Proving Positive Lyapunov Exponents: Beyond Independence

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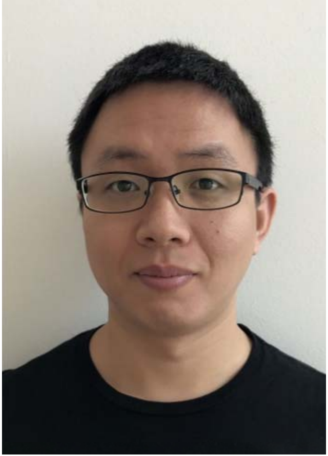

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Lyapunov Exponents of $\mathrm{SL}(2,\mathbb{R})$-Cocycles

Let us fix a compact metric space Ω, a continuous map $T : \Omega \to \Omega$, and an ergodic Borel probability measure μ. The triple (Ω, T, μ) is often referred to as the base dynamical system.

Given a continuous map $A : \Omega \to \mathrm{SL}(2,\mathbb{R})$, we consider the skew product

$$(T, A) : \Omega \times \mathbb{R}^2 \to \Omega \times \mathbb{R}^2,\ (\omega, v) \mapsto (T\omega, A(\omega)v)$$

For each $n \in \mathbb{Z}$, the map $A_n : \Omega \to \mathrm{SL}(2,\mathbb{R})$ is defined by $(T, A)^n = (T^n, A_n)$.

By Kingman's Subadditive Ergodic Theorem, there is a number $L(A) \geq 0$, called the Lyapunov exponent, such that

$$\begin{aligned} L(A) &= \inf_{n \geq 1} \frac{1}{n} \int \log \|A_n(\omega)\| \, d\mu(\omega) \\ &= \lim_{n \to \infty} \frac{1}{n} \int \log \|A_n(\omega)\| \, d\mu(\omega) \\ &= \lim_{n \to \infty} \frac{1}{n} \log \|A_n(\omega)\| \quad \text{for } \mu\text{-a.e. } \omega \end{aligned}$$

Naturally, we are interested in whether $L(A) > 0$ or $L(A) = 0$.

Lyapunov Exponents of One-Parameter Families of $\mathrm{SL}(2,\mathbb{R})$-Cocycles

Example

Consider Schrödinger operators

$$[H\psi](n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n)$$

in $\ell^2(\mathbb{Z})$, where the potential $V : \mathbb{Z} \to \mathbb{R}$ is dynamically defined, that is,

$$V(n) = f(T^n\omega)$$

with a base dynamical system (Ω, T, μ) as above and a continuous map $f : \Omega \to \mathbb{R}$. Then the solutions of the generalized eigenvalue equation

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n)$$

can be described via

$$\begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} = A_n^{(E)}(\omega) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}$$

with the Schrödinger cocycle generated by the map

$$A^{(E)}(\omega) = \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix}$$

Lyapunov Exponents of One-Parameter Families of $\mathrm{SL}(2,\mathbb{R})$-Cocycles

It is natural and customary to write $L(E)$ instead of $A^{(E)}$ for a Schrödinger cocycle. Since $L(E) > 0$ strongly indicates that the generalized eigenfunctions have exponential behavior, combining this with the existence of polynomially bounded generalized eigenfunctions spectrally almost everywhere, one expects spectral localization (i.e., pure point spectrum with exponentially decaying eigenfunctions) for μ-a.e. $\omega \in \Omega$ when $L(E) > 0$ holds for sufficiently many energies E. Let us denote the exceptional set of energies E by

$$\mathcal{Z} := \{E : L(E) = 0\}$$

Remark

(a) This connection holds almost always, by not always. In particular, there are examples with $\mathcal{Z} = \emptyset$, and yet for all $\omega \in \Omega$, the point spectrum of H is empty.

(b) Since countable sets cannot carry continuous spectral measures, one would want to embark on a proof of spectral localization by showing that $\mathcal{Z}$ is countable.

(c) For technical reasons, one generally desires to show that $\mathcal{Z}$ is discrete.

(d) This is a natural goal as requiring $\mathcal{Z} = \emptyset$ is too restrictive.

Positive Lyapunov Exponents for Schrödinger Cocycles: Examples

Let us discuss the goal of proving that $\mathcal{Z}$ is empty or at least small in several settings. We begin with the classical example, which is the simplest of them.

Example (The standard Anderson model)

The potential V is given by a realization of a sequence of independent identically distributed random variables. In our setting, this arises via the choices

- $\Omega = I^{\mathbb{Z}}$, where $I \subset \mathbb{R}$ is a compact interval
- $T : \Omega \to \Omega$ is the shift, $[T\omega](n) = \omega(n+1)$
- $\mu = \nu^{\mathbb{Z}}$, where ν is a probability measure supported by I (and $\# \operatorname{supp} \nu \geq 2$)
- $f : \Omega \to \mathbb{R}$, $f(\omega) = \omega(0)$

In this model one can show that $\mathcal{Z} = \emptyset$, and the proof is a straightforward application of Fürstenberg's Theorem about products of i.i.d. $\mathrm{SL}(2, \mathbb{R})$ matrices.

But what about more general f's? Fürstenberg's Theorem is then no longer applicable.

Positive Lyapunov Exponents for Schrödinger Cocycles: Examples

Example (The doubling map model)

The doubling map model is generated by

- $\Omega = \mathbb{T} = \mathbb{R}/\mathbb{Z}$
- $T : \Omega \to \Omega, \ T\omega = 2\omega$
- $\mu = \mathrm{Leb}$
- $f : \Omega \to \mathbb{R}$ suitable, sometimes taken to be $\lambda \cos(2\pi\omega)$

Strictly speaking, the map is non-invertible and hence the resulting operators are defined on $\ell^2(\mathbb{Z}_+)$, but this point is not essential to our discussion.

In this model it had been open whether $\mathcal{Z}$ is discrete, despite several attempts at proving this.

The binary expansion of $\omega \in \mathbb{T}$ shows that this model may be viewed as a full shift with a non-local sampling function. Thus, there is some underlying independence (as the measure is then a product measure), but the resulting cocycles are not products of i.i.d. $\mathrm{SL}(2, \mathbb{R})$ matrices and hence Fürstenberg's Theorem does not apply.

Some Known Results

Here are some of the known results about the exceptional set in the case of the doubling map (or related maps):

Theorem (Chulaevsky-Spencer 1995)

$\Omega = \mathbb{T}$, $T\omega = 2\omega$, $\mu = \mathrm{Leb}$, $g \in C^1(\mathbb{T})$ *non-constant*, $f = \lambda g$. *Then,* $\mathrm{Leb}\, \mathcal{Z} \to 0$ *as* $\lambda \to 0$.

Theorem (D.-Killip 2005)

$\Omega = \mathbb{T}$, $T\omega = 2\omega$, $\mu = \mathrm{Leb}$, f *non-constant. Then,* $\mathrm{Leb}\, \mathcal{Z} = 0$.

Theorem (Bourgain-Bourgain-Chang 2015)

$\Omega = \mathbb{T}$, $\mu = \mathrm{Leb}$, $f \in C^1(\mathbb{T})$ *non-constant. Then, for m large enough and* $T\omega = m\omega$, $\mathcal{Z} = \emptyset$.

The limitations on the sampling function, the base transformation, and/or the statement about the structure of $\mathcal{Z}$ are significant.

The Setting

Our work addresses the problem of proving positive Lyapunov exponents for Schrödinger operators with potentials generated by hyperbolic transformations.

The setting in which our results are established is very general. But we will for simplicity focus on the special case of the full shift in our discussion below, as this case was already out of reach prior to this work, and the main ingredients are easier to grasp in this setting.

The general setting assumes that

- (Ω, T) is a subshift of finite type that has a fixed point
- μ is a T-ergodic probability measure that has a local product structure

While we won't define all these notions in detail, we do point out that the assumptions are satisfied by

$$\Omega = \{1, \dots, \ell\}^{\mathbb{Z}}, \quad [T\omega](n) = \omega(n+1), \quad \mu = \nu^{\mathbb{Z}}$$

This allows one to study the Bernoulli Anderson model with more general sampling functions, as well as the doubling map.

To give another well-known example, we note that they are satisfied by the Arnold cat map

$$\Omega = \mathbb{T}^2, \quad T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mu = \mathrm{Leb}$$

The Main Results

Theorem (Avila-D.-Zhang)

Suppose f is Hölder continuous and non-constant. Then $\mathcal{Z}$ is discrete.

Theorem (Avila-D.-Zhang)

Suppose f is Hölder continuous, non-constant, and globally bunched or locally constant. Then $\mathcal{Z}$ is finite.

Theorem (Avila-D.-Zhang)

For each $\alpha \in (0,1]$, there is a dense G_δ subset $\mathcal{G}$ of $C^\alpha(\Omega,\mathbb{R})$ such that for each $f \in \mathcal{G}$, we have $\mathcal{Z} = \emptyset$.

Theorem (Avila-D.-Zhang)

For each $\alpha \in (0,1]$, consider the subspaces of $C^\alpha(\Omega,\mathbb{R})$ consisting of globally bunched or locally constant functions. For each of them, there is an open and dense subset $\mathcal{G}$ such that for every $f \in \mathcal{G}$, we have $\inf\{L(E) : E \in \mathbb{R}\} > 0$.

The Base Space

Recall that for simplicity we will consider the following special case:

$$\Omega = \{1, \dots, \ell\}^{\mathbb{Z}}, \quad [T\omega](n) = \omega(n+1), \quad \mu = \nu^{\mathbb{Z}}$$

We fix the following metric on Ω:

$$d(\omega, \tilde{\omega}) = e^{-N(\omega, \tilde{\omega})}$$

where

$$N(\omega, \tilde{\omega}) = \max\{N \geq 0 : \omega_n = \tilde{\omega}_n \text{ for all } |n| < N\}$$

Definition

The local stable set of a point $\omega \in \Omega$ is defined by

$$W^s_{\mathrm{loc}}(\omega) = \{\tilde{\omega} \in \Omega : \omega_n = \tilde{\omega}_n \text{ for } n \geq 0\}$$

and the local unstable set of ω is defined by

$$W^u_{\mathrm{loc}}(\omega) = \{\tilde{\omega} \in \Omega : \omega_n = \tilde{\omega}_n \text{ for } n \leq 0\}$$

Projectivization and Holonomies

Recall that a continuous map $A : \Omega \to \mathrm{SL}(2,\mathbb{R})$ induces the cocycle

$$(T,A) : \Omega \times \mathbb{R}^2 \to \Omega \times \mathbb{R}^2,\ (\omega, v) \mapsto (T\omega, A(\omega)v)$$

and that $A_n : \Omega \to \mathrm{SL}(2,\mathbb{R})$ is defined by $(T,A)^n = (T^n, A_n)$.

By linearity and invertibility of each $A(\omega)$, we can projectivize the second component and consider

$$(T,A) : \Omega \times \mathbb{RP}^1 \to \Omega \times \mathbb{RP}^1$$

Let us denote the fiber $\{\omega\} \times \mathbb{RP}^1$ by $\mathcal{E}_\omega$.

Definition

A stable holonomy h^s for A is a family of homeomorphisms $h^s_{\omega,\omega'} : \mathcal{E}_\omega \to \mathcal{E}_{\omega'}$, defined whenever ω and ω' belong to the same local stable set, satisfying

- **(i)** $h^s_{\omega',\omega''} \circ h^s_{\omega,\omega'} = h^s_{\omega,\omega''}$ and $h^s_{\omega,\omega} = \mathrm{id}$,
- **(ii)** $A(\omega') \circ h^s_{\omega,\omega'} = h^s_{T\omega,T\omega'} \circ A(\omega)$,
- **(iii)** $(\omega,\omega') \mapsto h^s_{\omega,\omega'}(\phi)$ is continuous when ω,ω' belong to the same local stable set, uniformly in ϕ.

An unstable holonomy $h^u_{\omega,\omega'} : \mathcal{E}_\omega \to \mathcal{E}_{\omega'}$ is defined analogously for pairs of points in the same unstable set.

Canonical Holonomies

A canonical way of producing stable and unstable holonomies is via suitable convergence properties of the matrix products $A_n(\omega)$.

Suppose that the limits

$$H^s_{\omega,\omega'} = \lim_{n\to\infty} A_n(\omega')^{-1} A_n(\omega), \quad H^u_{\omega,\omega'} = \lim_{n\to\infty} A_{-n}(\omega')^{-1} A_{-n}(\omega)$$

exist for ω, ω' in the same stable (resp., unstable) set.

Note that this will follow for example if A is locally constant or, in the case of Schrödinger cocycles, if $\|f\|_\infty$ is sufficiently small and E is in (a small neighborhood of) the spectrum.

The analogues of the properties (i)–(iii) above follow for $H^s_{\omega,\omega'}, H^u_{\omega,\omega'}$ directly from the construction, and this in turn implies (i)–(iii) for the induced maps $h^s_{\omega,\omega'}, h^u_{\omega,\omega'}$ obtained by projectivization.

Holonomies that arise in this way are called canonical holonomies of A.

Invariant Measures of Projective Cocycles

Consider a projective cocycle $(T, A) : \Omega \times \mathbb{RP}^1 \to \Omega \times \mathbb{RP}^1$ that has stable and unstable holonomies.

Definition

Suppose we are given a (T, A)-invariant probability measure m on $\Omega \times \mathbb{RP}^1$ that projects to μ in the first component. A disintegration of m along the fibers is a measurable family $\{m_\omega : \omega \in \Omega\}$ of probability measures on $\mathbb{RP}^1$ such that $m = \int m_\omega \, d\mu(\omega)$, that is,

$$m(D) = \int_\Omega m_\omega(\{z \in \mathbb{RP}^1 : (\omega, z) \in D\}) \, d\mu(\omega)$$

for each measurable set $D \subset \Omega \times \mathbb{RP}^1$.

Remark

Such a disintegration exists and is almost everywhere unique. A probability measure m on $\Omega \times \mathbb{RP}^1$ is (T, A)-invariant if and only if $A(\omega)_* m_\omega = m_{T\omega}$ for μ-almost every $\omega \in \Omega$.

Invariant Measures of Projective Cocycles

Definition

A (T, A)-invariant probability measure m on $\Omega \times \mathbb{RP}^1$ is called an *s-state* if it is invariant under the stable holonomies. That is, the disintegration $\{m_\omega : \omega \in \Omega\}$ satisfies

$$(h^s_{\omega,\omega'})_* m_\omega = m_{\omega'}$$

for μ-almost every $\omega \in \Omega$ and for every $\omega' \in W^s_{\mathrm{loc}}(\omega)$. In this case, we say that $\{m_\omega\}$ is *s-invariant*.

Similarly, m is called a *u-state* if it is invariant under the unstable holonomies,

$$(h^u_{\omega,\omega'})_* m_\omega = m_{\omega'}$$

for μ-almost every $\omega \in \Omega$ and for every $\omega' \in W^u_{\mathrm{loc}}(\omega))$. In this case, we say that $\{m_\omega\}$ is *u-invariant*.

A measure that is both an s-state and a u-state is called an *su-state*.

Zero Lyapunov Exponent and *su*-States

Proposition (Bonatti-Gómez-Mont-Viana)

Suppose A admits stable and unstable canonical holonomies. If $L(A) = 0$, then every (T, A)-invariant probability measure m on $\Omega \times \mathbb{RP}^1$ that projects to μ in the first component has a continuous, su-invariant disintegration $\{m_\omega : \omega \in \Omega\}$.

Remark

When the sampling function is not locally constant or uniformly small, one needs to work harder to establish a version of the result above. The key idea is then given by the realization that the vanishing of the Lyapunov exponent in itself implies a weak analogue of the existence of canonical holonomies.

Applying this to a Schrödinger cocycle $A^{(E)}$, the key idea that makes a direct connection to spectral theory possible is to

- pass to the conformal barycenter of the fiber measure m_ω
- identify the conformal barycenter as the Weyl-Titchmarsh function

Periodic Points of the Base Transformation and Schrödinger Spectra

Suppose that $\omega \in \Omega$ is a periodic point for T. That is, there is $p \in \mathbb{N}$ such that $T^p \omega = \omega$. It then follows that the associated potential is p-periodic as well:

$$V_\omega(n) = V_\omega(n + p), \quad n \in \mathbb{Z}$$

The spectrum of H_ω is well known to consist of bands. More precisely, there are p disjoint open intervals $I_1, \ldots, I_p$, so that $\sigma(H_\omega)$ is the closure of the union of the I_j's.

The matrix $A_p^{(E)}(\omega)$ is

- elliptic when $E \in I_j$ for some $j \in \{1, \ldots, p\}$
- hyperbolic when $E \notin \sigma(H_\omega)$

This can be rephrased as follows. The Weyl-Titchmarsh function $m_\omega(E)$ is

- imaginary when $E \in I_j$ for some $j \in \{1, \ldots, p\}$
- real when $E \notin \sigma(H_\omega)$

Exploiting the Connection

Suppose that $\omega_1, \omega_2 \in \Omega$ are periodic points for T with periods $p_1, p_2 \in \mathbb{N}$. By the discussion above, except for $2p_1 + 2p_2$ values of E (the endpoints of the two sets of intervals), we have that $m_{\omega_1}(E)$ is

- imaginary when $E \in \sigma(H_{\omega_1})$
- real when $E \notin \sigma(H_{\omega_1})$

and $m_{\omega_2}(E)$ is

- imaginary when $E \in \sigma(H_{\omega_2})$
- real when $E \notin \sigma(H_{\omega_2})$

Now choose the connector

$$\omega_c \in W^u_{\mathrm{loc}}(\omega_1) \cap W^s_{\mathrm{loc}}(\omega_2)$$

For every $E \in \mathcal{Z}$, we have probability measures $\{m_\omega^{(E)} : \omega \in \Omega\}$ on $\mathbb{RP}^1$ that are invariant under the cocycle and the holonomies.

Exploiting the Connection

Thus, the invariance properties imply that

- $m_{\omega_1}(E)$ and $m_{\omega_2}(E)$ are analytically related on $\mathcal{Z}$
- should $\mathcal{Z}$ (which is contained in the compact set $[-2 - \|f\|_\infty, 2 + \|f\|_\infty]$) be infinite, the relation extends globally
- and once that is the case, we find that

$$E \in \sigma(H_{\omega_1}) \Leftrightarrow E \in \sigma(H_{\omega_2})$$

We may deduce that $\mathcal{Z}$ being infinite has the consequence that all periodic spectra are the same!

Since T has a fixed point and non-constancy of f allows us to find a non-constant periodic potential of the form V_ω, this yields a contradiction due to the following result from inverse spectral theory:

Theorem
The spectrum of a periodic Schrödinger operator in $\ell^2(\mathbb{Z})$ consists of a single interval if and only if the potential is constant.

Concluding Remarks

We have explained the key ideas (in the special case of the full shift) leading to a proof of the second main result:

Theorem (Avila-D.-Zhang)

Suppose f is Hölder continuous, non-constant, and globally bunched or locally constant. Then $\mathcal{Z}$ is finite.

The first main result

Theorem (Avila-D.-Zhang)

Suppose f is Hölder continuous and non-constant. Then $\mathcal{Z}$ is discrete.

is obtained by working locally, initially in the vicinity of an accumulation point of $\mathcal{Z}$, and then extending along the band of a given periodic spectrum. This yields that the band in question must be a band in every other periodic spectrum. As the constant spectrum is an interval of length 4, and any non-constant periodic potential leads to a spectrum of total Lebesgue measure < 4, we again arrive at a contradiction.

We observe the crucial interplay between dynamics and spectral theory in the proof of each of these two results.

Thank You!

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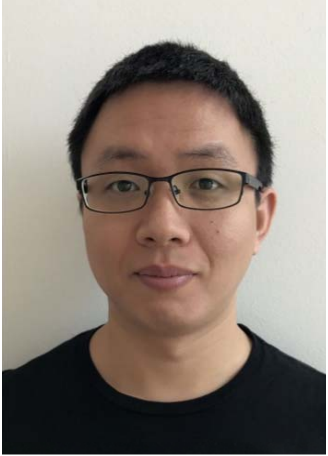

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