A geometric trapping approach to global regularity for 2D Navier-Stokes on manifolds

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Abstract

- We use frequency decomposition techniques to give a direct proof of global existence and regularity for the Navier-Stokes equations on two-dimensional Riemannian manifolds without boundary. The main tools include:
  - Mattingly and Sinai's method of geometric trapping on the torus.
  - Zaher Hani's refinement of multilinear estimates in the study of NLS.
  - Ideas from microlocal analysis.

Outline

Outline for section 1

Navier-Stokes

Recall the incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t U + \operatorname{div}(U \otimes U) - \nu \Delta_M U & = -\operatorname{grad} p \quad \text{in } M \\ \operatorname{div} U & = 0 \quad \text{in } M \\ U(0, \cdot) & = U_0 \quad \text{smooth} \end{cases}, \tag{1}$$

where:

- (M, g): closed, oriented, connected, compact smooth two-dimensional Riemannian manifold without boundary.
- $\nu > 0$: viscosity.
- Δ_M : any choice of Laplacian defined on vector fields (to be discussed).

History

- Navier-Stokes: too many to list.
- Global regularity for 2D N-S on flat spaces: well-known (Ladyzhenskaya, Fujita-Kato etc.).
    - Reason: enstrophy estimate (controlling the vorticity).

History

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- Global regularity for 2D N-S on flat spaces: well-known (Ladyzhenskaya, Fujita-Kato etc.).
  - Reason: enstrophy estimate (controlling the vorticity).
- In Mattingly and Sinai (1999) *An Elementary Proof of the Existence and Uniqueness Theorem for the Navier-Stokes Equations*: a simple proof of global regularity by directly working with Fourier coefficients.
  - Main idea: geometric trapping / maximum principle.
- In Pruess, Simonett, and Wilke (2020) *On the Navier-Stokes Equations on Surfaces*: local existence, and (assuming small data) global existence. Uses Fujita-Kato approach (heat semigroup etc.).

The Laplacian

Due to curvature, there are three canonical choices for the vector Laplacian:

- the Hodge-Laplacian $\Delta_H = -(d\delta + \delta d)$, where d is the exterior derivative (like gradient), and $\delta = -\mathrm{div}$ is the dual of d.
- the connection Laplacian (or *Bochner Laplacian*) $\Delta_B T := \mathrm{tr}\left(\nabla^2 T\right) = \nabla_i \nabla^i T$
  - $\Delta_B X = \Delta_H X + \mathrm{Ric}(X)$ (Weitzenbock formula, Ric: Ricci tensor)
- the deformation Laplacian $\Delta_D X = -2\mathrm{Def}^*\mathrm{Def}\, X = \Delta_H X + 2\,\mathrm{Ric}(X)$ for $\mathrm{div}\, X = 0$.

They differ by a smooth zeroth-order operator.

Main result

Theorem

Let (M, g) be a manifold as described above, and let Δ_M be any of the vector Laplacian operators Δ_H, Δ_B, or Δ_D on M.
Suppose that U_0 is a smooth vector field. Then there exists a unique global-in-time smooth solution $U : \mathbb{R} \to \mathfrak{X}(M)$ to the Navier-Stokes equation.

Obstacles on the sphere

Aynur: How to generalize Mattingly and Sinai's approach to the sphere?

- 1st approach: use the spherical harmonics (eigenfunctions) as replacement for $e^{i2\pi x}$. Does not work.
  - poor spectral localization of products on the sphere (unlike $e^{i2\pi\langle k_1,z\rangle}e^{i2\pi\langle k_2,z\rangle} = e^{i2\pi\langle k_1+k_2,z\rangle}$). Resulting frequency is bounded by triangle inequalities.
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Solution

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  - Instead of Holder's inequality on Fourier coefficients, we use multilinear estimates for eigenfunctions.
  - We find ourselves replicating the works of Zaher Hani, Nicolas Burq, Patrick Gérard, etc. from the study of non-linear Schrödinger equations. (Hani 2011; Burq, Gérard, and Tzvetkov 2005)
    - ★ Need to extend their estimates to handle more derivatives and the inverse Laplacian.

Generalizing to manifolds

How about general compact manifolds? There are 3 problems.

- Even poorer spectral localization (no triangle inequalities). The distribution of eigenvalues might no longer look like $\mathbb{N}$.
  - Instead of eigenspace projections, use spectral cutoffs. Pass between spectral cutoffs and eigenspace projections by a "Fourier trick".
  - Use Hani's refinement of multilinear estimates to handle the non-triangle regions. (main part of the proof)

Generalizing to manifolds

- There can be nontrivial harmonic 1-forms (nonzero Betti number). The vorticity equation alone does not fully describe N-S.
  - Use Hodge theory to find the correct vorticity formulation. There are cross-interactions between the second and third Hodge components (coexact and harmonic).

Generalizing to manifolds

- There can be nontrivial harmonic 1-forms (nonzero Betti number). The vorticity equation alone does not fully describe N-S.
  - Use Hodge theory to find the correct vorticity formulation. There are cross-interactions between the second and third Hodge components (coexact and harmonic).
- Ricci tensor is no longer a constant. So it does not commute with spectral cutoffs.
  - Use common ideas from microlocal analysis, like integration by parts and the method of stationary phase.

Outline for section 2

1. Introduction

2. The proof

Hodge theory

We assume all the standard results of Hodge theory:

- For any vector field (or function, or differential form) u, we have $u = \mathcal{P}_1 u + \mathcal{P}_2 u + \mathcal{P}_{\mathcal{H}} u = \text{exact} + \text{coexact} + \text{harmonic}$.
  - Range of $\mathcal{P}_{\mathcal{H}}$ is smooth and finite-dimensional (on which all Sobolev norms are equivalent). It is the frequency zero.
- Δ_H is bijective from $(1 - \mathcal{P}_{\mathcal{H}})\, H^{m+2}\Omega^k\,(M)$ to $(1 - \mathcal{P}_{\mathcal{H}})\, H^m\Omega^k\,(M)$, where $H^m\Omega^k$ = differential k-forms with coefficients in H^m. This defines the inverse Laplacian.

$$\|u\|_{H^m} \sim \|\mathcal{P}_{\mathcal{H}} u\|_{L^2} + \|(-\Delta_H)^{m/2}(1 - \mathcal{P}_{\mathcal{H}})u\|_{L^2}$$

Spectral cutoffs

- Define the **eigenspace projections** π_s such that $(-\Delta_H)\,\pi_s = s^2\pi_s$.
- Define the **frequency cutoff projections**

$$P_k = 1_{[k,k+1)}\left(\sqrt{-\Delta_H}\right) = \sum_{s\in\sigma\left(\sqrt{-\Delta}\right)\cap[k,k+1)} \pi_s$$

- Unlike π_s, P_k allows us to bypass problems with distribution of eigenvalues (Weyl's law).
  Disadvantage: $(-\Delta_H)^{-c}\,P_k \neq k^{-2c}P_k$. Luckily, there is a "Fourier trick" to relate π_s and P_k.

Vorticity

- Via the Riemannian metric g, the musical isomorphism identifies vector fields with 1-forms: $\flat X(Y) := g\left(X, Y\right)$, $g\left(\sharp\alpha, Y\right) = \alpha(Y)$ for vector fields X, Y and 1-form α.
- The vorticity ω is defined as $\omega := \star d\flat U$ where $\star$ is the Hodge star (turning gradient into divergence, and volume forms into scalars etc.).

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- The vorticity ω is defined as $\omega := \star d\flat U$ where $\star$ is the Hodge star (turning gradient into divergence, and volume forms into scalars etc.).
- ω being a scalar is crucial for the enstrophy estimate (unlike in 3D Navier-Stokes).
  - If we define $\operatorname{curl} f = -(\star df)^\sharp$, then

$$(1 - \mathcal{P}_{\mathcal{H}})\, U = \mathcal{P}_2 U = \operatorname{curl}(-\Delta)^{-1}\omega.$$

    Unlike on flat spaces, ω only controls the non-harmonic part of U.

Vorticity formulation

- Let λ_1 be the smallest nonzero eigenvalue of $\sqrt{-\Delta_H}$ (smallest frequency).
- Let $Z \subset \mathbb{N}_0 + \lambda_1$ be a finite subset selecting the modes included in the Galerkin approximation. Define $U_Z = P_Z U := \sum_{k \in Z} P_k U$.
- The truncated vorticity equation is

$$\begin{cases} U_Z &= \mathcal{P}_{\mathcal{H}} U_Z + \operatorname{curl}(-\Delta)^{-1} \omega_Z, \\ 0 &= \partial_t \omega_Z + P_Z \nabla_{U_Z} \omega_Z - \nu P_Z \star d \Delta_M \flat U_Z, \\ 0 &= \partial_t \mathcal{P}_{\mathcal{H}} U_Z + \mathcal{P}_{\mathcal{H}} \nabla_{U_Z} U_Z - \nu \mathcal{P}_{\mathcal{H}} \Delta_M U_Z, \end{cases} \tag{2}$$

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- Finite-dimensional ODE $\rightarrow$ smooth solution in local time.

Basic estimates

We have some basic estimates:

- *Energy inequality*: $\|U_Z(t)\|_{L^2} \leq \|U_Z(0)\|_{L^2}$.
- *Enstrophy estimate*: $\|\omega_Z(t)\|_{L^2} \lesssim_{\neg Z} (\|\omega_Z(0)\|_{L^2} + \|U_Z(0)\|_{L^2})\, e^{\nu C t}$ for some $C > 0$.
  - $\lesssim_{\neg Z}$ means the implied constant does not depend on Z.
  - enstrophy is non-increasing when $\Delta_M = \Delta_H$ $(F = 0)$, like on flat spaces.

$\rightarrow$ U_Z exists globally in time, by Picard's theorem.

A priori estimate

As $Z \uparrow \mathbb{N}_0 + \lambda_1$, we hope to recover the true Navier-Stokes solution. For smooth convergence, we will need the following $Z-$independent estimate:

Theorem

If for some $A_0 \in (0,\infty)$ and $r > 1$,

$$\|U_Z(0)\|_2 \leq A_0 \quad \textit{and} \quad \|P_k \omega_Z(0)\|_2 \leq \frac{A_0}{|k|^r} \; \forall k \in Z,$$

then

$$\|P_k \omega_Z(t)\|_2 \leq \frac{A^*(t)}{|k|^r} \; \forall t \geq 0, \forall k \in Z$$

for some smooth $A^*(t)$ depending on r, ν, M, A_0 and not Z.

- This just means Sobolev norms, if bounded at time 0, are smoothly controlled in time, independently of Z. It is enough for global regularity.

A priori estimate

- The estimate is local in time, so it is enough to fix $T > 0$, and show the Sobolev norm is controlled on $[0, T]$: $\|P_k \omega_Z(t)\|_2 \leq \frac{A_T^*}{|k|^r}$ for some $A_T^* > 1$ depending on r, ν, M, A_0, and T, but not on Z.
  - Note: The enstrophy estimate alone only guarantees $\|P_k \omega_Z(t)\|_2 \leq \frac{A_{T,Z}^*}{|k|^r}$ for some $A_{T,Z}^*$ that depends on Z. Still, we can use this to control small k.

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- Let K_0 be a large number to be chosen later. By the enstrophy estimate, $\forall k \leq K_0$: $\|P_k\omega_Z(t)\|_2 \leq \frac{B_{T,K_0}}{|k|^r}$ for some $B_{T,K_0} > A_0$ (recall: $\|P_k\omega_Z(0)\|_2 \leq \frac{A_0}{|k|^r}$ $\forall k \in Z$)
  - We claim that when K_0 is large enough, $\|P_k\omega_Z(t)\|_2 \leq \frac{B_{T,K_0}}{|k|^r}$ also holds for $k > K_0$. Why? What happens when K_0 gets large?

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Geometric trapping.

Geometric trapping

- We aim to show that the sequence $\left(\left\|P_k \omega_Z\left(t\right)\right\|_2\right)_{k\in\mathbb{N}_0+\lambda_1}$ remains trapped in

$$\mathfrak{S}\left(K_0\right) = \left\{\left(a_k\right)_{k\in\mathbb{N}_0+\lambda_1} : a_k \leq \frac{B_{T,K_0}}{\left|k\right|^r}\ \forall k \in \mathbb{N}_0 + \lambda_1\right\}$$

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- Certainly at time $t = 0$, the sequence lies in the set (as we picked $B_{T,K_0} > A_0$).
- If the sequence tries to escape and hit the boundary, there will be t_0 and $k_0 > K_0$ such that $\|P_{k_0}\omega_Z(t_0)\|_2 = \frac{B_{T,K_0}}{|k_0|^r}$ and $\|P_k \omega_Z(t_0)\|_2 \leq \frac{B_{T,K_0}}{|k|^r}$ for all other k.
  - If we can show that $\partial_t \left(\|P_{k_0}\omega_Z(t_0)\|_2^2 \right) < 0$, then the sequence remains trapped, and the a priori estimate is proven, and we have global regularity.

Geometric trapping

- We are left to show $\partial_t \left(\| P_{k_0} \omega_Z (t_0) \|_2^2 \right) < 0$. Note that $\| P_{k_0} \omega_Z (t_0) \|_2 = \frac{B_{T,K_0}}{|k_0|^r}$ implies

$$\| \Delta P_{k_0} \omega_Z (t_0) \|_2 \sim \frac{B_{T,K_0}}{|k_0|^{r-2}}$$

This should be the biggest power of k in the equation. It comes from the viscous term in Navier-Stokes. If we can show all other terms are dominated by the viscous term, then the vorticity equation roughly implies

$$\partial_t \left(\frac{1}{2} \| P_{k_0} \omega_Z (t_0) \|_2^2 \right) \approx \nu \langle\langle \Delta P_{k_0} \omega_Z (t_0), P_{k_0} \omega_Z (t_0) \rangle\rangle < 0$$

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  and we are done.

- **Summary**: We have reduced global regularity to viscous domination.

Viscous domination

To make things easier to follow, we remove any references to Navier-Stokes and make a self-contained statement.

Theorem

Let $w \in C^\infty(M)$ *and* $u \in \mathcal{P}_{\mathcal{H}}\mathfrak{X}(M)$. *Let* $A, B \geq 1$ *and* $k \in \mathbb{N}_0 + \lambda_1 + 1$. *Let* $r > 1$. *Assume that* $\pi_0 w = 0$ *and* $\|P_l w\|_2 \leq \frac{A}{|l|^r}$ *for all* $l \in \mathbb{N}_0 + \lambda_1$. *Assume also that* $\|w\|_2 + \|u\|_2 \leq B$. *Then*

$$\sum_{l_1, l_2 \in \mathbb{N}_0 + \lambda_1} \left\| P_k \left\langle \operatorname{curl} (-\Delta)^{-1} P_{l_1} w, \nabla P_{l_2} w \right\rangle \right\|_2 + \sum_{l \in \mathbb{N}_0 + \lambda_1} \left\| P_k \left\langle \mathcal{P}_{\mathcal{H}} u, \nabla P_l w \right\rangle \right\|_2 + \left\| P_k D^1 \mathcal{P}_{\mathcal{H}} u \right\|_2 + \sum_{l \in \mathbb{N}_0 + \lambda_1} \left\| P_k D^2 \operatorname{curl} (-\Delta)^{-1} P_l w \right\|_2 \lesssim_{M,r} \frac{AB}{|k|^{r - \frac{7}{4}}}$$

We note that D^j is schematic notation for any smooth differential operator of order j.

Bilinear estimate

To prove viscous domination, the first tool we need is a generalisation of the bilinear estimate from the study of NLS.

Lemma

For any $f, g \in L^2(M)$ and $l_1, l_2 \geq \lambda_1(M)$ and $a, b, c \in \mathbb{N}_0$, we have

$$\left\| \left(\nabla^a P_{l_1} f\right) * \left(\nabla^b (-\Delta)^{-c} P_{l_2} g\right) \right\|_2$$
$$\lesssim_{\neg l_1, \neg l_2} \min(l_1, l_2)^{\frac{1}{4}} l_1^a \| P_{l_1} f \|_2 \, l_2^{b-2c} \| P_{l_2} g \|_2$$

where $\left(\nabla^a P_{l_1} f\right) * \left(\nabla^b P_{l_2} g\right)$ is schematic for any contraction of the two tensors.

- The factor $\min(l_1, l_2)^{\frac{1}{4}}$ is not present on the torus, but is essentially sharp on the sphere.

Trilinear estimate

The second tool is an adaptation of the first, for distant regions of frequency interactions.

Lemma

For any $f_1, f_2, f_3 \in L^2(M)$; $a_1, b_1, a_2, b_2, a_3, b_3, J \in \mathbb{N}_0$ *and* $l_1 \geq l_2 \geq l_3 \geq \lambda_1(M)$ *such that* $l_1 = l_2 + Kl_3 + 2$ *for* $K > 1$, *we have*

$$\left| \int_M \left(\nabla^{a_1} (-\Delta)^{-b_1} P_{l_1} f_1 \right) * \left(\nabla^{a_2} (-\Delta)^{-b_2} P_{l_2} f_2 \right) * \left(\nabla^{a_3} (-\Delta)^{-b_3} P_{l_3} f_3 \right) \right|$$
$$\lesssim_{J, M, \neg l_1, \neg l_2, \neg l_3} \frac{l_3^{\frac{1}{4}}}{K^J} \prod_{j=1}^{3} l_j^{a_j - 2b_j} \left\| P_{l_j} f_j \right\|_2$$

- This essentially says that the distant regions are “negligible”.

Convective term

With the bilinear and trilinear estimate, we can now handle the main term in the problem of viscous domination.
Assuming $\|P_l w\|_2 \leq \frac{A}{|l|^r} \forall l$ and $\|w\|_2 = \big\| \|P_j w\|_2 \big\|_{l_j^2(\mathbb{N}_0+\lambda_1)} \leq B$, we show that for any k:

$$\sum_{l_1, l_2 \in \mathbb{N}_0+\lambda_1} \left\| P_k \left\langle \operatorname{curl} (-\Delta)^{-1} P_{l_1} w, \nabla P_{l_2} w \right\rangle \right\|_2 \lesssim \frac{AB}{k^{r-\frac{7}{4}}}$$

- Note that k, l_1, l_2 are the three “frequencies” interacting.
- Strategy: split into multiple scenarios for values of k, l_1, l_2. If the argument can not be closed, assume more conditions and split further.

Diagram

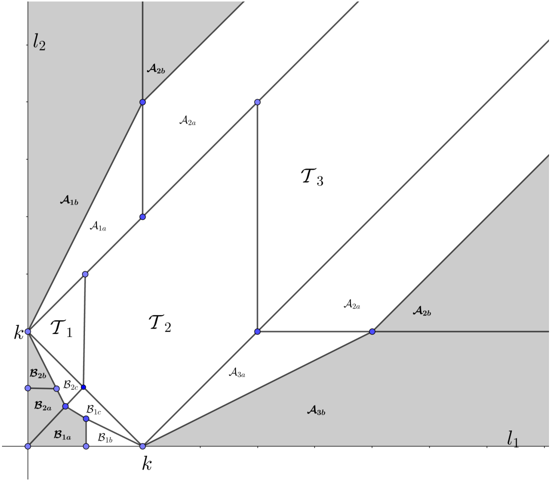

Figure: All the possible scenarios found through trial and error. Shaded regions are where the trilinear estimate is used. Example: $\mathcal{T}_2$ is defined by $|l_1 - l_2| \leq k \leq l_1 + l_2$, $\frac{k}{2} < l_1 \leq 2k$.

Example of a shaded region

- Assume $l_1 \geq k, l_2 \geq k, 2k+2 \leq |l_1 - l_2|$ (region $\mathcal{A}_{2b}$). Applying the trilinear estimate, for any J (chosen to be large), we can bound the sum by

$$\sum_{l_1} \sum_{l_2} l_1^{1/4} \frac{k^J}{|l_2 - l_1|^J} \frac{1}{l_1} \|P_{l_1} w\|_2 \, l_2 \, \|P_{l_2} w\|_2$$

$$\leq A k^J \sum_{l_1} \frac{1}{l_1^{3/4}} \|P_{l_1} w\|_2 \sum_{l_2} \frac{1}{|l_2 - l_1|^J} \cdot \frac{1}{l_2^{r-1}} \tag{3}$$

Choosing $J \in \mathbb{N}$ and $p \in (1, \infty)$ such that $Jp > 1, (r-1)\,p' > 1$ (possible since $r > 1$), we obtain:

$$(3) \lesssim A k^J \sum_{l_1} \frac{1}{l_1^{3/4}} \|P_{l_1} w\|_2 \frac{1}{k^{J-1/p}} \cdot \frac{1}{k^{r-1-1/p'}}$$

$$= A \frac{1}{k^{r-2}} \sum_{l_1} \frac{1}{l_1^{3/4}} \|P_{l_1} w\|_2 \lesssim \frac{AB}{k^{r-\frac{7}{4}}}$$

Harmonic term

Unlike the convective term, the harmonic term is very easy to handle. For any $m \in \mathbb{N}_0$:

$$k^{2m} \left\| P_k D^1 \mathcal{P}_{\mathcal{H}} u \right\|_2 \lesssim \| \mathcal{P}_{\mathcal{H}} u \|_{H^{2m+1}} \sim_m \| \mathcal{P}_{\mathcal{H}} u \|_2 \leq B$$

Linear terms

All the remaining terms that come from curvature, can be summarized by the following estimate:

Let $a, b \in \mathbb{N}_0$ such that $a - 2b \leq 1$. We write D_B^k as a schematic for a spatial differential operator of order k, such that any local coefficients $c(x)$ of D_B^k satisfy

$$\|c(x)\|_{C^m} \lesssim_m B$$

Then for all $k \in \mathbb{N}_0 + \lambda_1 + 1$,

$$\sum_{l \in \mathbb{N}_0 + \lambda_1} \left\| P_k \left(D_B^a \left(-\Delta\right)^{-b} P_l w \right) \right\|_2 \lesssim_{a,b,\neg k} \frac{AB}{k^{r-7/4}}.$$

Linear terms, critical region

Fix $\varepsilon \in \left(0, \frac{1}{2}\right)$. Handling the "critical region" $l \in [k - k^\varepsilon, k + k^\varepsilon]$ (where $l \sim_\varepsilon k$) is simple:

$$\sum_{l \in [k-k^\varepsilon, k+k^\varepsilon]} \left| \left\langle \left\langle D_B^a \left(-\Delta\right)^{-b} P_l w, P_k v_l \right\rangle \right\rangle \right|$$

$$\lesssim \sum_l l^{1/4} l^{a-2b} B \left\| P_l w \right\|_2$$

$$\sim_\varepsilon \sum_l \frac{AB}{k^{r-a+2b-\frac{1}{4}}}$$

$$\lesssim \frac{AB}{k^{r-a+2b-\frac{1}{4}-\varepsilon}} \lesssim \frac{AB}{k^{r-\frac{7}{4}}}$$

as $a - 2b \leq 1$ and $\varepsilon < \frac{1}{2}$.

Linear terms, distant region

Sketch:

- We pass from frequency cutoffs P_k to eigenspace projections π_s which diagonalize $(-\Delta)^{-b}$.
- We integrate by parts with commutators. We use the fact that $[D_B^a, -\Delta_H] = D_B^{a+1}$ (the principal symbol of Δ_H is a constant which commutes with the principal symbol of D_B^a).

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- We pass from frequency cutoffs P_k to eigenspace projections π_s which diagonalize $(-\Delta)^{-b}$.
- We integrate by parts with commutators. We use the fact that $[D_B^a, -\Delta_H] = D_B^{a+1}$ (the principal symbol of Δ_H is a constant which commutes with the principal symbol of D_B^a).
- Finally we use a "Fourier trick" to change from π_s back to P_k, which gives arbitrary decay $\frac{1}{k^\infty}$. Main idea of "Fourier trick": decompose a smooth symbol into multilinear pieces by the Fourier inversion theorem, and use the fact that the L^2 norm is modulation-independent:

$$\left\| \sum_z e^{i2\pi z\theta} \pi_{l+z} f \right\|_2 = \left\| \sum_z \pi_{l+z} f \right\|_2$$

Appendix: proving the trilinear estimate

To see that the bilinear estimate implies the trilinear estimate, we just need the Fourier trick, as well as the following integration by parts lemma:

For $i = 1, 2, 3, 4$, let $e_i \in C^\infty(M)$ be eigenfunctions where $(-\Delta) e_i = n_i^2 e_i$, and assume $n_1 \geq n_2 \geq n_3 \geq n_4 \geq 0$ and $n_1^2 \neq n_2^2 + n_3^2 + n_4^2$. Set $\mathcal{N} = \frac{1}{n_1^2 - n_2^2 - n_3^2 - n_4^2}$. Then, for any $a_1, a_2, a_3, a_4 \in \mathbb{N}_0$ and $m \in \mathbb{N}_1$, we have the schematic identity

$$\int_M (\nabla^{a_1} e_1) * (\nabla^{a_2} e_2) * (\nabla^{a_3} e_3) * (\nabla^{a_4} e_4)$$

$$= \mathcal{N}^m \sum_{\substack{b_2+b_3+b_4=2m \\ 0 \leq b_2, b_3, b_4 \leq m}} \int_M \nabla^{a_1} e_1 * \nabla^{a_2+b_2} e_2 * \nabla^{a_3+b_3} e_3 * \nabla^{a_4+b_4} e_4$$

$$+ \mathcal{N}^m \sum_{\substack{\sum_j c_j \leq \sum_j a_j + 2m-2 \\ 0 \leq c_j \leq a_j + m - 1 \ \forall j \neq 1 \\ c_1 \leq a_1}} \int_M T_{mc_1c_2c_3c_4} * \nabla^{c_1} e_1 * \nabla^{c_2} e_2 * \nabla^{c_3} e_3 * \nabla^{c_4} e_4$$

for some smooth tensors $T_{mc_1c_2c_3c_4}$.

Future

- How about Mattingly and Sinai's results regarding analytic solutions? (most likely to hold)
- How about manifolds with boundary, non-compact manifolds and exterior domains? (possibly non-trivial)
- Original goal of Aynur: how about other equations like SQG? (to be explored)

For Further Reading I

- Burq, Nicolas, Patrick Gérard, and Nikolay Tzvetkov (2005). "Multilinear Eigenfunction Estimates and Global Existence for the Three Dimensional Nonlinear Schrödinger Equations". In: *Annales scientifiques de l'École Normale Supérieure* 38.2, pp. 255–301. DOI: 10.1016/j.ansens.2004.11.003. URL: http://www.numdam.org/item/ASENS_2005_4_38_2_255_0/ (visited on 06/18/2020).
- Hani, Zaher (Nov. 15, 2011). *Global Well-Posedness of the Cubic Nonlinear Schr\"odinger Equation on Compact Manifolds without Boundary*. arXiv: 1008.2826 [math]. URL: http://arxiv.org/abs/1008.2826 (visited on 05/21/2020).
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- Pruess, Jan, Gieri Simonett, and Mathias Wilke (May 2, 2020). *On the Navier-Stokes Equations on Surfaces*. arXiv: 2005.00830 [math]. URL: http://arxiv.org/abs/2005.00830 (visited on 06/29/2020).

Thank you for listening.