The continuity of maximal operators in Sobolev and BV spaces

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Introduction

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This is an important object in analysis. Particularly, it is useful when proving pointwise convergence a.e.

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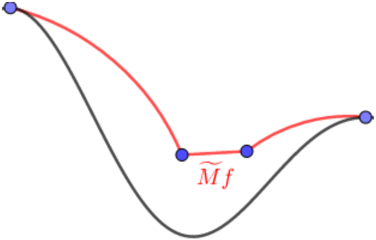

Figure: $\widetilde{M}f \geq f$

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is bounded from $W^{1,p}(\mathbb{R}^d)$ to itself, where $p > 1$. Later, Luiro proved the continuity of this map.

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However, we are interested at the derivative level behavior, therefore the natural map to analyze is

$$f \mapsto \nabla \widetilde{M}f.$$

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- We have that $f \mapsto \nabla M_c f$ is bounded from $W^{1,1}(\mathbb{R}) \to L^1(\mathbb{R})$ (Kurka 12').

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For $f \in W^{1,1}(\mathbb{R})$ we have that $\widetilde{M}f$ has no local maxima $\{\widetilde{M}f > f\}$.

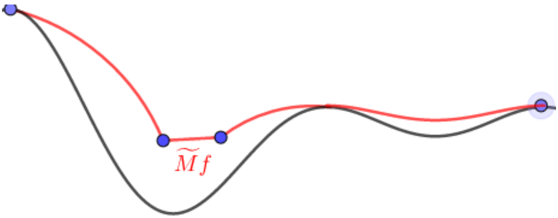

Figure: No local maxima

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The map $f \mapsto \nabla \widetilde{M} f$ is bounded in dimension 1 and in the radial case. However, it is not sublinear. An interesting follow up would be to determine if this map is continuous.

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Properties

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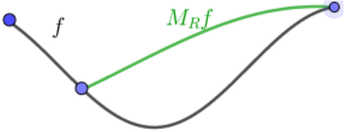

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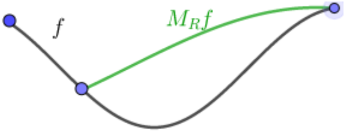

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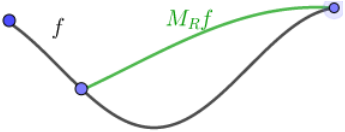

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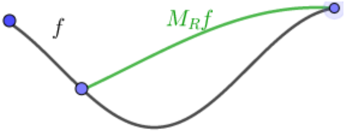

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After settling those properties the remaining argument is a carefully implemented measure theoretical argument based on a dichotomy and the **Brezis-Lieb Lemma**.

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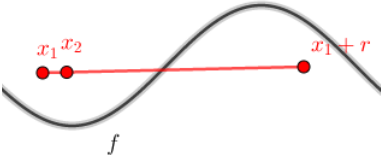

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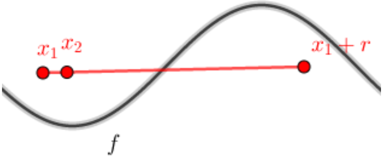

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from where we have $M_R f(x_2) > M_R f(x_1)$, from where we conclude.

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So, a main objective is to construct some suitable replacement for these M_R and M_L in higher dimensions.

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Establishing, thus, the first result of this kind (in the classical setting) for **higher dimensions**.

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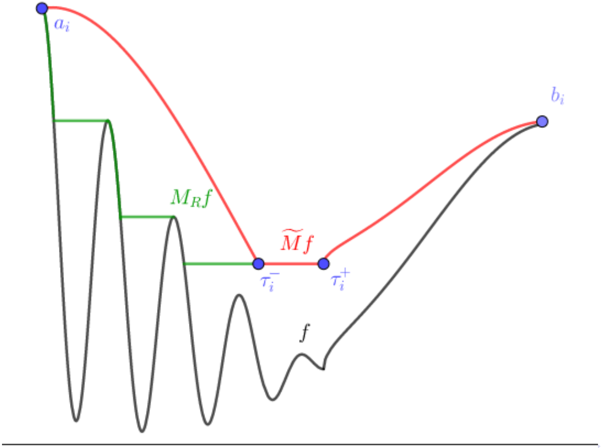

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The remaining of the proof is a technical measure theoretical argument, based on the **Brezis-Lieb lemma** and inspired in the one-dimensional result.

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$$M_\beta f(x) := \sup_{r>0} r^\beta \fint_{B(x,r)} |f|.$$

The boundedness of $f \mapsto \nabla M_\beta f$ from $W^{1,1}(\mathbb{R}^d)$ to $L^{\frac{d}{d-\beta}}(\mathbb{R}^d)$ was recently established by Weigt.

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Table 1. Endpoint continuity program

———	$W^{1,1}-$continuity; continuous setting	$BV-$continuity; continuous setting	$W^{1,1}-$continuity; discrete setting	$BV-$continuity; discrete setting
Centered classical maximal operator	OPEN: Question A	OPEN: Question C	YES[2]	OPEN: Question D
Uncentered classical maximal operator	YES: Thm 1	OPEN: Question B	YES[2]	YES: Thm 2
Centered fractional maximal operator	OPEN[1]: Question F	NO[1]: Thm 4	YES[3]	NO[1]: Thm 6
Uncentered fractional maximal operator	OPEN: Question E	NO: Thm 3	YES[3]	NO: Thm 5

Figure: Endpoint continuity program

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Centered classical maximal operator	OPEN: Question A	OPEN: Question C	YES[2]	OPEN: Question D ✓
Uncentered classical maximal operator	YES: Thm 1	OPEN: Question B	YES[2]	YES: Thm 2
Centered fractional maximal operator	OPEN[1]: Question F ✓	NO[1]: Thm 4	YES[3]	NO[1]: Thm 6
Uncentered fractional maximal operator	OPEN: Question E ✓	NO: Thm 3	YES[3]	NO: Thm 5

Figure: After Madrid and Beltran-Madrid

The $BV(\mathbb{R})$ setting

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Also, for $f \in BV(\mathbb{R})$, we have that $\widetilde{M}f$ is absolutely continuous.

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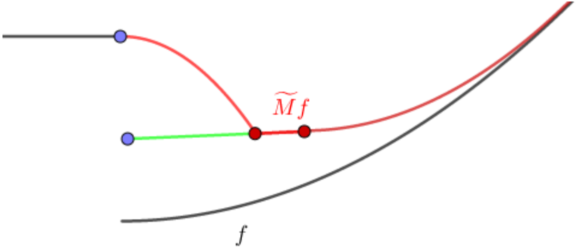

Figure: Discontinuity

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- Some classical formulas about the derivative $\nabla \widetilde{M} f$ do not hold.Those formulas are used in order to determine the pointwise convergence at the derivative level.
- Since f does not necessarily go to 0 at ∞ we need to consider the possibility of *good balls* with infinite radius.

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G-R., Kosz, 2020

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- Prove that $(\widetilde{M}f)' = 0$ in C.(flatness)
- (!!) Prove that $Var(\widetilde{M}f_j) \to Var(\widetilde{M}f)$.

We conclude by a measure theoretical argument based on Brezis-Lieb lemma.

The endpoint continuity program now and future goals

TABLE 1. Endpoint continuity program

	$W^{1,1}$–continuity; continuous setting	BV–continuity; continuous setting	$W^{1,1}$–continuity; discrete setting	BV–continuity; discrete setting
Centered classical maximal operator	OPEN	OPEN	YES[2]	YES[4]
Uncentered classical maximal operator	YES[1]	YES: Theorem 1	YES[2]	YES[1]
Centered fractional maximal operator	YES[5]	NO[1]	YES[3]	NO[1]
Uncentered fractional maximal operator	YES[4]	NO[1]	YES[3]	NO[1]

Figure: The program nowadays

Thank you for your attention