) = \frac{1}{2} \int_{\mathbb{T}^d} \Big(|\partial_t u|^2 + |u|^2 + |\nabla u|^2 \Big) \mathrm{d}x + \mathcal{V}(u),$$

with $\mathcal{V}(u) = \frac{1}{p+1} \int_{\mathbb{T}^d} u^{p+1} \mathrm{d}x$ or $\mathcal{V}(u) = \frac{1}{4} \int_{\mathbb{T}^d} (V * u^2) u^2 \mathrm{d}x.$

Then: Hamiltonian + Symplectic form $\to$ (NLW) and (HNLW).

Nonlinear wave equations

Question: Since (NLW) and (HNLW) exhibit a Hamiltonian structure, do they also have invariant Gibbs measures?

Three parts of the question:

(1) Can the Gibbs measure be constructed rigorously?

(2) What are the properties of the Gibbs measure?

(3) Can the invariance of the Gibbs measure be proven?

Similar questions can be posed for other Hamiltonian PDEs, such as nonlinear Schrödinger equations.

Overview of current results

Theorem. Existence and invariance of the Gibbs measure (after a renormalization).

Dim. & Nonlinearity	Wave	Schrödinger
$d = 1$, $\lvert u\rvert^{p-1}u$	Friedlander '85, Zhidkov '94	Bourgain '94
$d = 2$, $\lvert u\rvert^2 u$	Oh-Thomann '18	Bourgain '96
$d = 2$, $\lvert u\rvert^{p-1}u$		Deng-Nahmod-Yue '19
$d = 3$, $(V_\beta * \lvert u\rvert^2)u$	$\beta > 1$: Oh-Okamoto-Tolomeo '20 $\beta > 0$: B. '20	$\beta > 2$: Bourgain '97 $\beta > 1/2$: Feasible. $\beta > 0$: Open.
$d = 3$, $\lvert u\rvert^2 u$	Open	(Extremely) open

The (periodic) interaction potential $V_\beta\colon \mathbb{T}^3 \to \mathbb{R}$ behaves like $|x|^{-(3-\beta)}$.
Smaller $\beta \to$ higher difficulty.

Main result

$$\text{(HNLW)} \quad -\partial_t^2 u - u + \Delta_x u = {:}(V * u^2)u{:} \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^3.$$

- $V = V_\beta$ is a periodic version of $|x|^{-(3-\beta)}$.
- ${:}(V * u^2)u{:}$ is a renormalization of $(V * u^2)u$.

Theorem (B. '20). The Gibbs measure corresponding to (HNLW) exists and, for $0 < \beta < 1/2$, is *mutually singular* with respect to the Gaussian free field. Furthermore, it is invariant under (HNLW).

Remark. This is the only theorem on the invariance of a *singular* Gibbs measure for any dispersive equation.
The singularity heavily affects the global (but not local) dynamics.

B., Invariant Gibbs measures for the three-dimensional wave equation with a Hartree nonlinearity I/II (65 and 133 pages)

The Gibbs measure

Gaussian free field

$$\mathrm{d}g = \mathcal{Z}^{-1} \exp\Big(-\frac{1}{2} \int_{\mathbb{T}^d} |u|^2 + |\nabla u|^2 \mathrm{d}x \Big) \mathrm{d}u$$

For any $n \in \mathbb{Z}^d$, we define $\langle n \rangle = \sqrt{1 + |n|^2}$. By using the "Fourier" transformation

$$(g_n)_{n \in \mathbb{Z}^d} \mapsto u = \sum_{n \in \mathbb{Z}^d} \frac{g_n}{\langle n \rangle} e^{i\langle n, x \rangle},$$

we can view $\mathrm{d}g$ as the push-forward of the Gaussian measure

$$\bigotimes_{n \in \mathbb{Z}^d} \mathcal{Z}_n^{-1} \exp\Big(-\frac{|g_n|^2}{2} \Big) \mathrm{d}g_n.$$

Spatial regularity:

$$\sum_{n \in \mathbb{Z}^d} \frac{g_n}{\langle n \rangle} e^{i\langle n, x \rangle} \in C_x^s(\mathbb{T}^d) \text{ a.s.} \quad \Longleftrightarrow \quad s < 1 - \frac{d}{2}.$$

Gibbs measure and the Gaussian free field

The Gibbs measure is (formally) given by

$$\mathrm{d}\mu^{\otimes}(u, u_t) = \mathcal{Z}_0^{-1} \exp\big(- \mathcal{V}(u)\big) \exp\Big(- \frac{1}{2} \int_{\mathbb{T}^d} |u|^2 + |\nabla u|^2 \mathrm{d}x \Big) \mathrm{d}u$$
$$\otimes\, \mathcal{Z}_1^{-1} \exp\Big(- \frac{1}{2} \int_{\mathbb{T}^d} |\partial_t u|^2 \mathrm{d}x \Big) \mathrm{d}u_t.$$

Using our Gaussian free field, we (rigorously) define

$$\mathscr{g}^{\otimes} = \mathscr{g} \otimes \big((\langle \nabla \rangle_{\#} \mathscr{g} \big).$$

Idea: Show that the potential energy $\mathcal{V}$ is $\mathscr{g}$-a.s. finite and

$$\exp\big(- \mathcal{V}(u)\big) \in L^1(\mathscr{g}) \backslash \{0\}.$$

Then, we can (rigorously) define

$$\mathrm{d}\mu^{\otimes}/\mathrm{d}\mathscr{g}^{\otimes} = \mathcal{Z}^{-1} \exp\big(- \mathcal{V}(u)\big).$$

Gibbs measure and the Gaussian free field

Regularity: $s < 1 - d/2$.
Difficulty: If $d \geqslant 2$, then $s < 0$. $\quad \rightarrow \quad \mathcal{V}(u) = \infty$ g-a.s.

The previous idea can (only) be implemented if

$$\begin{aligned} \text{(i):} \ & d = 1, \\ \text{(ii):} \ & d = 2, \\ \text{(iii):} \ & d = 3 \text{ and } \beta > 1/2. \end{aligned}$$

In (ii) and (iii), the potential energy $\mathcal{V}$ needs to be renormalized. This is indicated by writing $:\!\mathcal{V}\!:$ instead of $\mathcal{V}$.

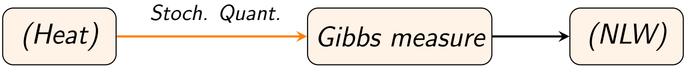
Remark. If $\mu^{\otimes} \ll g^{\otimes}$, we can use Gaussian initial data in the local theory for (NLW) and (HNLW).

Stochastic quantization

Stochastic quantization: The Gibbs measure is formally invariant under the stochastic heat equation

$$\text{(Heat)} \qquad \partial_t u + u - \Delta u = :u^3: + \sqrt{2}\eta, \qquad u(0) = \phi.$$

where η is space-time white noise.



Physics: Nelson ‘66, Parisi-Wu ‘81.
Mathematics: Da Prato-Debussche ‘03, Hairer-Matetski ‘15, Mourrat-Weber ‘17, Gubinelli-Hofmanová ‘18, ...

Variational approach: Barashkov-Gubinelli ‘18, ‘20. Similar spirit but relies on stochastic control theory.
$\rightarrow$ *Used here.*

Gibbs measure for the 3d Hartree nonlinearity

$$\mathcal{V}(u) = \frac{1}{4}\int_{\mathbb{T}^d} (V * u^2)u^2 \mathrm{d}x \quad \text{and} \quad V(x) \approx |x|^{-(3-\beta)}.$$

Theorem (B. '20, Measures). The Gibbs measure $\mu^{\otimes}$ corresponding to a (renormalization of) $\mathcal{V}$ exists and, for $0 < \beta < 1/2$, is mutually singular with respect to the Gaussian free field.
Furthermore, there exists a reference measure $\nu^{\otimes}$, an ambient probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and random functions $\bullet : \Omega \to C_x^{-1/2-\epsilon}(\mathbb{T}^3)$ and $\circ : \Omega \to C_x^{1/2+\beta-\epsilon}(\mathbb{T}^3)$ such that

$$\mu^{\otimes} \ll \nu^{\otimes}, \quad \nu^{\otimes} = \mathrm{Law}_{\mathbb{P}}\left(\bullet + \circ\right), \quad \text{and} \quad g^{\otimes} = \mathrm{Law}_{\mathbb{P}}(\bullet).$$

Remark. Oh-Okamoto-Tolomeo '20 independently obtained a similar result.

Why the "dots"? → Stay tuned!

Local dynamics

Local dynamics

Theorem (B. '20, LWP). On any small interval, (HNLW) is well-posed with high probability under the Gibbs measure.

Replace Gibbs measure → Reference measure $\nu^{\otimes} = \mathrm{Law}_{\mathbb{P}}(\bullet + \circ)$.

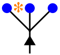
We write $u[t] = (u(t), \partial_t u(t))$. Recall that (HNLW) is given by

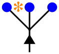
$$\begin{cases} -\partial_t^2 u - u + \Delta_x u = :(V * u^2)u: \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^3, \\ u[0] = \bullet + \circ. \end{cases}$$

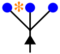
Deterministic critical regularity: $s_c = 1/2 - \beta$.
Regularity of $\bullet$: $s < -1/2$.

→ Almost a *full derivative* below the deterministic theory.

Ansatz

Ansatz: $u =$  $+$  $+ X + Y$, where:

-  is the linear evolution of •.
-  is the "first" Picard iterate, which solves

  $\big(-\partial_t^2 - 1 + \Delta\big)$  $=: \big(V * ($  $)^2\big)$  $:$

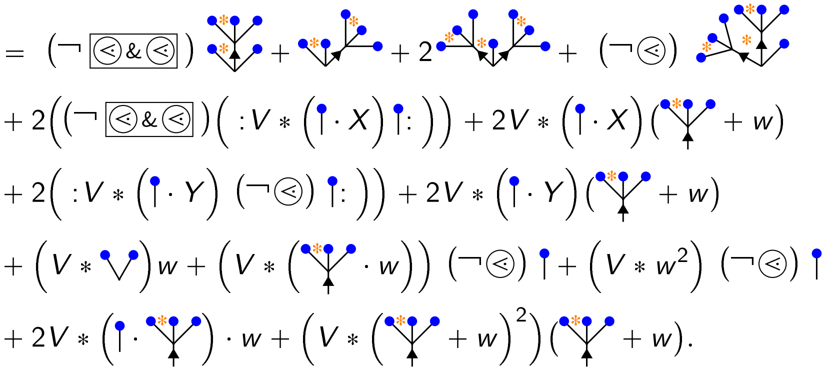
- X has regularity $1/2-$ but exhibits a *para-controlled structure*, as introduced by Gubinelli, Imkeller, and Perkowski.
- Y is a smooth remainder at regularity $1/2+$ which contains .

The threshold $s = 1/2$ determines whether the multiplication by  is well-defined.

Mourrat, Weber, Xu, Construction of Φ^4_3 diagrams for pedestrians.

Fleeting impression: Terms

$$(-\partial_t^2 - 1 + \Delta)Y$$



Message: (1) The evolution equations are quite involved.
(2) Stochastic diagrams are amazing!

Techniques

We combine ingredients from several different fields, such as:

- Gaussian hypercontractivity (Probability theory).
- Lattice point estimates (Number theory).
- Moment method (Random matrix theory).
- Multi-linear dispersive estimates (PDE).
- Multiple Itô integrals (Probability theory).
- Para-product estimates (Harmonic analysis).
- Para-controlled calculus (Stochastic PDE).
- Strichartz estimates (PDE).

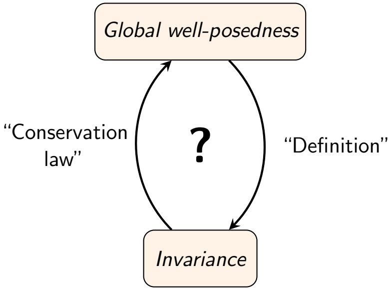
Global dynamics

Global well-posedness and invariance

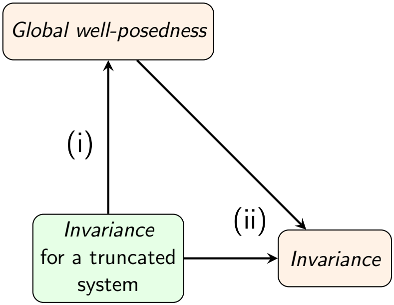
Recall: $\mu^{\otimes}$ is called invariant if

$$\text{Law}\left(u[0]\right) = \mu^{\otimes} \quad \Rightarrow \quad \text{Law}\left(u[t]\right) = \mu^{\otimes} \text{ for all } t \in \mathbb{R}.$$

Danger: Circular argument?



Solution: Bourgain!

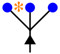


Singularity meets the dynamics

(1): The truncated Gibbs measures $\mu_N^{\otimes}$ only converge *weakly* to $\mu^{\otimes}$ but *not* in total-variation.

Serious difficulty, which requires *new ideas*.

(2): (Subtle) In our globalization argument, we want to use

$u =$  $+$  $+ X + Y.$

The Gaussian data • is only defined on the ambient probability space $(\Omega, \mathcal{F}, \mathbb{P})$. However, we cannot use $(\Omega, \mathcal{F}, \mathbb{P})$ and invariance-methods together.

Thank you

Questions