Uncertain signs

Emanuel Carneiro
(IMPA / ICTP)

Caltech & UCLA Joint Analysis Seminar
December 2020

joint work with Oscar E. Quesada-Herrera (IMPA)

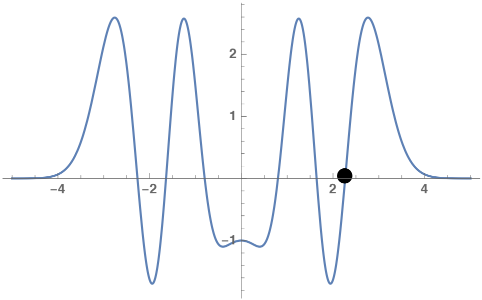
PART I

Prelude

Eventual non-negativity

- $f : \mathbb{R}^d \to \mathbb{R}$ is eventually non-negative if $f(x) \geq 0$ for sufficiently large $|x|$, and we define

$$r(f) := \inf\{r > 0 \ : \ f(x) \geq 0 \text{ for all } |x| \geq r\}.$$



$$f(x) = (x^{10} - 8x^8 + 15x^6 - x^4 - 2x^2 - 1)e^{-x^2}$$

An example

- Let us take a homogeneous polynomial, e.g. in dimension $d = 4$,

$$P(x_1, x_2, x_3, x_4) = \big(x_1^3 + x_1^2(x_2 - x_3) - x_1(x_2^2 + 2x_3^2) - x_2^3 + x_2^2 x_3 + 2x_2 x_3^2\big) x_4$$

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- Consider the family of functions:

$$\mathcal{A} = \left\{ \begin{array}{l} f \in L^1(\mathbb{R}^d)\backslash\{0\} \text{ continuous, real-valued; even;} \\ \\ \\ \end{array} \right.$$

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$$\mathbb{A} := \inf_{f \in \mathcal{A}} \sqrt{r(Pf)\, r(P\widehat{f})}$$

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- Tell me something about:

$$\mathbb{A} := \inf_{f \in \mathcal{A}} \sqrt{r(Pf)\, r(P\widehat{f})} = \sqrt{2}.$$

PART II

Uncertainty

Our guest of honor

Joseph Fourier
(1768 - 1830)

West face of the Eiffel tower

Fourier uncertainty

- Let $f \in L^1(\mathbb{R}^d)$. We define

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x)\, \mathrm{d}x \qquad (\xi \in \mathbb{R}^d).$$

- Plancherel: $\|f\|_{L^2(\mathbb{R}^d)} = \|\widehat{f}\|_{L^2(\mathbb{R}^d)}$.

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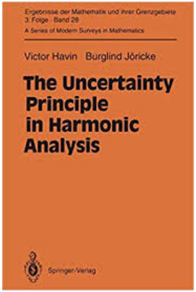
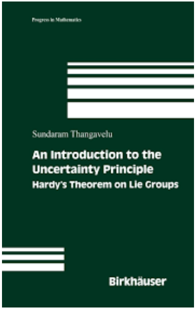
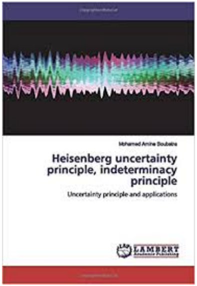
Fourier uncertainty: "one cannot have an unrestricted control of a function and its Fourier transform simultaneously."

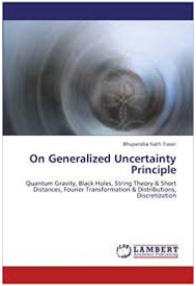
Subject is broad - no doubt about that...

Some books you may find in Amazon.com









Some examples

- Heisenberg (circa 1927):

$$\int_{\mathbb{R}^d} |f(x)|^2 \, \mathrm{d}x \leq \frac{4\pi}{d} \left(\int_{\mathbb{R}^d} |x|^2 \, |f(x)|^2 \mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^d} |\xi|^2 \, |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2}$$

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- Amrein-Berthier (1977): $E, F \subset \mathbb{R}^d$ of finite measure. Then

$$\int_{\mathbb{R}^d} |f(x)|^2 \, \mathrm{d}x \leq C(E, F, d) \left(\int_{E^c} |f(x)|^2 \, \mathrm{d}x + \int_{F^c} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \right).$$

PART III

Sign Fourier uncertainty

Sign uncertainty principle

Bourgain, Kahane and Clozel [Ann I Fourier, 2010]

- Consider the family:

$$\mathcal{A}_{+1}(d) = \begin{cases} f \in L^1(\mathbb{R}^d) \setminus \{0\} \text{ continuous, even, real-valued; } \widehat{f} \in L^1(\mathbb{R}^d); \\ \\ \\ \end{cases}$$

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$$\mathbb{A}_{+1}(d) := \inf_{f \in \mathcal{A}_{+1}(d)} \sqrt{r(f)\, r(\widehat{f})}.$$

(note that this is invariant under dilations $f_\delta(x) := f(\delta x)$).

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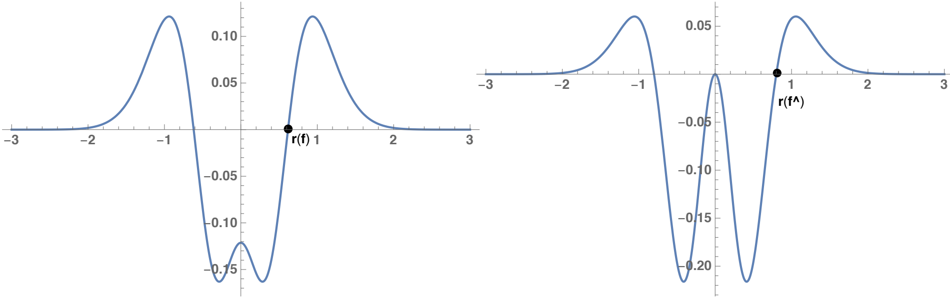
- They show:

$$\sqrt{\frac{d+2}{2\pi}} \geq \mathbb{A}_{+1}(d) \geq \sqrt{\frac{d}{2\pi e}}.$$

Pictures

Take, for example, $f(x) = e^{-\pi x^2/2} + e^{-2\pi x^2} - \left(\sqrt{2} + \frac{1}{\sqrt{2}}\right) e^{-\pi x^2}$.



Symmetrization: reducing to eigenfunctions

- The family is:

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$$w = f + \widehat{f}$$

It is clear that $w \in \mathcal{A}_{+1}(d)$ and $r(w) \leq r(f)$.

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- If you want

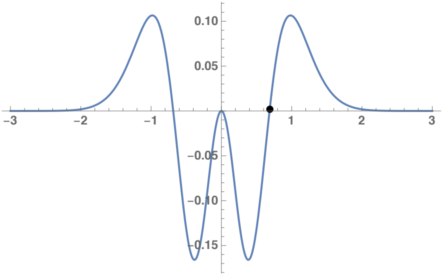
$$w^{\mathrm{rad}}(x) = \int_{SO(d)} w(Rx)\, \mathrm{d}\mu(R).$$

- Consider the (sub)-family:

$$\mathcal{A}^*_{+1}(d) = \left\{ \begin{array}{l} f \in L^1(\mathbb{R}^d) \setminus \{0\} \text{ continuous, even, real-valued ; } \widehat{f} = f; \\ f \text{ is eventually non-negative;} \\ f(0) = \int_{\mathbb{R}^d} f \leq 0. \end{array} \right\}.$$

- Then

$$\mathbb{A}_{+1}(d) = \mathbb{A}^*_{+1}(d) := \inf_{f \in \mathcal{A}^*_{+1}(d)} r(f).$$



$$f(x) = e^{-\pi x^2/2} + \sqrt{2}e^{-2\pi x^2} - (1+\sqrt{2})e^{-\pi x^2}$$

Proof

- The proof is quite simple: let $f \in \mathcal{A}^*_{+1}(d)$ and set $r = r(f)$. Let $x_+ = \max\{x, 0\}$ and $x_- = \max\{-x, 0\}$. Then

$$0 \geq \int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} f_+ - \int_{\mathbb{R}^d} f_-$$

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and therefore

$$\|f\|_\infty \leq \int_{\mathbb{R}^d} |f| = \int_{\mathbb{R}^d} f_+ + \int_{\mathbb{R}^d} f_- \leq 2\int_{B_r} f_- \leq 2\|f\|_\infty |B_r|.$$

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The conclusion is that

$$|B_r| \geq \frac{1}{2}.$$

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- Gonçalves, Oliveira e Silva, Steinerberger 2017: refined estimates and existence of extremizers.

A dual sign uncertainty principle

Cohn and Gonçalves [Invent. Math., 2019]

- Let $s = \{+1, -1\}$. Consider the family:

$$\mathcal{A}_s(d) = \left\{ \begin{array}{l} f \in L^1(\mathbb{R}^d) \setminus \{0\} \text{ continuous, even, real-valued ; } \widehat{f} \in L^1(\mathbb{R}^d); \\ \\ \\ \end{array} \right.$$

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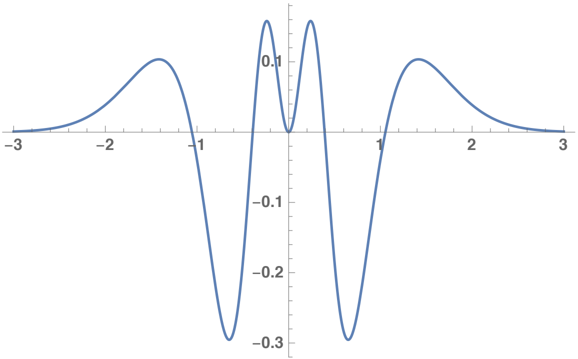
- Reduction to eigenfunctions: may assume that $\widehat{f} = sf$ above.
- They show:

$$C\sqrt{d} \geq \mathbb{A}_s(d) \geq c\sqrt{d}.$$

Pictures

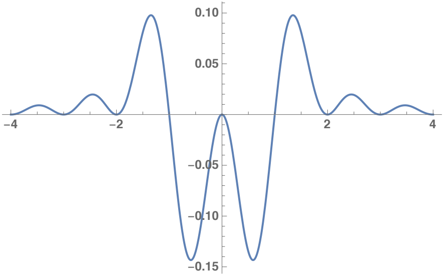
Example: $(\widehat{f} = -f)$

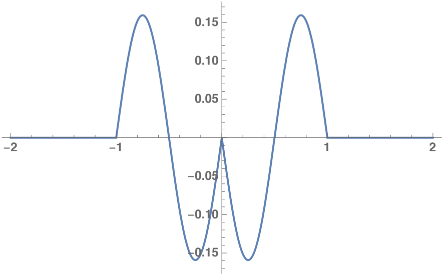
$$f(x) = e^{-\pi x^2/4} - 2e^{-4\pi x^2} + \left(\frac{1}{\sqrt{2}-1}\right)\left(e^{-\pi x^2/2} - \sqrt{2}e^{-2\pi x^2}\right).$$



Pictures

Example: $f(x) = \frac{\sin^2(\pi x)}{\pi^2(x^2-1)}$ and $\widehat{f}(x) = -\frac{1}{2\pi}\sin(2\pi|x|)\,\chi_{[-1,1]}(x)$.





(this turns out to be an extremal example!)

Sharp constants

Theorem

(i) *(Corollaries of Cohn and Elkies '03 ($d = 1$), Viazovska '17 ($d = 8$) and Cohn, Kumar, Miller, Radchenko and Viazovska '17 ($d = 24$))*

$$\mathbb{A}_{-1}(1) = 1 \; ; \; \mathbb{A}_{-1}(8) = \sqrt{2} \; ; \; \mathbb{A}_{-1}(24) = 2.$$

(ii) *(Cohn and Gonçalves '19)*

$$\mathbb{A}_{+1}(12) = \sqrt{2}.$$

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- It is conjectured that

$$\mathbb{A}_{-1}(2) = (4/3)^{1/4} \quad \textit{and} \quad \mathbb{A}_{+1}(1) = \frac{1}{\sqrt{1+\sqrt{5}}}.$$

  - Gonçalves, Oliveira e Silva and Ramos (preprint, 2020); extensions of the (± 1)- sign uncertainty to a suitable operator setting.

PART IV

Generalized sign Fourier uncertainty

A new point of view

- What about the other eigenvalues $\pm i$?

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- Goal: Investigate the situation where the signs of f and $\widehat{f}$ resonate with a given generic function P at infinity, given a suitable competing weighted integral condition.
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- Goal: Investigate the situation where the signs of f and $\widehat{f}$ resonate with a given generic function P at infinity, given a suitable competing weighted integral condition.
- All that happened before will be the case $P \equiv 1$.
- A measurable $g : \mathbb{R}^d \to \mathbb{R}$ is *eventually non-negative* if $g(x) \geq 0$ for sufficiently large $|x|$, and we define

$$r(g) := \inf\{r > 0 \;:\; g(x) \geq 0 \textit{ for all } |x| \geq r\}.$$

The function P

Let $P : \mathbb{R}^d \to \mathbb{R}$ be measurable function such that:

(P1) $P \in L^1_{\rm loc}(\mathbb{R}^d)$.

(P2) P is either even or odd. We let $\mathfrak{r} \in \{0, 1\}$ be such that

$$P(-x) = (-1)^{\mathfrak{r}} P(x).$$

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(P3) P is annihilating in the following sense: if $f \in L^1(\mathbb{R}^d)$ is a continuous eigenfunction of the Fourier transform such that Pf is eventually zero then $f \equiv 0$. (e.g. if $\{P(x) \neq 0\}$ is dense).

(P4) P is homogeneous. That is, there is a real number $\gamma > -d$ with

$$P(\delta x) = \delta^{\gamma} P(x)$$

for all $\delta > 0$ and $x \in \mathbb{R}^d$.

The generalized setup

Let $s \in \{+1, -1\}$. Consider the family:

$$\mathcal{A}_s(P; d) = \left\{ \begin{array}{l} f \in L^1(\mathbb{R}^d) \setminus \{0\} \text{ continuous, real-valued; } f(-x) = (-1)^{\mathfrak{r}} f(x); \\ \\ \\ \\ \end{array} \right.$$

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$$\mathbb{A}_s(P;d) = \inf_{f \in \mathcal{A}_s(P;d)} \sqrt{r(Pf)\ r\big(s(-i)^{\mathfrak{r}} P\widehat{f}\big)}.$$

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- We may consider

$$w = f + s(-i)^{\mathfrak{r}} \widehat{f}$$

Then $w \in \mathbb{A}_s(P;d)$ (uses (P3)) and $r(Pw) \leq r(Pf)$.

Generalized eigenfunction problem

$$\mathcal{A}_s^*(P;d) = \begin{cases} f \in L^1(\mathbb{R}^d) \setminus \{0\} \text{ continuous, real-valued; } \widehat{f} = s\, i^{\mathfrak{r}} f; \\ Pf \in L^1(\mathbb{R}^d); \\ \\ \\ \end{cases}$$

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- We may now forget conditions (P3) and (P4).
- $P_1 \equiv 1$ and $P_2(x) = 1$ for all $x \in \mathbb{R} \setminus \{a_n\}_{n \in \mathbb{Z}}$, $P_2(a_n) = -1$, where $|a_n| \to \infty$. Any function $f \in \mathcal{A}_s^*(P_2;1)$ will necessarily have zeros at a_n for $n \geq n_0$ (even problems when $P = 0$ a.e. are not trivial!).

Examples

$$\mathcal{A}_s^*(P;d) = \left\{ \begin{array}{l} f \in L^1(\mathbb{R}^d) \setminus \{0\} \text{ continuous, real-valued; } \widehat{f} = s\, i^{\mathfrak{r}} f; \\ Pf \in L^1(\mathbb{R}^d); \\ Pf \text{ is eventually non-negative;} \\ \int_{\mathbb{R}^d} Pf \leq 0. \end{array} \right\}.$$

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- $P(x_1, x_2, x_3, x_4) = (x_1^3 + x_1^2(x_2 - x_3) - x_1(x_2^2 + 2x_3^2) - x_2^3 + x_2^2 x_3 + 2x_2 x_3^2)\, x_4$

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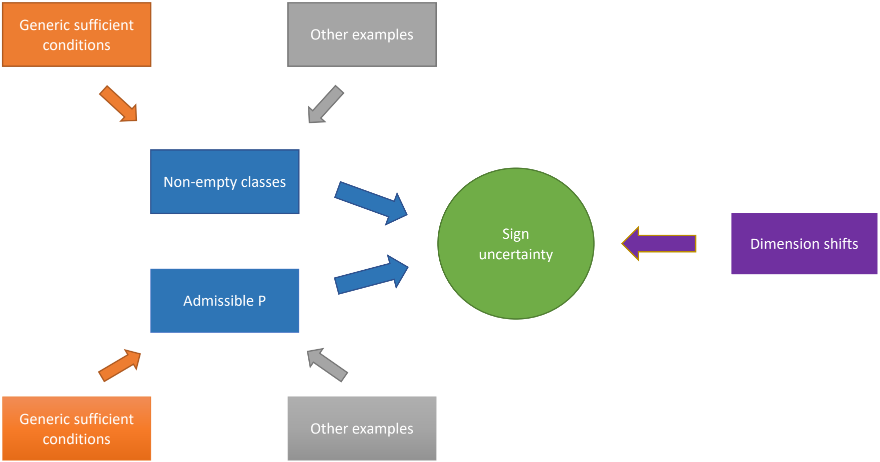
  We will show, for instance, that $\mathbb{A}_{+1}^*(P;4) = \sqrt{2}$.

The path


Generic sufficient conditions
Other examples
Non-empty classes
Sign uncertainty
Dimension shifts
Admissible P
Generic sufficient conditions
Other examples

Non-empty classes

Theorem (Non-empty classes)

Let P be such that $P\, e^{-\lambda\pi|\cdot|^2} \in L^1(\mathbb{R}^d)$ for all $\lambda > 0$. Assume that $P = H \cdot Q$, where $H : \mathbb{R}^d \to \mathbb{R}$ is a homogeneous and harmonic polynomial of degree $\ell \geq 0$, and $Q : \mathbb{R}^d \to \mathbb{R}$ is eventually non-negative. Then $\mathcal{A}_s^*(P; d)$ is non-empty.

Admissible functions

Definition (Admissible functions)

P is admissible if there there exists $1 \leq q \leq \infty$ and a positive constant $C = C(P; d; q)$ such that:

(i) *For all $f \in L^1(\mathbb{R}^d)$, with $\widehat{f} = \pm i^{\mathfrak{r}} f$ and $Pf \in L^1(\mathbb{R}^d)$, we have*

$$||f||_q \leq C\,\|Pf\|_1. \tag{1}$$

(ii) *If $q > 1$ then $P \in L^{q'}_{\mathrm{loc}}(\mathbb{R}^d)$. If $q = 1$ we have $\lim_{r\to 0^+} \|P\|_{L^\infty(B_r)} = 0$.*

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Theorem (Sufficient conditions for admissibility)

Let P be such that the sub-level set $A_\lambda = \{x \in \mathbb{R}^d \,:\, |P(x)| \leq \lambda\}$ has finite Lebesgue measure for some $\lambda > 0$. Then inequality (1) holds with $q = 1$. In particular, P is admissible with respect to $q = \infty$.

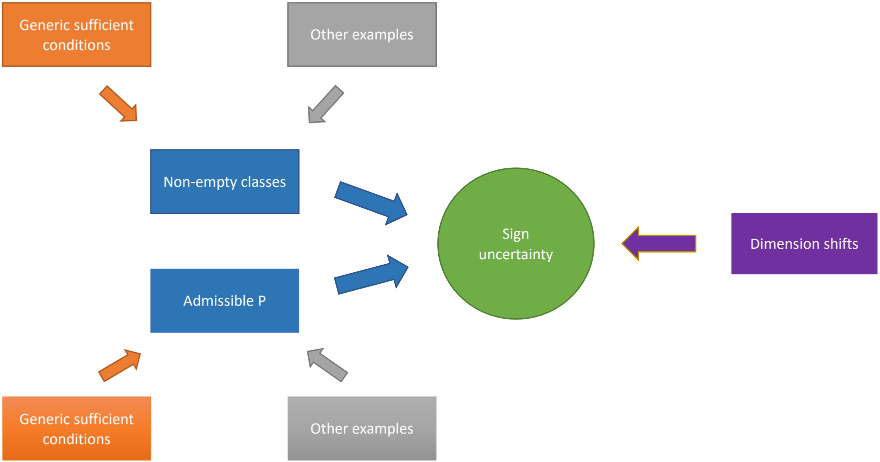
Sign uncertainty

Theorem

Assume that the class $\mathcal{A}_s^*(P; d)$ is non-empty and that P is admissible with respect to an exponent $1 \leq q \leq \infty$. Then there exists a positive constant $C^* = C^*(P; d; q)$ such that

$$\mathbb{A}_s^*(P; d) \geq C^*.$$

The path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Theorem (Dimension shifts)

Let $\ell \geq 0$ *and* $\mathfrak{r}(\ell) \in \{0, 1\}$ *be such that* $\mathfrak{r}(\ell) \equiv \ell \pmod 2$. *Let* $P : \mathbb{R}^{d+2\ell} \to \mathbb{R}$ *be a radial function verifying* (P3). *Write* $P(x) = P_0(|x|)$. *Let* $\widetilde{P} : \mathbb{R}^d \to \mathbb{R}$ *be of the form*

$$\widetilde{P}(x) = H(x)\, P_0(|x|)\, Q(x),$$

where $H : \mathbb{R}^d \to \mathbb{R}$ *is a non-zero homogeneous and harmonic polynomial of degree* ℓ *and* $Q : \mathbb{R}^d \to \mathbb{R}$ *is an even non-negative function, homogeneous of degree* 0. *If* $\mathcal{A}_s^*(P; d + 2\ell)$ *is non-empty, then* $\mathcal{A}^*_{s(-1)^{(\mathfrak{r}(\ell)+\ell)/2}}\big(\widetilde{P}; d\big)$ *is also non-empty and*

$$\mathbb{A}_s^*(P; d + 2\ell) \geq \mathbb{A}^*_{s(-1)^{(\mathfrak{r}(\ell)+\ell)/2}}\big(\widetilde{P}; d\big).$$

If P *has a bounded sub-level set,* $Q \equiv 1$ *and* $H \in O(d)(x_1 x_2 \dots x_\ell)$ $(0 \leq \ell \leq d)$*, the equality holds.*

Useful: $Q = |x|^\ell \,\mathrm{sgn}(H)/H$, when $P(x) = |x|^\gamma$, $\gamma < 0$.

Corollary (Sharp constants)

Let $\mathfrak{r}(\ell) \in \{0, 1\}$ *be such that* $\mathfrak{r}(\ell) \equiv \ell \pmod 2$. *Then*

$$\mathbb{A}_{(-1)^{(\mathfrak{r}(\ell)+\ell+2)/2}}(R(x_1 \ldots x_\ell)\,; 8 - 2\ell) = \sqrt{2}, \quad 0 \le \ell \le 2; \quad R \in O(8 - 2\ell);$$

$$\mathbb{A}_{(-1)^{(\mathfrak{r}(\ell)+\ell)/2}}(R(x_1 \ldots x_\ell)\,; 12 - 2\ell) = \sqrt{2}, \quad 0 \le \ell \le 4; \quad R \in O(12 - 2\ell);$$

$$\mathbb{A}_{(-1)^{(\mathfrak{r}(\ell)+\ell+2)/2}}(R(x_1 \ldots x_\ell)\,; 24 - 2\ell) = 2, \quad 0 \le \ell \le 8; \quad R \in O(24 - 2\ell);.$$

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Thank you for your kind attention!!